1 Notation

We will work on $L^2(a,b)$, where $-\infty < a < b < \infty$. The symbol $AC(a,b)$ denotes all functions from $L^2(a,b)$ that are absolutely continuous on every compact subset of (a,b). Similarly $AC^2(a,b)$ means all functions from $L^2(a,b)$ that are differentiable and theirs derivative is absolutely continuous on every compact subset of (a,b).

2 First derivative

Because $C_0^\infty(a,b)$ is dense, the perpendicular subspace $C_0^\infty(a,b)^\perp$ is zero. Thus for $f \in L^2(a,b)$

$$(f,u) = 0, \forall u \in C_0^\infty(a,b) \quad \Rightarrow f = 0 \tag{1}$$

Now suppose that for some $f \in L^2(a,b)$ and $\forall u \in C_0^\infty(a,b)$

$$(f,u') = \int_a^b \overline{f(x)} u'(x) dx = 0 \tag{2}$$

We will show that this implies $f = C$ almost everywhere on (a,b) for some $C \in \mathbb{C}$. This implication can be interpreted in terms of distributions[1]. To show it directly let us define the following function (cap shaped function)

$$\omega_\epsilon(x) = \begin{cases} C_\epsilon \exp(-\frac{\epsilon^2}{\epsilon^2 - |x|^2}), & |x| \le \epsilon \\ 0 & |x| \ge \epsilon \end{cases} \tag{3}$$

where C_ϵ is chosen in order to $\int_{-\infty}^{\infty} \omega_\epsilon(x) dx = 1$. It can be shown that $\omega_\epsilon \in C_0^\infty(\mathbb{R})$. If we now use the shifted version $\tilde{\omega}_\epsilon(x) = \omega_\epsilon(x - (a+b)/2)$ for $0 < \epsilon < (b-a)/2$ then $\int_a^b \tilde{\omega}_\epsilon(x) dx = 1$ and $\tilde{\omega}_\epsilon \in C_0^\infty(a,b)$.

For any $\varphi \in C_0^\infty(a,b)$ the function

$$\psi(x) = \int_a^x \left[\varphi(y) - \tilde{\omega}_\epsilon(y) \int_a^b \varphi(t) dt \right] dy \in C_0^\infty(a,b). \tag{4}$$

This means

$$0 = (f, \psi') = (f, \varphi - \tilde{\omega}_\epsilon \int_a^b \varphi(t) dt) = (f, \varphi) - (f, \tilde{\omega}_\epsilon) \int_a^b \varphi(t) dt. \tag{5}$$

Consequently, writing $(f, \tilde{\omega}_\epsilon) = \overline{C}$, we obtain

$$(f, \varphi) = \overline{C} \int_a^b \varphi(t) dt = (C, \varphi). \tag{6}$$

[1]Because f is also in $L^1(a,b)$ (Minkowski inequality) it represents a regular generalised function $f \in \mathscr{D}'$ in the sense of generalized functions (see 2.5 in [1]). The equation (2) thus means that $(f',u) = -(f,u') = 0$ for all $u \in \mathscr{D} \equiv C_0^\infty(a,b)$. This implies that $f = \text{const.}$ in $\mathscr{D}'$ (see 6.3 (e)). Since f is locally integrable it together with the Du Bois Reymonds Lemma gives that there exists $C \in \mathbb{C}$ such that $f(x) = C$ a.e. on (a,b).

This holds for every $\varphi \in C_0^\infty(a,b)$ and thus $f = C$ almost everywhere on (a,b).

This can help us to determine the adjoint operator to the operator defined as

$$D(P_0) = C_0^\infty(a,b), \qquad P_0 f = -if'. \tag{7}$$

P_0 is densely defined and symmetric. Let now determine its adjoint P_0^*. The domain of P_0^* is given by those $g \in L^2(a,b)$ for which exists $f \in L^2(a,b)$ such that for all $u \in D(P_0)$

$$(f,u) = (g, P_0 u). \tag{8}$$

Since f is also in $L^1(a,b)$, we can define

$$w(x) = \int_a^x f(y)dy. \tag{9}$$

Relation (27) can be rewritten as

$$0 = (g, -iu') - (f,u) = \int_a^b -i\overline{g(x)}u'(x) - \overline{f(x)}u(x)dx = \tag{10}$$

$$= \int_a^b -i\overline{g(x)}u'(x) + \overline{w(x)}u'(x)dx - [\overline{w(x)}u(x)]_a^b = \tag{11}$$

$$= \int_a^b (-i\overline{g(x)} + \overline{w(x)})u'(x)dx. \tag{12}$$

This in the sense of previous discussion implies $g(x) = iw(x) + C$ for some $C \in \mathbb{C}$ almost everywhere (a,b). It means that g is differentiable almost everywhere and is an integral of its derivative. Therefore g is absolutely continuous: $g \in AC(a,b)$. For such functions, per partes can be performed

$$(g, -iu') = -[i\overline{g(x)}u(x)]_a^b + i(g', u) = (-ig', u) \tag{13}$$

We can summarize obtained result

$$D(P_0^*) = \{g \in AC(a,b) | g' \in L^2(a,b)\}, \qquad P_0^* g = -ig'. \tag{14}$$

3 Second derivative

To do the similar analysis for the free Laplacian operator we will use more complicated approach. Let start with an integral operator K with the kernel

$$k(x,y) = |x-y|\,\eta(x-y), \tag{15}$$

where $\eta(x)$ is a $C^\infty(\mathbb{R})$ function such that

$$\eta(x) = \begin{cases} 1, & |x| \le \epsilon/2 \\ 0, & |x| \ge \epsilon \end{cases} \tag{16}$$

Such function can be obtained as a convolution of an interval indicator $\mathbb{1}_{(-3/4\epsilon, 3/4\epsilon)}(x)$ and the cap function $\omega_{\epsilon/4}(x)$ (given by (3)). The operator K is Hilbert-Schmidt on $L^2(a,b)$ if $-\infty < a < b < \infty$. The kernel $k(x,y)$ is infinitely differentiable except for $x = y$ and $k(x,y) = 0$ for $|x-y| \geq \epsilon$. Adjoint operator K^* to K is given by the kernel $k^*(x,y) = \overline{k(y,x)}$.

Let $w \in C_0^\infty(a+2\epsilon, b-2\epsilon)$ and let $u = Kw$. Obviously $u(x) = 0$ outside the interval $(a+\epsilon, b-\epsilon)$. To determine the derivative of u we rewrite the definition relation:

$$u(x) = \int_a^b k(x,y)w(y)dy = \int_a^x (x-y)\eta(x-y)w(y)dy - \int_x^b (x-y)\eta(x-y)w(y)dy. \tag{17}$$

Both integrals have continuous and infinitely differentiable inner parts. We can apply the usual formula for differentiation and obtain

$$u'(x) = |x-x|\,\eta(x-x)w(x) + \int_a^x [\eta(x-y) + (x-y)\eta'(x-y)]w(y)dy + \tag{18}$$

$$+ |x-x|\,\eta(x-x)w(x) - \int_x^b [\eta(x-y) + (x-y)\eta'(x-y)]w(y)dy = \tag{19}$$

$$= \int_a^b \mathrm{sign}(x-y)[\eta(x-y) + (x-y)\eta'(x-y)]w(y)dy = \tag{20}$$

$$= \int_a^b \mathrm{sign}(x-y)k'(x,y)w(y)dy, \tag{21}$$

where the symbol $k'(x,y)$ stands for the continuous part of the derivative of $k(x,y)$. $k'(x,y)$ is infinitely differentiable, $k'(x,y) = 0$ for $|x-y| \geq \epsilon$ and $k'(x,y) = 1$ for $|x-y| \leq \epsilon/2$. The integral operator with the kernel $k'(x,y)$ we denote by K'.

To determine the second derivative of u we use the same trick,

$$u''(x) = k'(x,x)w(x) + \int_a^x \partial_x k'(x,y)w(y)dy + \tag{22}$$

$$+k'(x,x)w(x) - \int_x^b \partial_x k'(x,y)w(y)dy = \tag{23}$$

$$= 2w(x) + \int_a^b k''(x,y)w(y)dy. \tag{24}$$

By $k''(x,y)$ we mean the function

$$k''(x,y) = \begin{cases} \partial_x k'(x,y), & y < x \\ 0, & y = x \\ -\partial_x k'(x,y), & y > x, \end{cases} \tag{25}$$

where $\partial_x k'(x,y)$ stands for the partial derivative of $k'(x,y)$ with respect to x. Since $\partial_x k'(x,y) = 0$ for $|x-y| \leq \epsilon/2$, $k''(x,y)$ is continuous and infinitely differentiable. Because $k''(x,y) = 0$ for $|x-y| \geq \epsilon$, $u''(x) = 0$ outside the interval $(a+\epsilon, b-\epsilon)$. Integral operator with the kernel $k''(x,y)$ we denote by K''. Adjoint operator K''^* to K'' is given by the kernel $k''^*(x,y) = \overline{k''(y,x)}$.

If we summarize the above construction we obtained for every $w \in C_0^\infty(a+2\epsilon, b-2\epsilon)$ the function $u = Kw$ that is $C_0^\infty(a+\epsilon, b-\epsilon)$ The two first derivatives of u are given by $u' = K'w$ resp. $u'' = 2w + K''w$. The higher derivatives of u are easily given by the derivatives of w and $K''w$ where the latter can be performed on the integral kernel under the integral sign.

Now we can proceed to evaluation of the adjoint operator to the free laplacian defined as

$$D(T_0) = C_0^\infty(a,b), \qquad T_0 f = -f''. \tag{26}$$

The domain of T_0^* is given by those $g \in L^2(a,b)$ for which exists $f \in L^2(a,b)$ such that for all $u \in D(T_0)$

$$(f,u) = (g, T_0 u). \tag{27}$$

Let us now look only on u given by $u = Kw$ with the above defined integral operator K and $w \in C_0^\infty(a+2\epsilon, b-2\epsilon)$. The definition relation leads to

$$(f, Kw) = (K^* f, w) = -(g, 2w + K''w) = -(2g + K''^* g, w). \tag{28}$$

This holds for all $w \in C_0^\infty(a+2\epsilon, b-2\epsilon)$, so $K^* f = -2g - K''^* g$ almost everywhere on $(a+2\epsilon, b-2\epsilon)$. It means that

$$g = -\frac{1}{2}\left(K^* f + K''^* g\right) \tag{29}$$

Note that previous analysis implies that $K^* f$ has at least 2 derivatives with the second be $(K^* f)'' = 2f + K''^* f$. We can therefore claim that g is almost everywhere on $(a+2\epsilon, b-2\epsilon)$ equal to function that is twice differentiable with all derivatives in $L^2(a,b)$. Since this is true for all $\epsilon > 0$, $g \in AC^2(a,b)$. Now we can perform per partes integration and obtain

$$(f, -u'') = -[\overline{f(x)} u'(x)]_a^b + (f', u') = [\overline{f'(x)} u(x)]_a^b - (f'', u). \tag{30}$$

The result can be summarized as

$$D(T_0^*) = \{g \in AC^2(a,b) | g'' \in L^2(a,b)\}, \quad T_0^* g = -g''. \tag{31}$$

References

[1] Vladimirov: Equations of mathematical physics