The Pauli equation with non-Hermitian $\mathcal{PT}$-symmetric boundary conditions

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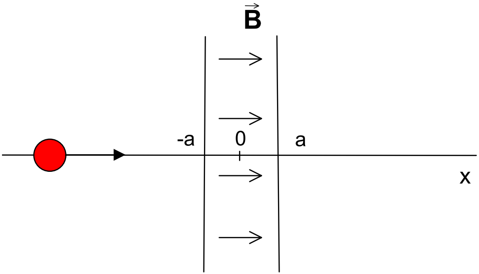
May 17, 2011

Joint work with David Krejčiřík and Petr Siegl

The Motivation

Particle passing through a constant magnetic field

- simple $\mathcal{PT}$-symmetric model - possibility to find the metric operator?
- investigation of scattering



The scattering

[HCKrSi10]

Scattering system in the reflectionless regime

can be described by

- effective Schrödinger eq. in a bounded interval
- complex Robin boundary conditions

$\mathcal{PT}$-symmetric potential $\Rightarrow$ $\mathcal{PT}$-symmetric quantum problem

Influence of the spectrum

Complex points in the spectrum correspond to the loss of perfect-transmission energies

[HCKrSi10] Hernandez-Coronado, Krejčiřík, Siegl, Perfect transmission scattering as a $\mathcal{PT}$-symmetric spectral problem, Preprint, 2010, arXiv:1011.4281

The Hamiltonian in a magnetic field

Hilbert space: $L^2((-a,a),dx)\otimes\mathbb{C}^2$

Hamiltonian

$$H=\begin{pmatrix}-\Delta+c & 0\\ 0 & -\Delta-c\end{pmatrix}$$

$$\mathrm{Dom}(H)=$$

$$\left\{\Psi\in W^{2,2}((-a,a))\,\middle|\,\begin{matrix}\Psi'(a)+A\Psi(a)=0\\ \Psi'(-a)-\overline{A}\Psi(-a)=0\end{matrix},\mathrm{A}\in\mathbb{C}^{2,2}\right\}$$

Spinors

$\Psi=\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix},\qquad \Phi=\begin{pmatrix}\phi_+\\ \phi_-\end{pmatrix}$

Properties of the Hamiltonian

Sesquilinear form

$$\begin{aligned} h[\Phi, \Psi] = (\Phi, H\Psi) &= h_0[\Phi, \Psi] + h_1[\Phi, \Psi] \\ h_0[\Phi, \Psi] &= (\Phi', \Psi') \\ h_1[\Phi, \Psi] &= c(\phi_+, \psi_+) - c(\phi_-, \psi_-) \\ &\quad + \overline{\Phi(a)} A \Psi(a) + \overline{\Phi(-a)}\, \overline{A} \Psi(-a) \end{aligned}$$

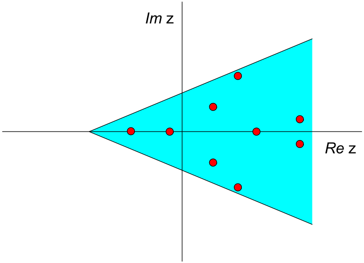
Fact

h_0 corresponds to the operator $-\Delta$ with Neumann boundary conditions.

Sectoriality

Definition

The form t is said to be sectorial if $\Theta(t)$ is a subset of a sector of the form $|\arg(\eta - \gamma)| \leq \Theta$ where $0 \leq \Theta < \frac{\pi}{2}$ and $\gamma \in \mathbb{R}$.



Sectoriality

Definition

Let t be a sectorial form in $\mathscr{H}$. A form t' in $\mathscr{H}$ is said to be relatively bounded with respect to t (or t-bounded), if $\mathrm{D}(t') \supset \mathrm{D}(t)$ and

$$|t'[u]| \leq a \|u\|^2 + b\,|t[u]| \tag{1}$$

where $u \in \mathrm{D}(t)$ and a,b are nonnegative constants.

Theorem

Let t be a sectorial form in $\mathscr{H}$ and let t' be t-bounded with $b < 1$ in (1). Then $t + t'$ is sectorial. $t + t'$ is closed if and only if t is closed.

Proposition

h_1 is h_0-bounded with bound < 1.

The first representation theorem

Theorem

Let $t[u,v]$ be a densely defined, closed, sectorial sesquilinear form in $\mathscr{H}$. There exist an m-sectorial operator T such that

i) $\mathrm{D}(T) \subset \mathrm{D}(t)$ *and* $t[u,v] = (u, Tv)$ *for every* $u \in \mathrm{D}(t)$ *and* $v \in \mathrm{D}(T)$*;*

ii) $\mathrm{D}(T)$ *is a core of* t*;*

iii) *if* $v \in \mathrm{D}(t), w \in \mathscr{H}$ *and* $t[u,v] = (u,w)$ *holds for every* u *belonging to a core of* t*, then* $v \in \mathrm{D}(T)$ *and* $Tv = w$*.*

The m-sectorial operator T is uniquely determined by the condition i).

Selfadjointness

Proposition

$h[\Psi]$ is real if and only if $A = A^\dagger$.

Symmetries of H

- $\mathcal{PT}$-symmetric by definition
- $\mathcal{P}$-self-adjoint if $A = A^T$
- self-adjoint if $A = A^\dagger$

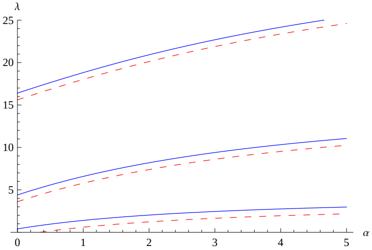
$A = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$

Spectrum of H

$$\begin{aligned} &(2\alpha k_+ \cos(2ak_+) - (k_+^2 - \alpha^2) \sin(2ak_+)) \\ &(2\alpha k_- \cos(2ak_-) - (k_-^2 - \alpha^2) \sin(2ak_-)) = 0 \end{aligned} \quad ,$$

where $k_- := \sqrt{\lambda - c}$ and $k_+ := \sqrt{\lambda + c}$

$A = A^\dagger \Rightarrow$ spectrum is real.

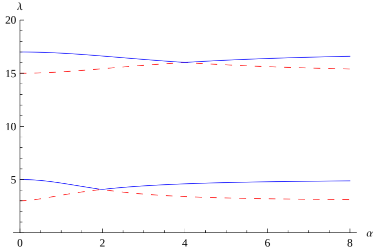


$A = \begin{pmatrix} 0 & \mathrm{i}\alpha \\ -\mathrm{i}\alpha & 0 \end{pmatrix}$

Spectrum of H

$$\tan(ak_+)\cot(ak_-) + \tan(ak_-)\cot(ak_+) = -\frac{k_+^2 k_-^2 + \alpha^4}{\alpha^2}$$

$A = A^\dagger \Rightarrow$ spectrum is real.



$A = \begin{pmatrix} \mathrm{i}\alpha+\beta & 0 \\ 0 & \mathrm{i}\alpha+\beta \end{pmatrix}$

Spectrum of H

$$\left(-2\beta k_- \cos(2ak_-) + (k_-^2 - \alpha^2 - \beta^2)\sin(2ak_-)\right)$$
$$\left(-2\beta k_+ \cos(2ak_+) + (k_+^2 - \alpha^2 - \beta^2)\sin(2ak_+)\right) = 0$$

Studied for fixed β [KrSi10]

- $\beta = 0$
- $\beta > 0$
- $\beta < 0$

[KrSi10] Krejčiřík, Siegl, $\mathcal{PT}$-symmetric models in curved manifolds, Journal of Physics A: Mathematical and Theoretical, 2010

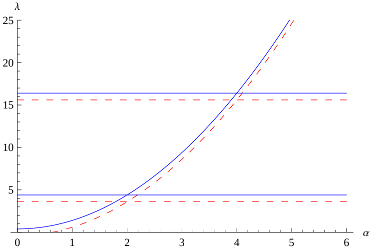
$$A = \begin{pmatrix} i\alpha+\beta & 0 \\ 0 & i\alpha+\beta \end{pmatrix}$$

Spectrum of H for $\beta = 0$

[KrBiZn06]

$$(k_-^2 - \alpha^2)(k_+^2 - \alpha^2)\sin(2ak_-)\sin(2ak_+) = 0$$

$$\lambda_{j,\pm} = \begin{cases} \alpha^2 \pm c, \\ \left(\frac{j\pi}{2a}\right)^2 \pm c \end{cases}$$



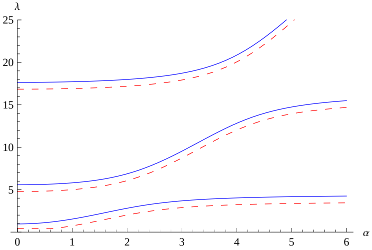
[KrBiZn06] 2006 Krejčiřík, Bíla, Znojil, *Journal of Physics A: Mathematical and General* 39

$$A = \begin{pmatrix} i\alpha+\beta & 0 \\ 0 & i\alpha+\beta \end{pmatrix}$$

Spectrum of H for $\beta > 0$

$$\left(-2\beta k_- \cos(2ak_-) + (k_-^2 - \alpha^2 - \beta^2)\sin(2ak_-)\right)$$
$$\left(-2\beta k_+ \cos(2ak_+) + (k_+^2 - \alpha^2 - \beta^2)\sin(2ak_+)\right) = 0$$

Spectrum is real.

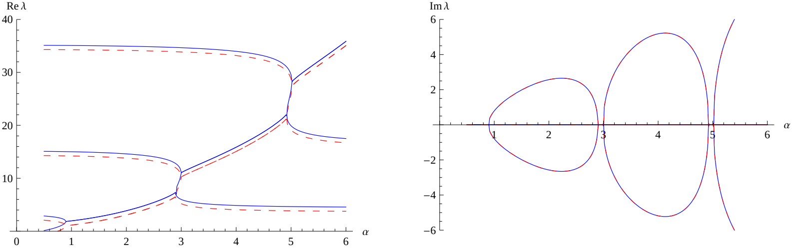


$$A = \begin{pmatrix} \mathrm{i}\alpha+\beta & 0 \\ 0 & \mathrm{i}\alpha+\beta \end{pmatrix}$$

Spectrum of H for $\beta < 0$

$$\left(-2\beta k_- \cos(2ak_-) + (k_-^2 - \alpha^2 - \beta^2)\sin(2ak_-)\right)$$
$$\left(-2\beta k_+ \cos(2ak_+) + (k_+^2 - \alpha^2 - \beta^2)\sin(2ak_+)\right) = 0$$

Complex-conjugated pairs of eigenvalues appear in the spectrum.

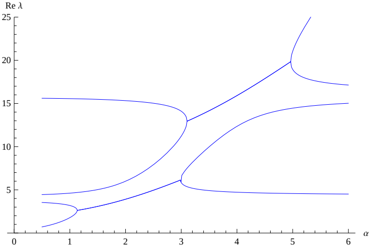


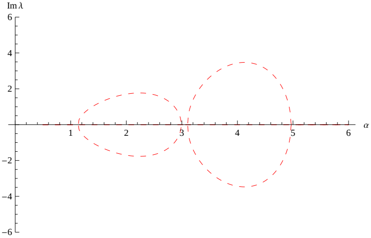
$A = \begin{pmatrix} 0 & i\alpha \\ i\alpha & 0 \end{pmatrix}$

Spectrum of H

$$\tan(ak_+)\cot(ak_-) + \tan(ak_-)\cot(ak_+) = \frac{k_+^2 k_-^2 + \alpha^4}{4\alpha^2 k_- k_+}$$

Complex-conjugated pairs of eigenvalues appear in the spectrum.





Conclusions

Results

- effects of a constant magnetic field on the spectrum
- reality of spectrum for wide range of boundary conditions

Further research

? examination of all possible cases of boundary conditions

? proofs of reality of these eigenvalues

? existence and construction of the metric operator