Název práce: **PT symetrická verze supersymetrické kvantové mechaniky**

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Abstrakt: Představíme základní myšlenky $\mathcal{PT}$-symetrie a supersymetrie a popíšeme nový model supersymetrie, který umožňuje studium $\mathcal{PT}$-symetrických systémů. Teoretické formulace podpoříme konkrétními příklady.

Klíčová slova: $\mathcal{PT}$-symetrie, pseudo-hermitovost, supersymetrie, pseudo-supersymetrie.

Title: **PT symmetric version of supersymmetric quantum mechanics**

Author: Petr Siegl

Abstract: We describe key ideas of $\mathcal{PT}$-symmetry and supersymmetry. A new model of supersymmetry enabling the examination of $\mathcal{PT}$-symmetric systems is proposed. The theoretical formulation is supported by concrete examples.

Key words: $\mathcal{PT}$-symmetry, pseudo-Hermiticity, supersymmetry, pseudo-supersymmetry.

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Contents

1 Introduction

$\mathcal{PT}$-symmetry is a very new conception which gives rise to questions in the foundations of Quantum Mechanics. Traditionally, only self-adjoint operators are treated as admissible observables. However, $\mathcal{PT}$-symmetric operators may possess a real spectrum and new questions arise [1, 2, 3]. In their light a new, so-called $\mathcal{PT}$-symmetric Quantum Mechanics has been proposed by C. Bender et al [4].

The reality of the spectrum is not the only important property of self-adjoint operators. Further aspects of the formulation of $\mathcal{PT}$-symmetric Quantum Mechanics had to be developed concerning new physical scalar product, time evolution etc. In this context, one may return to the older, nice review of Quantum Mechanics by Scholtz et al [5] to find a few key answers to the related questions.

$\mathcal{PT}$-symmetry itself may be easily understood either as a standard pseudo-Hermiticity property of operators [6] or, if needed, its suitable alternatives [7]. Its current applications within Quantum Mechanics [4, 8, 9, 10, 11] may also find further close parallels in quantum cosmology [6], in classical magnetohydrodynamics [12] etc. The aim of our present work is to review and to describe certain particularly interesting applications of $\mathcal{PT}$-symmetry in the context of supersymmetry.

The latter concept itself appeared in the context of physics in 1971 [13]. Unfortunately, the ambitious predictions based on the standard assumptions of Hermiticity and leading to the existence of the bosonic-fermionic multiplets still wait for their experimental verification. In this sense we feel strongly motivated by the new possibilities opened by the new mathematical $\mathcal{PT}-$symmetric framework.

All four parts of our text lead us from a brief formulation of the theoretical idea to the explicit and concrete examples. The appropriately modified infinite square well and harmonic oscillator are used as an illustration of the characteristic features of the theory.

2 $\mathcal{PT}$-symmetry

2.1 The origin of $\mathcal{PT}$-symmetry

In 1998, Bender and Boettcher [4] demonstrated numerically that the spectrum of a non-Hermitian Hamiltonian

$$H = -\frac{d^2}{dx^2} + x^2(ix)^\nu, \quad \nu \in \mathcal{R}^+ \tag{1}$$

is real, positive and discrete. Later, this result has been rigorously proved by P. Dorey et al [14]. The initial impulse for examination of such Hamiltonians (1) was apparently given by D. Bessis who numerically studied (1) with $\nu = 1$ and conjectured that the spectrum is real and positive. Bender and Boettcher suggested that the spectral properties of (1) have roots in $\mathcal{PT}$-symmetry of H,

$$[\mathcal{PT}, H] = 0. \tag{2}$$

In physics, the parity $\mathcal{P}$ reflects the spatial symmetry, while the complex conjugation $\mathcal{T}$ represents the time symmetry. Formally, $\mathcal{P}$ and $\mathcal{T}$ satisfy $\mathcal{P}^2 = I$, $T^2 = I$ and

$$(\mathcal{P}(\alpha\psi + \varphi))(x) = \alpha\psi(-x) + \varphi(-x), \quad (\mathcal{T}(\alpha\psi + \varphi))(x) = \alpha^*\psi^*(x) + \varphi^*(x), \tag{3}$$

where $\psi, \varphi \in \mathcal{H}$, $\alpha \in \mathcal{C}$. From the mathematical point of view, the transition from Hermiticity to the $\mathcal{PT}$-symmetry is not too drastic since the eigenvalues of H are real or coming in the complex conjugate pairs. An eigenvalue E of H corresponding to the eigenvector ψ_E is real, if we have

$$\mathcal{PT}\psi_E = \psi_E. \tag{4}$$

It is easy to prove this proposition. Let us take the vector $\mathcal{PT}\psi_E$ and with the help of (2), (4) we arrive at the relations

$$H\mathcal{PT}\psi_E = \mathcal{PT}H\psi_E = E^*\mathcal{PT}\psi_E \tag{5}$$

and

$$H\mathcal{PT}\psi_E = E\psi_E = E\mathcal{PT}\psi_E. \tag{6}$$

This implies that $E = E^*$ [15].

Of course, the symmetry of eigenvectors is not ensured in general case. The reality of eigenvalues is guaranteed in the case of the symmetry of Hamiltonian and, simultaneously, of the symmetry of eigenvectors. One often speaks about unbroken $\mathcal{PT}$-symmetry [16].

2.2 Pseudo-Hermiticity

Let $\mathcal{H}$ be a separable Hilbert space. Then a linear operator $A : \mathcal{H} \to \mathcal{H}$ is said to be pseudo-Hermitian if there exists an invertible, bounded, self-adjoint operator η: $\mathcal{H} \to \mathcal{H}$ satisfying [6, 17]

$$A^\dagger = \eta A \eta^{-1}. \tag{7}$$

The class of $\mathcal{PT}$-symmetric Hamiltonians

$$H = -\frac{d^2}{dx^2} + V(x), \tag{8}$$

where the potential V is $\mathcal{PT}$-symmetric, is $\mathcal{P}$-pseudo-Hermitian,

$$H^\dagger = -\frac{d^2}{dx^2} + V^*(x) = \mathcal{P}H\mathcal{P} = \mathcal{P}H\mathcal{P}^{-1}. \tag{9}$$

Nevertheless, $\mathcal{P}$-pseudo-Hermiticity and $\mathcal{PT}$-symmetry are distinct properties. Consider the non-Hermitian Hamiltonians

$$H_1 := P^2 + x^2 P, \quad H_2 := P^2 + i(x^2 P + P x^2), \tag{10}$$

where P is a momentum operator. H_1 is $\mathcal{PT}$-symmetric, but it is not $\mathcal{P}$-pseudo-Hermitian, whereas H_2 is P-pseudo-Hermitian and not $\mathcal{PT}$-symmetric [6]. Although H_1 is not $\mathcal{P}$-pseudo-Hermitian, the existence of operator η for which H_1 is η-pseudo-Hermitian, is not excluded. The relation between $\mathcal{PT}$-symmetric and pseudo-Hermitian Hamiltonians is studied in [18].

We restrict ourselves to the operators having a discrete nondegenerate spectrum and we assume that their eigenvectors form a biorthonormal set

$$\begin{gathered} A|n\rangle = E_n|n\rangle, \quad A^\dagger|n\rangle\rangle = E_n^*|n\rangle\rangle \\ \langle\langle n|m\rangle = \delta_{mn}, \end{gathered} \tag{11}$$

which is complete [6]

$$\sum_n |n\rangle\langle\langle n| = \sum_n |n\rangle\rangle\langle n| = I. \tag{12}$$

Theorem 1. Let A be an operator with a discrete spectrum and a complete biorthonormal set of eigenvectors. Then A is pseudo-Hermitian if and only if one of the following conditions hold

1. The spectrum of H is real

2. The complex eigenvalues come in a complex conjugate pairs and the multiplicity of complex conjugate eigenvalues are the same.

The proof may be found in [6]. The consequence of this theorem is that every $\mathcal{PT}$-symmetric operator with a discrete spectrum and a complete biorthonormal set of eigenvectors is pseudo-Hermitian.

When we return to the the original concept of $\mathcal{PT}$-symmetry, the symbol $\mathcal{T}$ denotes the complex conjugation and $\mathcal{P}$ denotes the parity. The transition from the $\mathcal{PT}$ symmetry to the $\mathcal{P}$-pseudo-Hermiticity and then to the η-pseudo-Hermiticity [6] may be understood as a shift from parity $\mathcal{P}$, with properties $\mathcal{P}^2 = I,\ \mathcal{P} = \mathcal{P}^{-1} = \mathcal{P}^\dagger$, to the Hermitian operator η. A next step lies in admitting non-Hermitian version of η

$$H^\dagger = \mathcal{R} H \mathcal{R}^{-1}, \tag{13}$$

where $\mathcal{R} \neq \mathcal{R}^\dagger$. The last formulation is proposed in [7] and the properties of such Hamiltonians are studied with the help of schematic examples there.

2.3 Quasi-Hermiticity

The most important subset of pseudo-Hermitian operators, studied in [5], are quasi-Hermitian operators. A linear operator $A : \mathcal{H} \to \mathcal{H}$ is said to be quasi-Hermitian if there exists an invertible, bounded, self-adjoint, positive operator Θ: $\mathcal{H} \to \mathcal{H}$ satisfying $A^\dagger = \Theta A \Theta^{-1}$ [5, 17]. An important theorem was proved in [15].

Theorem 2. Let A be an operator with a discrete spectrum and a complete biorthonormal set of eigenvectors. Then the a spectrum of A is real if and only A is quasi-Hermitian.

Let us take a quasi-Hermitian operator A. By definition, there exists an invertible, bounded and positive operator Θ such that $A^\dagger = \Theta A \Theta^{-1}$. We define the quadratic form $\langle\cdot,\cdot\rangle_\Theta := \langle\cdot,\Theta\cdot\rangle$. This form satisfies the requirements for being a scalar product and we may consider a Hilbert space $(\mathcal{H},\langle\cdot,\cdot\rangle_\Theta)$. The operator Θ may be called a metric operator. The operator A satisfies

$$\langle\psi, A\varphi\rangle_\Theta = \langle\psi, \Theta A\varphi\rangle = \langle\psi, A^\dagger\Theta\varphi\rangle = \langle A\psi, \Theta\varphi\rangle = \langle A\psi, \varphi\rangle_\Theta, \tag{14}$$

for arbitrary ψ, $\varphi \in \mathrm{Dom}(A)$ [5]. Hence, the eigenvalues of the quasi-Hermitian are real. However, the non-trivial existence of a positive metric operator is ensured by the Theorem 2.

In the pseudo-Hermitian case, it is used instead of the metric Θ so-called indefinite metric η. The associated quadratic form cannot be used as the scalar product. Nevertheless, existence of the positive metric operator for the pseudo-Hermitian case is not excluded. We may search for such metric operator in the form

$$\Theta = \sum_n |n\rangle\rangle s_n \langle\langle n|, \tag{15}$$

where $s_n > 0$ [15].

2.4 Examples

2.4.1 Harmonic oscillator and $\mathcal{PT}$-symmetry

Let us take the $\mathcal{PT}$-symmetric the Hamiltonian H with the potential V

$$H = -\frac{d^2}{dx^2} + (x+i\varepsilon)^2, \quad V(x) = (x+i\varepsilon)^2. \tag{16}$$

In order to solve the eigenvalue problem we follow a formal transformation from x to $y = x + i\varepsilon$ and we arrive at the differential equation

$$-\frac{d^2\varphi(y)}{dy^2} + (y^2 - E)\psi = 0. \tag{17}$$

This is equivalent to the standard harmonic oscillator eigenvalue problem. Hence, the eigenvalues and eigenvectors are known,

$$E_n = 2n+1, \quad \psi_n(x) = C_n e^{-\frac{(x+i\varepsilon)^2}{2}} H_n(x+i\varepsilon), \quad n \in \mathcal{N}_0. \tag{18}$$

C_n is a normalization constant.

2.4.2 $\mathcal{PT}$-symmetric square well

The $\mathcal{PT}$-symmetric square well [8], [9]

$$V(x) = \begin{cases} iZ, & -1 < x < 0 \\ -iZ, & 0 < x < 1 \\ \infty, & |x| > 1. \end{cases} \tag{19}$$

is a next example of the $\mathcal{PT}$-symmetric system. Its Schrödinger equation

$$-\psi''(x) + V(x)\psi(x) = E\psi(x), \quad x \in (-1, 1) \tag{20}$$

with boundary conditions $\psi(\pm 1) = 0$, may be solved for the real eigenvalue E. If φ is a solution of (20) for the real eigenvalue E, then $\mathcal{PT}\varphi$ is a solution of (20) for the same eigenvalue E. Therefore, we may search for the $\mathcal{PT}$-symmetric solution. The solution ψ, compatible with boundary conditions, may be written as

$$\psi(x) = \begin{cases} K_L \sinh[\kappa^*(1+x)], & -1 < x < 0 \\ K_R \sinh[\kappa(1-x)], & 0 < x < 1, \end{cases} \tag{21}$$

where

$$\kappa^2 = -E - iZ = (s - it)^2, \quad Z = 2st, \quad E = t^2 - s^2. \tag{22}$$

The continuity of ψ together with its first derivative in the origin imposes the conditions

$$\begin{gathered} \tfrac{K_R}{K_L} = \tfrac{\sinh \kappa^*}{\sinh \kappa}, \\ \kappa \coth \kappa + \kappa^* \coth \kappa^* = 0. \end{gathered} \tag{23}$$

We are permitted to require, according to $\mathcal{PT}$-symmetry of ψ,

$$\begin{gathered} \psi(0-) = \psi(0+) = \alpha \\ \partial_x \psi(0-) = \partial_x \psi(0+) = i\beta, \end{gathered} \tag{24}$$

where α, β are real parameters and $(0\pm)$ denotes $\lim_{x \to 0_\pm}$. Then ψ may be rewritten as

$$\psi(x) = \begin{cases} \frac{\alpha}{\sinh \kappa^*} \sinh[\kappa^*(1+x)], & -1 < x < 0 \\ \frac{\alpha}{\sinh \kappa} \sinh[\kappa(1-x)], & 0 < x < 1. \end{cases} \tag{25}$$

With the help of (22), the matching condition (23) may be put in the form

$$s \sinh 2s + t \sin 2t = 0. \tag{26}$$

This equation, together with the definition $Z = 2st$, can be numerically solved. After rescaling $t \to T = \frac{2t}{\pi}$, we see that in a Hermitian limit $Z \to 0$, the spectrum becomes standard $E_n \sim n^2$. The rescaling $s \sinh 2s = 2\sinh^2 S$ enables us to write the condition (26) in the form

$$4\sinh^2 S = -\pi T \sin(\pi T). \tag{27}$$

When we express S from (27), (22) and plot the curves $X(T), Y(Z,T)$, we see that

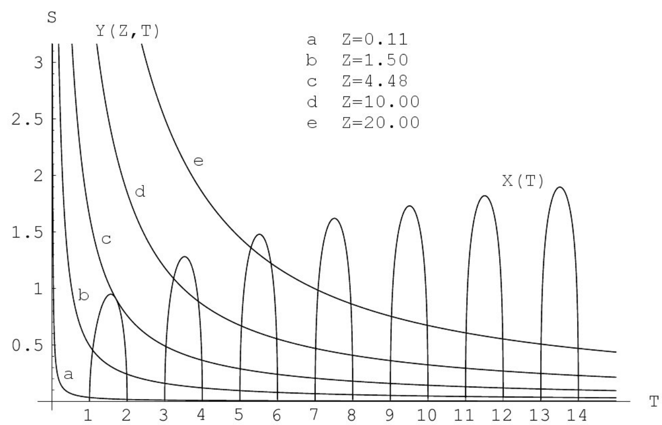

Figure 1: The curves $S = X(T)$ and $S = Y(Z,T)$ in a $S - T$ plane

the character of the spectrum changes fundamentally above critical values of Z. The first critical value $Z_0 \doteq 4,4748$ was determined with the highest accuracy in [9]. It may seem from the Figure 1 that two energy levels are vanishing above Z_0. However, Schrödinger equation (20) must be solved for complex eigenvalues. According to the $\mathcal{PT}$-symmetry of the Hamiltonian we take two complex conjugate levels E_0 and E_1,

$$E_0 = E - i\varepsilon, \quad E_1 = E + i\varepsilon. \tag{28}$$

Schrödinger equation (20) reads

$$\psi''(x) = \begin{cases} (k_L^{(n)})^2 \psi_n, & -1 < x < 0 \\ (k_R^{(n)})^2 \psi_n, & 0 < x < 1, \end{cases} \tag{29}$$

where

$$
\begin{aligned}
(k_R^{(0)})^2 &= -E + i\varepsilon - iZ = \kappa^2 = (s - it)^2, \\
(k_R^{(1)})^2 &= -E - i\varepsilon - iZ = \lambda^2 = (p - iq)^2, \\
(k_L^{(0)})^2 &= -E + i\varepsilon + iZ = (\lambda^*)^2, \\
(k_L^{(1)})^2 &= -E - i\varepsilon + iZ = (\kappa^*)^2.
\end{aligned}
\tag{30}
$$

The solution obeying the boundary conditions may be expressed as

$$
\psi_0(x) = \begin{cases} K_L \sinh[\lambda^*(1+x)], & -1 < x < 0 \\ K_R \sinh[\kappa(1-x)], & 0 < x < 1, \end{cases}
$$

$$
\psi_1(x) = \begin{cases} L_L \sinh[\kappa^*(1+x)], & -1 < x < 0 \\ L_R \sinh[\lambda(1-x)], & 0 < x < 1. \end{cases}
\tag{31}
$$

The matching conditions at $x = 0$

$$
\begin{aligned}
L_R \sinh\lambda &= L_L \sinh\kappa^*, \\
\lambda L_R \cosh\lambda &= -\kappa^* L_L \cosh\kappa^*, \\
K_R \sinh\kappa &= K_L \sinh\lambda^*, \\
\kappa K_R \cosh\kappa &= -\lambda^* K_L \cosh\lambda^*,
\end{aligned}
\tag{32}
$$

are defining relations for coefficients K_R and L_R in terms of arbitrary K_L, L_L. The intertwining relation between λ and κ results in

$$
\lambda \coth\lambda + \kappa^* \coth\kappa^* = 0. \tag{33}
$$

Hence, κ and λ define the energies E_0 and E_1. Once we express E, ε, Z in terms of s, t, p, q we get

$$
E = t^2 - s^2 = q^2 - p^2, \quad \varepsilon = pq - st, \quad Z = pq + st. \tag{34}
$$

We re-parametrize

$$
s = k \sinh\alpha, \quad t = k \cosh\alpha, \quad p = k \sinh\beta, \quad q = k \cosh\beta \tag{35}
$$

and eliminate

$$
k = \sqrt{\frac{2Z}{\sinh 2\alpha + \sinh 2\beta}}. \tag{36}
$$

We see that, the solution of the Schrödinger equation is completely determined by two real parameters α and β for which the condition (33) in appropriate form must be satisfied.

3 Supersymmetry

3.1 Superalgebra

Let G be a monoid with binary operation '$\cdot$' : $G \times G \to G$. A G-graded algebra A is a linear vector space over field $\mathcal{C}$ endowed with a bilinear binary operation $[\cdot,\cdot] : A \times A \to A$ enabling the decomposition to a direct sum

$$A = \bigoplus_{i \in G} A_i \tag{37}$$

such that

$$[A_m, A_n] \subset A_{m.n}. \tag{38}$$

Elements of A_n are called homogeneous elements of degree n. In physics, the term superalgebra refers to a $\mathcal{Z}_2$-graded algebra

$$A = A_0 \oplus A_1 \tag{39}$$

with a bilinear binary operation $[\cdot,\cdot] : A \times A \to A$, called a Lie superbracket or a supercommutator, satisfying

$$[x, y] = -(-1)^{|x||y|}[y, x] \tag{40}$$

and the super Jacobi identity

$$(-1)^{|z||x|}[x, [y, z]] + (-1)^{|x||y|}[y, [z, x]] + (-1)^{|y||z|}[z, [x, y]] = 0, \tag{41}$$

where x, y, z are homogeneous elements and $|x|$ denotes the degree of x, i.e.

$$|x| = \begin{cases} 0, & x \in A_0 \\ 1, & x \in A_1. \end{cases}$$

Since for all $x, y, z \in A_0$ the superbracket becomes the standard Lie bracket (commutator) $[x, y] = -[y, x]$ as well as the super Jacobi identity becomes the standard Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \tag{42}$$

the so-called even subalgebra A_0 forms a standard Lie algebra. We may create a Lie superalgebra from a given associative $\mathcal{Z}_2$-graded algebra A with product '$\cdot$' by defining the superbracket on homogenous elements

$$[x, y] = x \cdot y - (-1)^{|x||y|} y \cdot x \tag{43}$$

and extending this definition by linearity to all other elements.

The commutator is denoted $[\cdot,\cdot]$, the anticommutator $\{\cdot,\cdot\}$

$$[A, B] = AB - BA, \quad \{A, B\} = AB + BA. \tag{44}$$

3.2 Schrödinger's factorization method

The method, usually connected with Schrödinger, was already used by Dirac [19] before by solving the eigenvalue problem for the one dimensional harmonic oscillator [20]. In fact, origin dates back to the nineteenth century, namely to Darboux [21].

If $u(x)$ satisfies $-u''(x) + [V(x) - \varepsilon]u(x) = 0$ and if $-\theta''(x) + V(x)\theta(x) = 0$, then

$$\tilde{u}(x) := \left(-\frac{d}{dx} + \frac{\theta'(x)}{\theta(x)} \right) u(x) \tag{45}$$

solves $-\tilde{u}''(x) + [\tilde{V}(x) - \varepsilon]\tilde{u}(x) = 0$ for

$$\tilde{V}(x) := V(x) + \left(\frac{\theta'(x)}{\theta(x)} \right)'. \tag{46}$$

Inspired by this, we describe the following procedure [13]. Let H_1 be a Hamiltonian

$$H_1 = -\frac{d^2}{dx^2} + V_1(x). \tag{47}$$

We factorize H_1 using ansatz

$$H_1 = A^\dagger A, \tag{48}$$

$$A = \frac{d}{dx} + W(x), \quad A^\dagger = -\frac{d}{dx} + W(x). \tag{49}$$

Hence

$$H_1 = A^\dagger A = -\frac{d^2}{dx^2} + W^2(x) - W'(x) \tag{50}$$

and we identify potential V_1 with

$$V_1(x) = W^2(x) - W'(x). \tag{51}$$

We construct a new Hamiltonian H_2

$$H_2 = AA^\dagger = -\frac{d^2}{dx^2} + W^2(x) + W'(x) \tag{52}$$

and we define a new potential V_2

$$V_2(x) = W^2(x) + W'(x). \tag{53}$$

Equation (51) is actually the definition of the function $W(x)$, usually referred to as the superpotential. The potentials V_1 and V_2 are known as supersymmetric partner potentials. The superpotential W may be found conveniently, if we know the ground state $E_0 = 0$ wave function $\psi_0(x)$ which has no nodes. We require

$$A\psi_0 = 0 \Rightarrow H_1 = A^\dagger A\psi_0 = 0, \tag{54}$$

and it yields

$$W(x) = -\frac{\psi_0'(x)}{\psi_0(x)}. \tag{55}$$

If the ground state energy E_0 does not equal zero, we use Hamiltonian $(H_1 - E_0)$ and follow the former procedure. Once we denote $E_n^{(1,2)}$ the energy eigenvalues of $H_{1,2}$ and $\psi_n^{(1,2)}$ the corresponding eigenfunctions, the Schrödinger equation

$$H_1\psi_n^{(1)} = A^\dagger A\psi_n^{(1)} = E_n^{(1)}\psi_n^{(1)} \tag{56}$$

implies

$$H_2(A\psi_n^{(1)}) = AA^\dagger A\psi_n^{(1)} = E_n^{(1)}(A\psi_n^{(1)}) \tag{57}$$

and similarly

$$H_2\psi_n^{(2)} = AA^\dagger\psi_n^{(2)} = E_n^{(2)}\psi_n^{(2)} \tag{58}$$

implies

$$H_1(A^\dagger\psi_n^{(2)}) = A^\dagger AA^\dagger\psi_n^{(2)} = E_n^{(2)}(A^\dagger\psi_n^{(2)}). \tag{59}$$

Since $E_0^{(1)} = 0$ and $A\psi_0^{(1)} = 0$ eigenvalues and eigenfunctions satisfy

$$E_0^{(1)} = 0, \quad E_n^{(2)} = E_{n+1}^{(1)}, \tag{60}$$

$$\psi_n^{(2)} = \frac{1}{\sqrt{E_{n+1}^{(1)}}} A\psi_{n+1}^{(1)}, \tag{61}$$

$$\psi_{n+1}^{(1)} = \frac{1}{\sqrt{E_n^{(2)}}} A^\dagger \psi_n^{(2)}. \tag{62}$$

We see that the energy eigenvalues and the wave functions of H_1 and H_2 are related and operator A converts the eigenfunction of H_1 into the eigenfunction of H_2 with the same energy and $A^\dagger$ does it conversely. The ground state wave function $\psi_0^{(1)}$ is annihilated by A.

3.3 Supersymmetric quantum mechanics

We construct new, so-called supersymmetric, Hamiltonian

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \tag{63}$$

and so-called supercharges

$$Q = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}. \tag{64}$$

It is easy to verify that these operators obey the commutation and anticommutation relations

$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0, \quad \{Q, Q^\dagger\} = H, \quad [H, Q] = [H, Q^\dagger] = 0. \tag{65}$$

In a more general setting we may assume that the Hamiltonian $H \neq 0$ is a self-adjoint operator acting a Hilbert space $\mathcal{H}$ and the quantum mechanical system $(\mathcal{H}, H)$ is then called supersymmetric if there exists a finite number of non-self-adjoint operators $Q_1, ..., Q_M$ on $\mathcal{H}$ such that

$$\{Q_i, Q_j^\dagger\} = \delta_{ij} H, \quad \{Q_i, Q_j\} = 0, \quad i, j \in \{1, ..., M\}. \tag{66}$$

The operators $Q_1, ..., Q_M$ are called supercharges [22].

Important consequences of this definition (see Appendix A.1 for details) are that H has a non-negative spectrum and the eigenvectors may written in the form

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{67}$$

For the most important case with $M = 1$, where $Q_1 \equiv Q$, operators H and $Q, Q^\dagger$ are homogenous elements of a superalgebra $A = A_0 \oplus A_1$, where $A_0 =$span$\{H\}$ and $A_1 =$span$\{Q, Q^\dagger\}$. The superbracket $[\cdot,\cdot]_s$ is defined by

$$[x, y]_s = xy - (-1)^{|x||y|} yx \tag{68}$$

and we may verify that this yields the multiplication table (65).

3.4 Examples

3.4.1 Harmonic oscillator

The simplest example is the one dimensional harmonic oscillator,

$$H = -\frac{d^2}{dx^2} + x^2, \quad V(x) = x^2. \tag{69}$$

The eigenvalue problem of this system is well-known,

$$E_n = 2n + 1, \quad \psi_n = C_n e^{-\frac{x^2}{2}} H_n(x), \quad H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \quad n \in \mathcal{N}_0, \tag{70}$$

C_n is normalization constant, H_n are Hermite polynomials. Since $E_0 = 1$, we modify H,

$$H_1 := -\frac{d^2}{dx^2} + V_1(x) = -\frac{d^2}{dx^2} + x^2 - 1, \quad V_1(x) := V(x) - E_1 = x^2 - 1 \tag{71}$$

$$\psi_n^{(1)} := \psi_n, \quad E_n^{(1)} := E_n - E_0 = 2n, \quad n \in \mathcal{N}_0. \tag{72}$$

Superpotential W is given by (55) and the partner potential V_2 by (53)

$$W(x) = x, \quad V_2(x) = x^2 + 1. \tag{73}$$

We receive the eigenfunctions $\psi_n^{(2)}$ and eigenvalues $E_n^{(2)}$ from (61) and (60)

$$\psi_n^{(2)} = C_n e^{-\frac{x^2}{2}} H_n(x) = \psi_n^{(1)}, \quad E_n^{(2)} = 2n + 2. \tag{74}$$

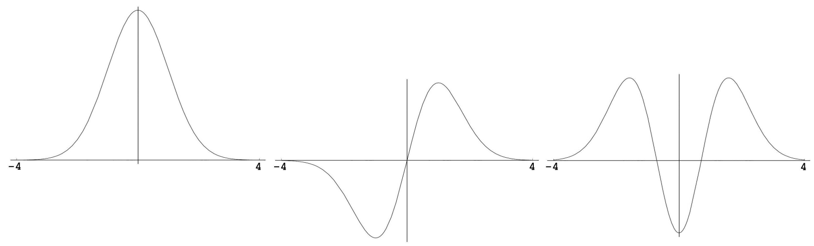

Figure 2: Harmonic oscillator eigenfunctions ψ_0, ψ_1, ψ_2

3.4.2 Superpartners of square well

Let us take the square well

$$V(x) = \begin{cases} 0, & |x| < 1 \\ \infty, & |x| > 1, \end{cases} \tag{75}$$

$$H = -\frac{d^2}{dx^2} + V(x). \tag{76}$$

The eigenvalues and eigenstates are well-known

$$E_n = \frac{\pi^2}{4} n^2, \quad \psi_n(x) = C_n \sin[n\frac{\pi}{2}(x-1)], \; n \in \mathcal{N}, \tag{77}$$

and C_n stands for a normalization constant. With regard to the non-zero ground state energy of the system, we shift the energy scale and we modify the eigenvalues and eigenstates,

$$H_1 := H - E_1 = -\frac{d^2}{dx^2} + V(x) - E_1, \quad V_1(x) := V(x) - E_1, \tag{78}$$

$$E_n^{(1)} := E_{n+1} - E_1 = \frac{\pi^2}{4}((n+1)^2 - 1), \quad \psi_n^{(1)} := \psi_{n+1}, \quad n \in \mathcal{N}_0. \tag{79}$$

The ground state $\psi_0^{(1)}$ belongs to the zero ground state energy

$$\psi_0^{(1)} = C_0 \cos(\frac{\pi}{2}x), \quad E_0^{(1)} = 0. \tag{80}$$

In conformity with the factorization method we find the superpotential W with the help of relation (55)

$$W(x) = \frac{\pi}{2} \tan(\frac{\pi}{2}x). \tag{81}$$

Supersymmetric partner potential V_2 is given by (53)

$$V_2(x) = \frac{\pi^2}{4}\frac{1+\sin^2(\frac{\pi}{2}x)}{\cos^2(\frac{\pi}{2}x)}. \tag{82}$$

Eigenfunctions $\psi_n^{(2)}$ of the supersymetric Hamiltonian H_2

$$H_2 = -\frac{d^2}{dx^2} + V_2(x) \tag{83}$$

may be obtained by applying A to $\psi_{n+1}^{(1)}$ (61) and eigenvalues $E_n^{(2)}$ are given by (60)

$$\begin{aligned}\psi_n^{(2)} = C_{n+1}\tfrac{\pi}{2}\tfrac{1}{\sqrt{(n+1)(n+3)}}((n+2)\cos[(n+2)\tfrac{\pi}{2}(x-1)]+ \\ +\tan(\tfrac{\pi}{2}x)\sin[(n+2)\tfrac{\pi}{2}(x-1)]),\end{aligned} \tag{84}$$

$$E_n^{(2)} = \frac{\pi^2}{4}(n+1)(n+3). \tag{85}$$

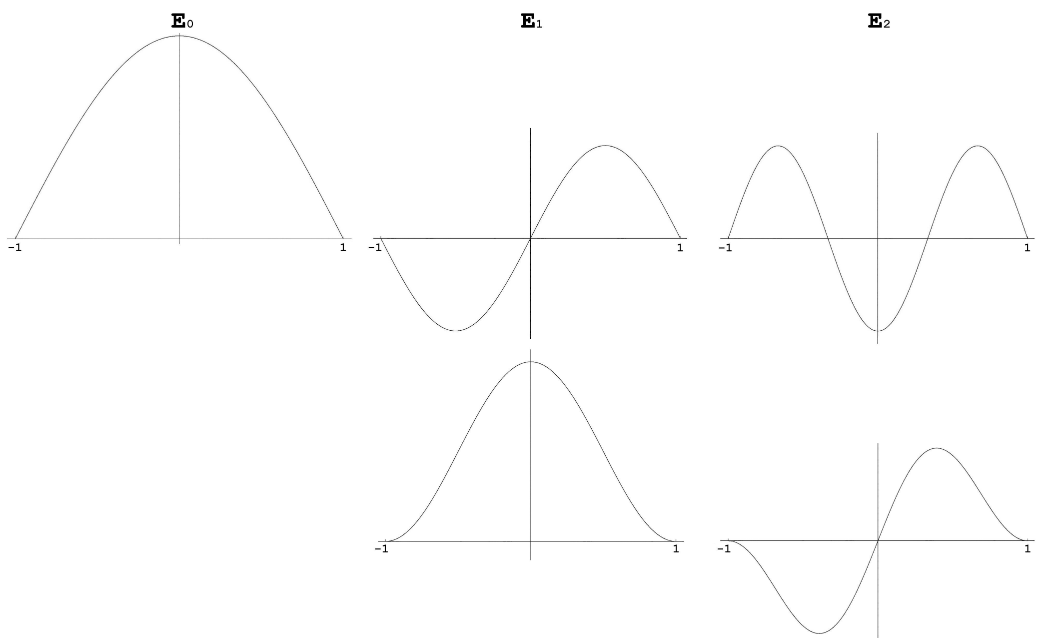

Figure 3: SUSY square well eigenfunctions corresponding to E_0, E_1, E_2

4 $\mathcal{PT}$-symmetry and supersymmetry

4.1 $\mathcal{PT}$-symmetric supersymmetry

$\mathcal{PT}$-symmetric systems possess usually the complex potential V. Therefore the straight application of the factorization method may lead to the inconsistent results. The factorization $A^\dagger A$ does not allow identification (51) for the complex superpotential W. The possible solution, used for example in [24], is stating the factorization in the form

$$A = \tfrac{d}{dx} + W(x), \quad \bar{A} = -\tfrac{d}{dx} + W(x),$$
$$H_1 = \bar{A}A, \quad H_2 = A\bar{A}, \tag{86}$$

i.e. $A, \bar{A}$ are not related by the Hermitian conjugation. Nevertheless, the example of searching for the SUSY partners to the $\mathcal{PT}$-symmetric square well [24] shows that $A, \bar{A}$ exchange states of H_1 and H_2 and this property is considered to be most important for the SUSY system. We describe the generalizations of SUSY QM which enable us to study the $\mathcal{PT}$-symmetric systems.

The quantum mechanical system $(\mathcal{H}, H)$ is called $\mathcal{PT}$-supersymmetric [25] if there exists operators $Q,\ \bar{Q}$ such that

$$\{Q, \bar{Q}\} = H, \quad \{Q, Q\} = \{\bar{Q}, \bar{Q}\} = 0. \tag{87}$$

The commutation rules

$$[H, Q] = [H, \tilde{Q}] = 0 \tag{88}$$

are satisfied. Hamiltonian and supercharges are represented by

$$Q = \begin{pmatrix} 0 & 0 \\ \mathcal{T}A & 0 \end{pmatrix}, \quad \bar{Q} = \begin{pmatrix} 0 & \bar{A}\mathcal{T} \\ 0 & 0 \end{pmatrix},$$
$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} = \begin{pmatrix} \bar{A}A & 0 \\ 0 & \mathcal{T}A\bar{A}\mathcal{T} \end{pmatrix}, \tag{89}$$

where $A, \bar{A}$ coincide with those of (86). Therefore operators $Q,\ \bar{Q}$ are not intertwined by Hermitian conjugation. The relations

$$\mathcal{T}AH_1 = H_2\mathcal{T}A, \quad \bar{A}\mathcal{T}H_2 = H_1\bar{A}\mathcal{T}, \tag{90}$$

show that the eigenfunctions of H_1 are converted to those of H_2 by $\mathcal{T}A$ and conversely by $\bar{A}\mathcal{T}$. Possible exceptions are the states which are annihilated by $\mathcal{T}A$ or $\bar{A}\mathcal{T}$.

4.2 Pseudo-supersymmetry

Pseudo-supersymmetry is a different generalization of SUSY in which supercharges are formally related by so-called pseudo-Hermitian conjugation $X^\ddagger = \eta^{-1}X^\dagger\eta$ [6].

Let H be a η-pseudo-Hermitian Hamiltonian. The quantum mechanical system $(\mathcal{H}, H)$ is then called pseudo-supersymmetric if there exists a finite number of operators $\mathcal{Q}_1$,...,$\mathcal{Q}_M$ and an operator K on $\mathcal{H}$ such that

$$\begin{gathered}\{\mathcal{Q}_i, \mathcal{Q}_j^\ddagger\} = \delta_{ij}H, \quad \{\mathcal{Q}_i, \mathcal{Q}_j\} = \{\mathcal{Q}_i^\ddagger, \mathcal{Q}_j^\ddagger\} = 0, \quad i, j \in \{1, ..., M\} \\ K = K^\dagger = K^{-1}, \quad [\eta, K] = 0, \quad \{\mathcal{Q}, K\} = 0.\end{gathered} \tag{91}$$

We will restrict ourselves on the special case $M = 1$ and we denote $\mathcal{Q}_1 \equiv Q$. Analogously to the case of standard SUSY QM system, operators $H, K, \mathcal{Q}, \eta$ may be represented by

$$\begin{gathered}H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}, \quad K = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \\ \mathcal{Q} = \begin{pmatrix} 0 & 0 \\ A^\# & 0 \end{pmatrix}, \quad \mathcal{Q}^\ddagger = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2, \end{pmatrix} \\ H_1 = A^\# A, \quad H_2 = AA^\#,\end{gathered} \tag{92}$$

where $A^\# = \eta_2^{-1}A^\dagger\eta_1$. $H_{1,2}$ are $\eta_{1,2}$-pseudo-Hermitian and they satisfy

$$AH_1 = H_2A, \quad A^\# H_2 = H_1 A^\#. \tag{93}$$

Therefore A maps the eigenvector of H_1 to that of H_2 and vice versa $A^\#$ maps the eigenvector of H_2 to that of H_1. The only possible exceptions of the eigenvectors that are annihilated by A or $A^\#$.

If we search for the supersymmetric partners of a chosen Hamiltonian, the pseudo-supersymmetry yields a variety of systems. We have η_1-pseudo-Hermitian Hamiltonian H_1 and we are permitted to select η_2 and try to find H_2. Operators η_1, η_2 determine the form of A, $A^{\#}$. In case of the $\mathcal{P}$-pseudo-Hermitian system, we may require $\eta_2 = -\mathcal{P}$, i.e. H_2 to be $\mathcal{P}$-pseudo-Hermitian. Under certain conditions, this choice allows us to express A and $A^{\#}$ in the standard form,

$$A = \frac{d}{dx} + W(x), \quad A^{\#} = -\frac{d}{dx} + W(x). \tag{94}$$

More explicitly,

$$\eta = \begin{pmatrix} \mathcal{P} & 0 \\ 0 & -\mathcal{P} \end{pmatrix} \Rightarrow A^{\#} = -\frac{d}{dx} - W^*(-x). \tag{95}$$

Therefore the factorization is successful exactly in the form (94) if and only if the superpotential W, obtained from the relation (51), satisfies

$$Re\,W(-x) = -Re\,W(x), \quad Im\,W(-x) = Im\,W(x). \tag{96}$$

Although the pseudo-Hermiticity does not involve all $\mathcal{PT}$-symmetric systems, the above definition facilitates construction of many pseudo-supersymmetric systems [26].

4.3 Nonlinear supersymmetry

We investigate a spiked $\mathcal{PT}$-symmetric oscillator [11, 27]

$$H^{(\alpha)} = -\frac{d^2}{dx^2} + (x - i\varepsilon)^2 + \frac{\alpha^2 - \frac{1}{4}}{(x - i\varepsilon)^2}, \tag{97}$$

where $\alpha > 0$, $\varepsilon > 0$. To solve the eigenvalue problem, we use a transformation $y = x - i\varepsilon$. This leads to the solution in terms of Laguerre polynomials

$$\psi_n^{(q\alpha)} = C_n.(x - i\varepsilon)^{q\alpha + \frac{1}{2}} e^{-\frac{(x-i\varepsilon)^2}{2}} L_n^{(q\alpha)}((x - i\varepsilon)^2), \tag{98}$$

and the spectrum numbered by the integer $n \in \mathcal{N}_0$ and so called quasi-parity $q = \pm 1$

$$E_n^{(q\alpha)} = 4n + 2 + 2q\alpha \tag{99}$$

In following, we consider $\alpha \neq 0, 1, 2, ...$. In the pseudo-supersymmetry framework, we find the superpotential from the ground state $\psi_0^{(-\alpha)}$

$$W^{(\alpha)}(x) = x - i\varepsilon + \frac{\alpha - \frac{1}{2}}{x - i\varepsilon}. \tag{100}$$

We see that it fulfils the requirement (96). The factorization (94) works and the partner Hamiltonian is $\mathcal{P}$-pseudo-Hermitian,

$$\begin{aligned} H_1 &= -\tfrac{d^2}{dx^2} + (x - i\varepsilon)^2 + \tfrac{\alpha^2 - \frac{1}{4}}{(x-i\varepsilon)^2} + 2\alpha - 2, \\ H_2 &= -\tfrac{d^2}{dx^2} + (x - i\varepsilon)^2 + \tfrac{(\alpha-1)^2 - \frac{1}{4}}{(x-i\varepsilon)^2} + 2\alpha. \end{aligned} \tag{101}$$

The energy levels of superpartners are

$$E_n^{(q\alpha)(1)} = 4n + 2\alpha(q+1), \quad E_n^{(q\alpha)(2)} = 4n + 4 + 2\alpha(q+1), \quad n \in \mathcal{N}_0 \tag{102}$$

and the eigenvectors of H_2 may be obtained from (61).

When we examine the action of operators

$$A^{(\gamma)} = \frac{d}{dx} + W^{(\gamma)}(x), \quad A^{\#(\gamma)} = -\frac{d}{dx} + W^{(\gamma)}(x) \tag{103}$$

on the eigenvectors we arrive at the annihilation and creation operators for the spiked $\mathcal{PT}$-symmetric oscillator

$$A(\alpha) = A^{(-\gamma-1)}A^{(\gamma)}, \quad B(\alpha) = A^{\#(-\gamma)}A^{\#(\gamma-1)}, \tag{104}$$

where $\alpha = |\gamma|$ and

$$\begin{aligned} A(\alpha)\psi_{n+1}^{(\gamma)} = C(n,\gamma)\psi_n^{(\gamma)}, \quad B(\alpha)\psi_n^{(\gamma)} = C(n,\gamma)\psi_{n+1}^{(\gamma)}, \\ C(n,\gamma) = -4\sqrt{(n+1)(n+1+\gamma)}. \end{aligned} \tag{105}$$

Hamiltonian $H^{(\alpha)}$ may be factorized

$$H^{(\alpha)} = \frac{1}{8}[A(\alpha)B(\alpha) - B(\alpha)A(\alpha)]. \tag{106}$$

It satisfies the intertwining relations

$$[A(\alpha), H^{(\alpha)}] = 4A(\alpha), \quad [H^{(\alpha)}, B(\alpha)] = 4B(\alpha). \tag{107}$$

New supersymmetry was introduced in [28], $A, \bar{A}$ (86) and $H_{1,2}$ are replaced by $A(\alpha), B(\alpha)$ and $G_{1,2}$,

$$Q = \begin{pmatrix} 0 & 0 \\ A(\alpha) & 0 \end{pmatrix}, \quad \bar{Q} = \begin{pmatrix} 0 & B(\alpha) \\ 0 & 0 \end{pmatrix},$$
$$G = \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}. \tag{108}$$

We put

$$G_1 = B(\alpha)A(\alpha), \quad G_2 = A(\alpha)B(\alpha), \tag{109}$$

and conclude that

$$\{Q, \bar{Q}\} = G, \quad \{Q, Q\} = \{\bar{Q}, \bar{Q}\} = 0, \quad [G, Q] = [G, \bar{Q}] = 0. \tag{110}$$

This result may be interpreted in the second-derivative supersymmetry SSUSY framework [29]. In this approach operators $A, \bar{A}$ have the second-derivative realization. In place of the Hamiltonian one uses so-called quasi-Hamiltonian $\mathcal{K}$ which is the fourth-order differential operator,

$$A = \left(-\frac{d}{dx} + W_1\right)\left(-\frac{d}{dx} + W_2\right), \quad \bar{A} = \left(\frac{d}{dx} + W_2\right)\left(\frac{d}{dx} + W_1\right), \tag{111}$$

where $W_{1,2}$ are two superpotentials. $\mathcal{K}$ may be related to the square of Hamiltonian under certain conditions,

$$\mathcal{K} = (H + a)^2 + d, \tag{112}$$

a, d are constants. This is known as a polynomial SUSY.

4.4 Examples

4.4.1 $\mathcal{PT}$-symmetric harmonic oscillators as superpartners

We consider Hamiltonian

$$H_1 = -\frac{d^2}{dx^2} + (x + i\varepsilon)^2 - 1, \tag{113}$$

for which eigenvalues and eigenvectors are already known (18).

The relation (55) yields $W(x) = x + i\varepsilon$. Since W satisfies the conditions (96) the proposed scheme of pseudo-supersymmetry for $\eta_1 = \mathcal{P}$, $\eta_2 = -\mathcal{P}$ works in conformity with (94). The partner Hamiltonian, its eigenvalues and eigenfunctions are

$$H_2 = -\tfrac{d^2}{dx^2} + (x + i\varepsilon)^2 + 1,$$
$$E_n^{(1)} = 2(n+1), \quad \psi_n^{(2)} = C_{n+1}\sqrt{2(n+1)}e^{-\frac{(x+i\varepsilon)^2}{2}}H_n(x+i\varepsilon), \quad n \in \mathcal{N}_0. \tag{114}$$

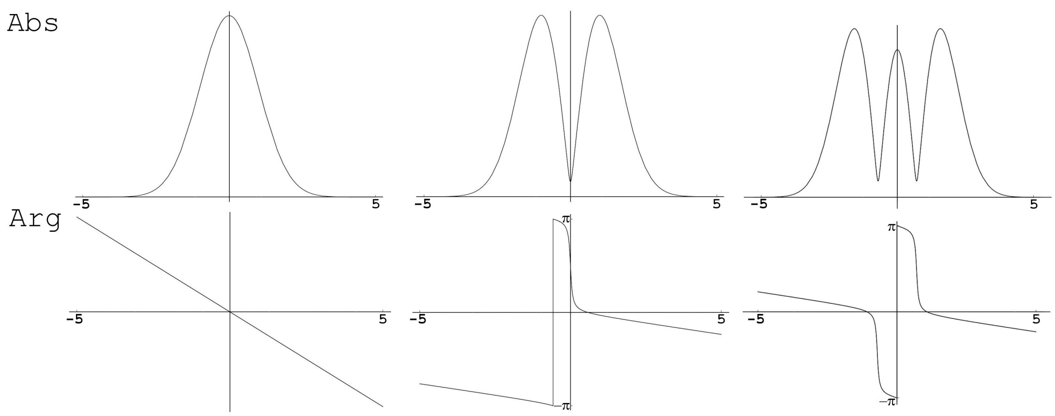

Figure 4: PsSUSY $\mathcal{PT}$-symmetric oscillator, $\varepsilon = 0.05$, eigenfunctions ψ_0, ψ_1, ψ_2.

4.4.2 Superpartners of $\mathcal{PT}$-symmetric square well

The supersymmetric construction for $Z > Z_0$ is investigated in [24]. Let us contemplate the $Z < Z_0$ case only. Our starting point is the Hamiltonian

$$H_1 = -\frac{d^2}{dx^2} + V(x) - E_0, \tag{115}$$

where E_0 is determined by (27) and

$$V(x) = \begin{cases} iZ, & -1 < x < 0 \\ -iZ, & 0 < x < 1 \\ \infty, & |x| > 1. \end{cases} \tag{116}$$

Eigenvalues given by (27) are shifted and eigenvectors are identical with ψ_n in (25),

$$E_n^{(1)} = E_n - E_0, \quad \psi_n^{(1)} = \psi_n. \tag{117}$$

The relation (55) yields

$$W(x) = \begin{cases} -\kappa_0^* \coth[\kappa_0^*(1+x)], & -1 < x < 0 \\ \kappa_0 \coth[\kappa_0(1-x)], & 0 < x < 1, \end{cases} \tag{118}$$

where $\kappa_0 = E_0 - iZ$. It meets requirements (96), therefore the factorization (94) is possible. The explicit form of $\mathcal{P}$-pseudo-Hermitian H_2 may be obtained from (92),

$$H_2 = -\frac{d^2}{dx^2} + V_2(x), \tag{119}$$

where the potential

$$V_2(x) = \begin{cases} \frac{\kappa_0^2 \cosh^2[\kappa_0(1+x)]+1}{\sinh^2[\kappa_0(1+x)]}, & -1 < x < 0 \\ \frac{(\kappa_0^*)^2 \cosh^2[\kappa_0^*(1-x)]+1}{\sinh^2[\kappa_0^*(1-x)]}, & 0 < x < 1 \\ \infty, & |x| > 1. \end{cases} \tag{120}$$

The eigenvalues of H_2 are $E_n^{(2)} = E_{n+1} - E_0$ and the eigenvectors read

$$\psi_n^{(2)} = \begin{cases} \frac{\alpha_{n+1} \sinh[\kappa_{n+1}^*(1+x)]}{\sinh \kappa_{n+1}^*} \{\kappa_{n+1}^* \coth[\kappa_{n+1}^*(1+x)] - \kappa_0^* \coth[\kappa_0^*(1+x)]\} \\ \\ \frac{\alpha_{n+1} \sinh[\kappa_{n+1}(1-x)]}{\sinh \kappa_{n+1}} \{\kappa_{n+1} \coth[\kappa_{n+1}(1-x)] - \kappa_0 \coth[\kappa_0(1-x)]\}. \end{cases} \tag{121}$$

When we investigate $\mathcal{PT}$-symmetric oscillators and square well in the $\mathcal{PT}$-symetric supersymmetry framework, we arrive at the similar results. In fact, the only change concerns the complex conjugation of V_2 and $\psi_n^{(2)}$.

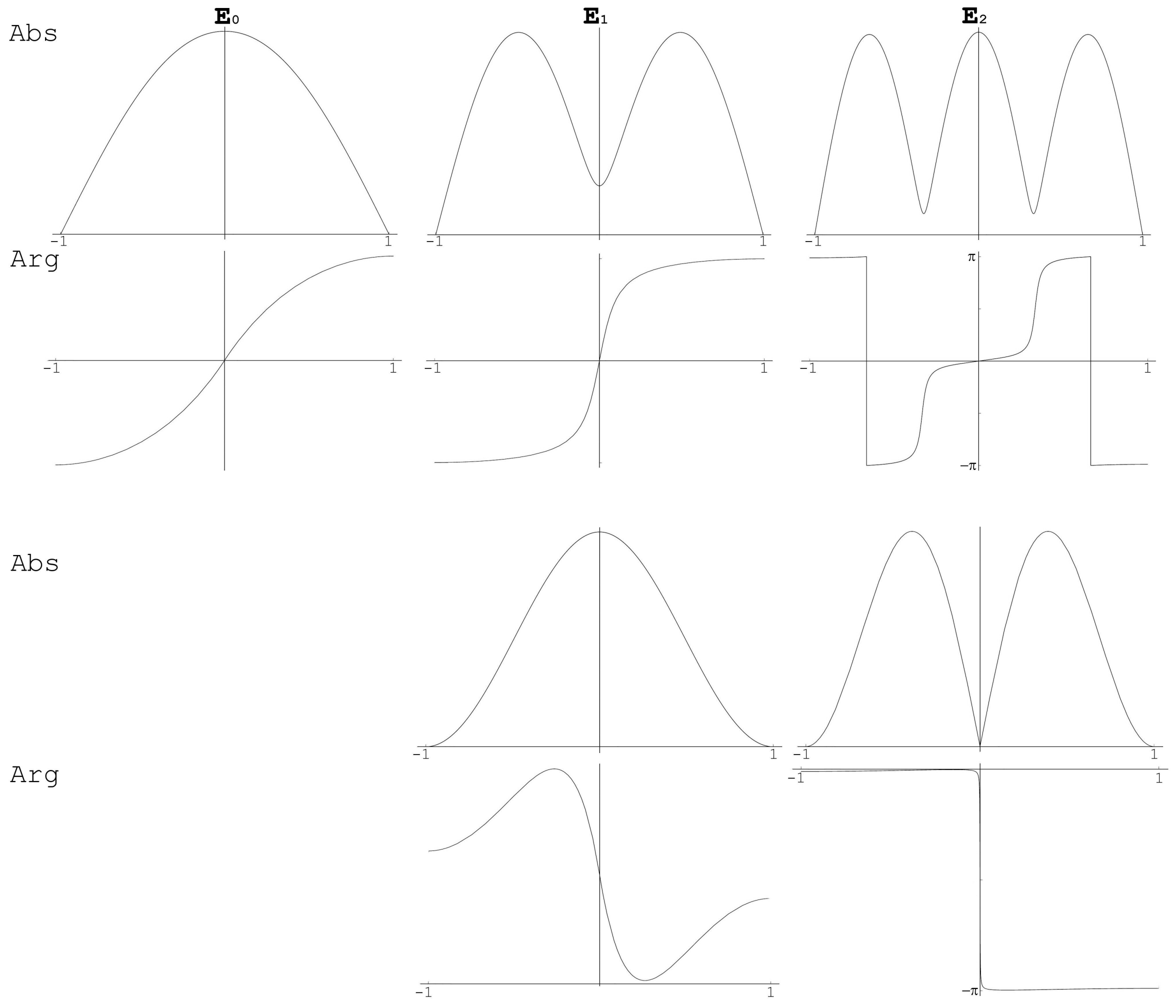

Figure 5: PsSUSY $\mathcal{PT}$-symmetric square well, Z=1.5, eigenfunctions corresponding to E_0, E_1, E_2.

5 Conclusions

We presented generalized models of SUSY which may describe $\mathcal{PT}$-symmetric systems consistently. We mentioned the possible extension of the second-derivative supersymmetry with the help of the spiked $\mathcal{PT}$-symmetric oscillator example. The pseudo-supersymmetric construction for the $\mathcal{PT}$-symmetric harmonic oscillators yields only shifted system, similarly, like in the standard Hermitian case. However, superpartners of $\mathcal{PT}$-symmetric square well represent the non-trivial solvable model (120). Further examples are solved and attempts of new physical interpretation are proposed in [26, 10, 29].

Although we work with unbounded operators, we do not concentrate on their domains of definition, our main goal is to show the basic principles of the $\mathcal{PT}$-symmetry, SUSY and their combinations. We do not prove all propositions and theorems, nevertheless the appropriate references are presented.

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A Appendix

A.1 Some mathematical aspects the Hermitian formulation of SUSY

The definition (66) implies

$$\{Q_i^\dagger, Q_j^\dagger\} = 0, \quad [H, Q_i] = [H, Q_i^\dagger] = 0, \quad i, j \in \{1, ..., M\} \tag{122}$$

and the defining relations (66) do not allow self-adjoint supercharges unless $H = 0$. Relations (66) yield the existence of a bounded Hermitian operator K, $K \neq \pm I$, called a Klein operator or a Witten parity operator [22], with properties

$$K^2 = I, \quad \{K, Q_i\} = 0, \quad i \in \{1, ..., M\}. \tag{123}$$

Let us pick up the most important special case with $M = 1$ and denote $Q_1 \equiv Q$. We create self-adjoint operators q_1, q_2 from $Q, Q^\dagger$,

$$q_1 = \frac{1}{2}(Q + Q^\dagger), \quad q_2 = \frac{i}{2}(Q^\dagger - Q), \tag{124}$$

$$Q = q_1 + iq_2, \quad Q^\dagger = q_1 - iq_2. \tag{125}$$

We see from $\{Q, Q\} = 0$ that

$$0 = Q^2 = (q_1 + iq_2)^2 \Rightarrow q_1^2 = q_2^2, \quad \{q_1, q_2\} = 0. \tag{126}$$

Relation $\{Q, Q^\dagger\} = H$ yields, with the use of (125) and (126),

$$H = \{Q, Q^\dagger\} = 2q_1^2 + 2q_2^2 = 4q_1^2 = 4q_2^2 \tag{127}$$

and

$$[H, q_1] = [H, q_2] = 0. \tag{128}$$

An important consequence of (127) is that H has a non-negative spectrum,

$$(\psi, H\psi) = (\psi, 4q_1^2\psi) = \|2q_1\psi\|^2 \geq 0. \tag{129}$$

Since $K^2 = I$, the only admissible eigenvalues of K are ± 1. Every $\psi \in \mathcal{H}$ can be written in following form

$$\psi = \frac{1}{2}(\psi + K\psi) + \frac{1}{2}(\psi - K\psi) \tag{130}$$

and therefore if we denote

$$\mathcal{H}_1 = \{\psi \in \mathcal{H} | K\psi = \psi\}, \quad \mathcal{H}_2 = \{\psi \in \mathcal{H} | K\psi = -\psi\}, \tag{131}$$

we arrive at the direct sum decomposition of $\mathcal{H}$ to the two non-trivial ($K \neq \pm I$) subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2. \tag{132}$$

We write

$$\psi = \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \psi_1 \in \mathcal{H}_1, \ \psi_2 \in \mathcal{H}_2 \tag{133}$$

and

$$K = \begin{pmatrix} I_1 & 0 \\ 0 & -I_2 \end{pmatrix}, \tag{134}$$

where I_1 and I_2 are identity operators on $\mathcal{H}_1$ and $\mathcal{H}_2$. This partitioned notation facilitates discussing the operators commuting or anticommuting with K. Indeed, for every operator

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{135}$$

$$[X, K] = 0 \Leftrightarrow X = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad \{X, K\} = 0 \Leftrightarrow X = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}. \tag{136}$$

Since q_1 and q_2 are anticommuting with K it follows from (136) that

$$q_1 = \frac{1}{2}\begin{pmatrix} 0 & A^\dagger \\ A & 0 \end{pmatrix}, \quad q_2 = \frac{1}{2}\begin{pmatrix} 0 & B^\dagger \\ B & 0 \end{pmatrix} \tag{137}$$

(the factor $\frac{1}{2}$ is chosen only for convenience). Hence, the relation between H and q_1 (127) implies

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix}. \tag{138}$$

We decompose A to $A = a_1 + ia_2$ and B to $B = b_1 + ib_2$, where a_1, a_2, b_1, b_2 are self-adjoint operators,

$$q_1 = \frac{1}{2}\begin{pmatrix} 0 & a_1 - ia_2 \\ a_1 + ia_2 & 0 \end{pmatrix}, \quad q_2 = \frac{1}{2}\begin{pmatrix} 0 & b_1 - ib_2 \\ b_1 + ib_2 & 0 \end{pmatrix} \tag{139}$$

and we determine q_2 from relations (126) up to an overall sign

$$q_2 = \frac{1}{2}\begin{pmatrix} 0 & -a_2 - ia_1 \\ -a_2 + ia_1 & 0 \end{pmatrix}. \tag{140}$$

We return to our supercharges in (66)

$$Q = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}. \tag{141}$$

Once we take $\psi \in \mathcal{H}$ and apply Q we get

$$Q\psi = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} A^\dagger\psi_2 \\ 0 \end{pmatrix}. \tag{142}$$

We take an eigenvector ψ of H belonging to energy $E \geq 0$, $H\psi = E\psi$, we apply q_1 on the Schrödinger equation and with the help of (128) we have

$$(Hq_1)\psi = E(q_1\psi) \tag{143}$$

and for the ground state $E = 0$

$$0 = (\psi, H\psi) = \|2q_1\psi\|^2 \Rightarrow q_1\psi = 0. \tag{144}$$

Hence, we see the degeneracy of energy levels with the only exception $E = 0$. The corresponding eigenvectors for the eigenvalue E are ψ and $q_1\psi$. If $\psi \in \mathcal{H}_1$, then $q_1\psi \in \mathcal{H}_2$ and vice versa, if $\psi \in \mathcal{H}_2$, then $q_1\psi \in \mathcal{H}_1$. Operators H and Q, $Q^\dagger$ are homogenous elements of a superalgebra $A = A_0 \oplus A_1$, where A_0 =span$\{H\}$ and A_1 =span$\{Q, Q^\dagger\}$. The superbracket $[\cdot,\cdot]_s$ is defined by

$$[x, y]_s = xy - (-1)^{|x||y|}yx, \tag{145}$$

i.e. with the help of (66), (122)

$$[Q,Q]_s = QQ + QQ = \{Q,Q\} = 0, \quad [Q^\dagger, Q^\dagger]_s = Q^\dagger Q^\dagger + Q^\dagger Q^\dagger = \{Q^\dagger, Q^\dagger\} = 0,$$

$$[H,Q]_s = HQ - QH = [H,Q] = 0, \quad [H,Q^\dagger]_s = HQ^\dagger - Q^\dagger H = [H,Q^\dagger] = 0, \quad (146)$$

$$[Q,Q^\dagger]_s = QQ^\dagger + Q^\dagger Q = \{Q,Q^\dagger\} = H.$$

In short, we have the multiplication table (65) based on both commutators and anticommutators.

A.2 Physical interpretation of SUSY based on the harmonic oscillator

A very elegant technique to solve the harmonic oscillator eigenvalue problem may use lowering (annihilation) and raising (creation) operators $b, b^\dagger$ [19].

$$H = -\frac{d^2}{dx^2} + x^2, \quad b = \frac{d}{dx} + x, \quad b^\dagger = -\frac{d}{dx} + x, \quad (147)$$

$$H = \frac{1}{2}\{b, b^\dagger\}. \quad (148)$$

The creation and annihilation operators obey commutation relation

$$[b, b^\dagger] = I \quad (149)$$

and if we consider associated bosonic number operator $N_b = b^\dagger b$ we get

$$[N_b, b] = -b, \quad [N_b, b^\dagger] = b^\dagger. \quad (150)$$

We may express

$$H = N_b + I. \quad (151)$$

The method proposed by Dirac requires

$$b\psi_b^{(0)} = 0. \quad (152)$$

The n particle state is then given by

$$\psi_b^{(n)} = \frac{1}{\sqrt{n!}} b^\dagger \psi_b^{(0)}. \tag{153}$$

Hamiltonian of SUSY harmonic oscillator H as well as supercharges $Q, Q^\dagger$ may be expressed in terms of the bosonic operator b and the fermionic f, where the fermionic annihilation and creation operators are represented by

$$f = \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f^\dagger = \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{154}$$

and obeying

$$\{f^\dagger, f\} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \{f^\dagger, f^\dagger\} = \{f, f\} = 0, \quad [f, f^\dagger] = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{155}$$

$Q = f \otimes b^\dagger$ and $Q^\dagger = f^\dagger \otimes b$, hence

$$H = \{Q, Q^\dagger\} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \left(-\frac{d^2}{dx^2} + x^2 \right) - [f, f^\dagger] \otimes I. \tag{156}$$

If we introduce the fermion number operator $N_f = f^\dagger f$, we see from anticommutation relations (155)

$$N_f^2 = N_f \tag{157}$$

and therefore, the only admittable eigenvalues of N_f are 0 and 1.

The supercharge changes a fermion into a boson and when we remark the relations (62), (74), we see that Q does not change the energy of the state. The boson-fermion degeneracy is characteristic for SUSY theories and it has been already shown as a result of the algebraic formulation of SUSY.

For the general case of SUSY quantum mechanics, supercharges $Q, Q^\dagger$ are construct from $A, A^\dagger$ instead of $a, a^\dagger$ and the description of the bosonic sector is not so simple [30].