Mathematical modelling of elastoplasticity

Jan Valdman

School of Engineering and Natural Sciences
University of Iceland, Reykjavik
email: jan.valdman@gmail.com

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Outline

Explaining papers to theory and numerics:

- Carsten Carstensen, Martin Brokate, Jan Valdman, A quasi-static boundary value problem in multi-surface elastoplasticity. I: Analysis. Math. Methods Appl. Sci. 27, No.14, 1697-1710 (2004)
- Carsten Carstensen, Martin Brokate, Jan Valdman, A quasi-static boundary value problem in multi-surface elastoplasticity. II: Numerical solution. Math. Methods Appl. Sci. 28, No.8, 881-901 (2005)
- Andreas Hofinger, Jan Valdman, Numerical solution of the two-yield elastoplastic minimization problem. Computing 81, No. 1, 35-52 (2007)

Elastoplasticity solver can be downloaded at

http://www.mathworks.com/matlabcentral/fileexchange/authors/37756

The tensile test

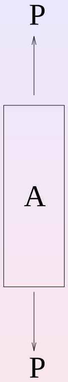


Figure: The tensile test: an increasing stress $\sigma = P/A$ is applied to the specimen.

The tensile test: stress-strain relation

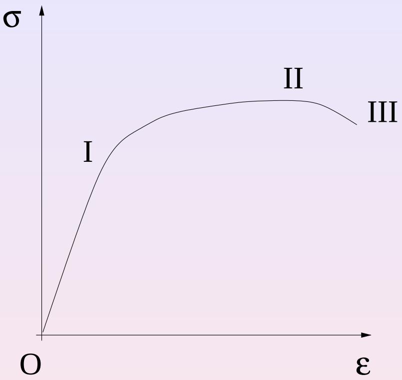
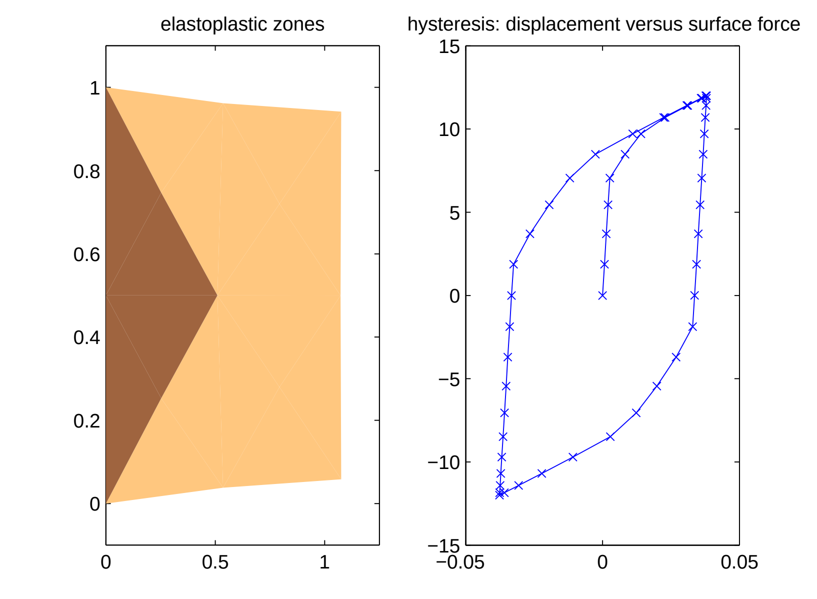


Figure: The tensile test: the resulting stress-strain relation.

- elasticity in the region $O - I$
- plasticity with hardening after the elastic limit (point I)
- softening after necking (point II) untill fractures occur (point III)

Time dependent 2D problem in Matlab



Rheological elements

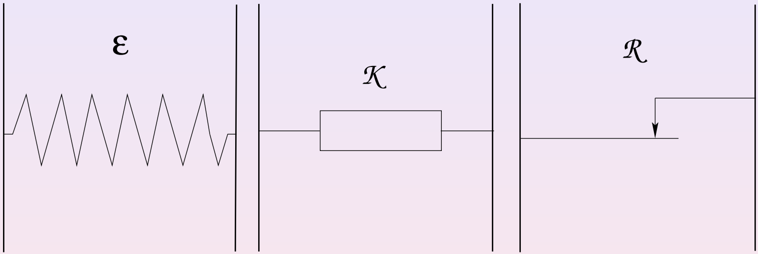


Figure: The elastic, kinematic and rigid-plastic element.

Rheological elements

Every element is characterized by its (internal) stress and strain tensors. We denote the stress by σ and the strain by ε.

The elastic element

$$\sigma = \mathbb{C}\varepsilon$$

The kinematic element

$$\sigma = \mathcal{H}\varepsilon,$$

where $\mathcal{H}$ is a positive definite matrix, for instance $\mathcal{H} = h\mathbb{I}$, where $h > 0$ is a hardening coefficient and $\mathbb{I}$ represents the identical matrix.

Rheological elements

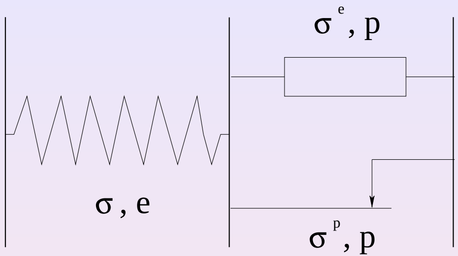
The rigid-plastic element

$$\sigma \in Z$$
$$\langle \dot{\varepsilon}, q - \sigma \rangle \leq 0 \qquad \text{for all } q \in Z$$

with a convex set $Z \subset \mathbb{R}^{d \times d}_{sym}$.

Example: 1D

Kinematic hardening model



$$\varepsilon = \mathrm{e} + p$$

$$\sigma = \sigma^e + \sigma^p$$

$$\sigma^e = \mathcal{H} p$$

$$\sigma = \mathbb{C}\,\mathrm{e}$$

$$\sigma^p \in Z$$

$$\langle \dot{p}, q - \sigma^p \rangle \leq 0 \quad \text{for all } q \in Z.$$

Hysteresis property of the kinematic hardening model

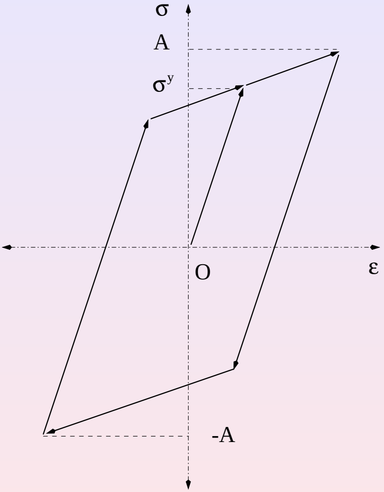


Figure: Stress-strain relation in case of linear kinematic hardening model and the cyclic stress $\sigma = A\sin(t)$.

Motivation for the multi-yield model

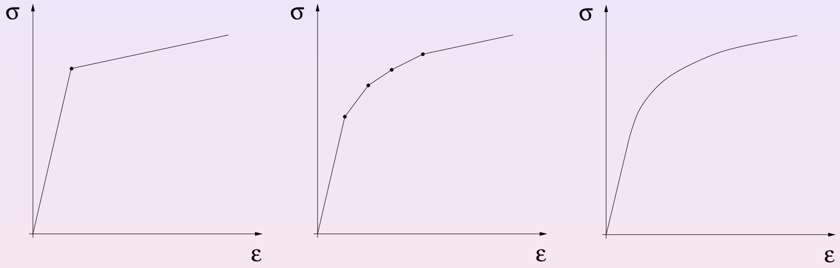
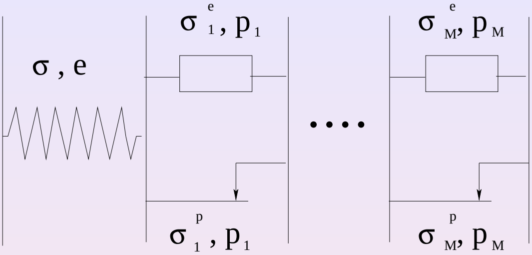


Figure: single-yield (left), multi-yield (middle) and realistic model (right) - stress-strain relation.

The M-yield hardening model



$$\varepsilon = \mathrm{e} + p, \quad p = \sum_{r=1}^{M} p_r,$$

$$\sigma = \sigma_r^e + \sigma_r^p \quad \text{for all } r = 1, \dots, M,$$

$$\sigma_r^p \in Z_r,$$

$$\langle \dot{p}_r, q_r - \sigma_r^p \rangle \leq 0 \quad \text{for all } q_r \in Z_r, r = 1, \dots, M,$$

$$\sigma = \mathbb{C}\,\mathrm{e},$$

$$\sigma_r^e = \mathcal{H}_r p_r, \quad r = 1, \dots, M.$$

Hysteresis property of the 2-yield hardening model

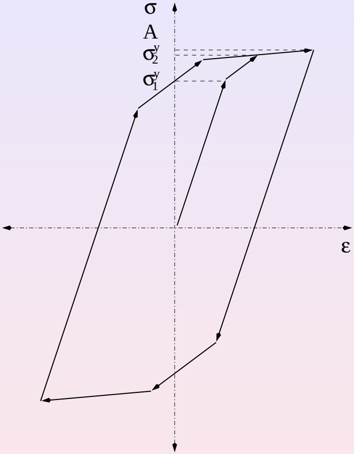


Figure: Stress-strain relation in case of two-yield model and cyclic stress $\sigma = A\sin(t)$.

Books on hysteresis

- Visintin, A., Differential models of hysteresis, Springer, 1994
- Brokate, M. and Sprekels, J., Hysteresis and Phase Transitions, Springer-Verlag New York, 1996
- Krejčí, P., Hysteresis, Convexity and Dissipation in Hyperbolic Equations, GAKUTO International Series, Mathematical Sciences and Applications, 1996

Yield criterion

von Mises criterion

$$Z = \{\sigma \in \mathbb{R}^{d\times d}_{sym} : ||\operatorname{dev}\sigma||_F \leq \sigma^y\},$$

where $||\cdot||_F$ denotes the Frobenius matrix norm $||a||_F^2 = a:a = \sum_{i,j=1}^d a_{ij}^2$,
$\operatorname{dev}\sigma = \sigma - \frac{1}{d}\operatorname{tr}(\sigma)\mathbb{I}$ is the deviatoric operator (deviator),
$\operatorname{tr}\sigma = \sigma:\mathbb{I}$ is the trace operator.

Dissipation functional

Lemma

Let $(\dot{p}, \sigma^p) \in \mathbb{R}^{d\times d}_{sym} \times \mathbb{R}^{d\times d}_{sym}$. *Then*

$$\sigma^p \in Z, \quad \dot{p} : (\tau - \sigma^p) \leq 0 \quad \textit{for all } \tau \in Z \quad (*)$$

together with $\operatorname{tr} \dot{p} = 0$ *hold if and only if*

$$\sigma^p : (q - \dot{p}) \leq \mathcal{D}(q) - \mathcal{D}(\dot{p}) \quad \forall q \in \mathbb{R}^{d\times d}_{sym}, \quad (**)$$

where $\mathcal{D} : \mathbb{R}^{d\times d}_{sym} \to \mathbb{R} \cup \{\infty\}$,

$$\mathcal{D}(q) = \begin{cases} \sigma^y ||q|| & \textit{if } \operatorname{tr} q = 0, \\ +\infty & \textit{otherwise.} \end{cases}$$

Proof: together only implication $(*) \Rightarrow (**)$.

Some convex analysis

Definition (indicator function)

For any set $Z \subset X$, the *indicator function* I_Z of Z is defined by

$$I_Z(x) = \begin{cases} 0 & \text{if } x \in Z, \\ +\infty & \text{if } x \notin Z. \end{cases} \tag{1}$$

Definition (subdifferential)

Let f be a convex function on X. For any $x \in X$ the *subdifferential* $\partial f(x)$ of x is the possibly empty subset of X^* defined by

$$\partial f(x) = \{x^* \in X^* : \langle x^*, y - x \rangle \leq f(y) - f(x) \quad \forall y \in X\}. \tag{2}$$

It means that

$$\dot{p} \in \partial I_Z(\sigma^p).$$

Some convex analysis

Definition (conjugate function)

For a function $f : X \to [-\infty, \infty]$ we define the *conjugate function* $f^* : X^* \to [-\infty, \infty]$ by

$$f^*(x^*) = \sup_{x \in X} (\langle x^*, x \rangle - f(x)). \tag{3}$$

Lemma

Let X be a Banach space, $f : X \to [-\infty, \infty]$ be a proper, convex, lower semicontinuous function. Then

$$x^* \in \partial f(x) \Leftrightarrow x \in \partial f^*(x^*). \tag{4}$$

Therefore,

$$\dot{p} \in \partial I_Z(\sigma^p) \Leftrightarrow \sigma^p \in \partial I_Z^*(\dot{p})$$

and

$$D(\cdot) := I_Z^*(\cdot).$$

Equilibrium and its weak formulation

The equilibrium between external and internal forces is given by

$$\operatorname{div} \sigma(x,t) + f(x,t) = 0, \quad x \in \Omega, \quad t \in (0,T). \tag{5}$$

With the assumption of small deformations

$$\varepsilon(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right),$$

the variational formulation of (25) becomes (why?)

$$\int_\Omega \sigma : \varepsilon(v) \, \mathrm{d}x = \int_\Omega f \cdot v \, \mathrm{d}x + \int_{\Gamma_N} g \cdot v \, \mathrm{d}s, \tag{6}$$

valid for all $t \in [0,T]$ and all $v \in H^1_D(\Omega)$.

Weak formulation of rigid-plastic elements

We express constitutive laws

$$\sigma_r^p : (q_r - \dot{p}_r) \leq \mathcal{D}_r(q_r) - \mathcal{D}_r(\dot{p}_r) \quad \forall\, q_r \in Q\,,\; r \in I\,, \tag{7}$$

where (note that we only consider arguments with zero trace here)

$$\mathcal{D}_r(q_r) = \sigma_r^y ||q_r||\,.$$

The integral form of (7) over Ω is given by

$$\int_\Omega \sigma_r^p : (q_r - \dot{p}_r)\,\mathrm{d}x \leq \int_\Omega \mathcal{D}_r(q_r)\,\mathrm{d}x - \int_\Omega \mathcal{D}_r(\dot{p}_r)\,\mathrm{d}x. \tag{8}$$

Variational inequality

We sum the inequalities (8) over r
and subtract (6) in which we equivalently replace v by $v - \dot{u}$
to obtain

$$\int_\Omega \sigma : (\varepsilon(v) - \sum_{r\in I} q_r))\,\mathrm{d}x - \int_\Omega \sigma : (\varepsilon(\dot{u}) - \sum_{r\in I} \dot{p}_r)\,\mathrm{d}x + \sum_{r\in I} \int_\Omega \sigma_r^e : (q_r - \dot{p}_r)\,\mathrm{d}x$$

$$+ \sum_{r\in I} \int_\Omega \mathcal{D}_r(q_r)\,\mathrm{d}x - \sum_{r\in I} \int_\Omega \mathcal{D}_r(\dot{p}_r)\,\mathrm{d}x - \int_\Omega f \cdot (v - \dot{u})\,\mathrm{d}x - \int_{\Gamma_N} g \cdot (v - \dot{u})\,\mathrm{d}s \geq 0.$$

Next, we eliminate

$$\sigma = \mathbb{C}(\varepsilon(u) - p), \quad \sigma_r^e = \mathcal{H}_r p_r.$$

Variational inequality

We collect vectors of functions

$$w = (u, (p_r)_{r\in I}), \quad z = (v, (q_r)_{r\in I}).$$

to obtain

Problem (BVP of quasi-static multi-surface elastoplasticity)

For given $\ell \in H^1(0,T;\mathcal{H}^*)$ with $\ell(0) = 0$,
find $w \in H^1(0,T;\mathcal{H})$ with $w(0) = 0$, such that

$$a(w(t), z - \dot{w}(t)) + \psi(z) - \psi(\dot{w}(t)) \geq \langle \ell(t), z - \dot{w}(t) \rangle, \quad \text{for all } z \in \mathcal{H},$$

holds for almost all $t \in (0,T)$.

Variational inequality

A bilinear form $a(\cdot,\cdot)$, a linear functional $\ell(\cdot)$ and a nonlinear functional $\psi(\cdot)$ are defined as

$$a : \mathcal{H} \times \mathcal{H} \to \mathbb{R}, \quad a(w,z) = \int_\Omega \mathbb{C}(\varepsilon(u) - \sum_{r\in I} p_r) : (\varepsilon(v) - \sum_{r\in I} q_r)\,\mathrm{d}x + $$

$$+ \sum_{r\in I} \int_\Omega \mathcal{H}_r p_r : q_r\,\mathrm{d}x,$$

$$\ell(t) : \mathcal{H} \to \mathbb{R}, \quad \langle \ell(t), z\rangle = \int_\Omega f(t)\cdot v\,\mathrm{d}x + \int_{\Gamma_N} g(t)\cdot v\,\mathrm{d}s,$$

$$\psi : \mathcal{H} \to \mathbb{R}, \quad \psi(z) = \sum_{r\in I} \int_\Omega \mathcal{D}_r(q_r)\,\mathrm{d}x.$$

and $\mathcal{H} = H^1_D(\Omega) \times \prod_{r\in I} Q$.

Literature

- Glowinskii, R., Lions J. L. and Trémolières R., Numerical analysis of Variational Inequalities, North-Holland, Amsterdam, 1981
- Han, W. and Reddy, B., Plasticity: Mathematical Theory and Numerical Analysis, Springer-Verlag New York, 1999

Material assumptions

We pose the natural assumption that the elastic and hardening tensors are symmetric and positive definite,

$$\begin{aligned} \xi : \mathbb{C}\lambda = \mathbb{C}\xi : \lambda \quad &\text{for all } \xi, \lambda \in \mathbb{R}^{d\times d}, \\ \xi : \mathcal{H}_r\lambda = \mathcal{H}_r\xi : \lambda \quad &\text{for all } \xi, \lambda \in \mathbb{R}^{d\times d}, r = 1, \dots, M, \end{aligned} \tag{9}$$

and there exist constants $c, h_r > 0$ such that

$$\begin{aligned} \mathbb{C}\xi : \xi \geq c||\xi||^2 \quad &\text{for all } \xi \in \mathbb{R}^{d\times d}, \\ \mathcal{H}_r\xi : \xi \geq h_r||\xi||^2 \quad &\text{for all } \xi \in \mathbb{R}^{d\times d}, r = 1, \dots, M. \end{aligned} \tag{10}$$

Abstract theorem on solvability

Theorem

Assume that (9) and (10) hold, let $\ell \in H^1(0, T; \mathcal{H}^*)$ *with* $\ell(0) = 0$*. Then there exists a unique solution* $w \in H^1(0, T; \mathcal{H})$ *of BVP of quasi-static multi-surface elastoplasticity.*

based on

Abstract theorem on solvability

Theorem (Han, Reddy, 1999)

Let $\mathcal{H}$ be a Hilbert space, $a : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bilinear form that is symmetric, bounded, and $\mathcal{H}$-elliptic; $\ell \in H^1(0, T; \mathcal{H}^*)$ with $\ell(0) = 0$; and $\psi : \mathcal{H} \to \mathbb{R}$ nonnegative, convex, positively homogeneous, and Lipschitz continuous. Then there exists a unique $w \in H^1(0, T; \mathcal{H})$ with $w(0) = 0$ which satisfies the variational inequality

$$a(w(t), z - \dot{w}(t)) + \psi(z) - \psi(\dot{w}(t)) \geq \langle \ell(t), z - \dot{w}(t) \rangle \,, \quad \text{for all } z \in \mathcal{H},$$

for almost all $t \in (0, T)$.

Remark on ellipticity

To prove that

$$a(w,z) = \int_\Omega \mathbb{C}(\varepsilon(u) - \sum_{r\in I} p_r) : (\varepsilon(v) - \sum_{r\in I} q_r)\,\mathrm{d}x + \sum_{r\in I}\int_\Omega \mathcal{H}_r p_r : q_r\,\mathrm{d}x,$$

is elliptic, the following partial result is important:

Problem

To determine the largest constant $k(M)$, $M \in \mathcal{N}$, such that

$$\left(x_0 - \sum_{r=1}^{M} x_r\right)^2 + \sum_{r=1}^{M} x_r^2 \geq k(M) \sum_{r=0}^{M} x_r^2 \qquad (11)$$

holds for all $x_0, x_1, \ldots, x_M \in \mathbb{R}$.

Algebraic inequality

We refolmulate

$$\left(x_0 - \sum_{r=1}^{M} x_r\right)^2 + \sum_{r=1}^{M} x_r^2 = x^T A x, \tag{12}$$

where

$$A = D + a \otimes a\,, \quad D = \mathrm{diag}(0, 1, \ldots, 1)\,, \quad a = (1, -1, \ldots, -1)\,. \tag{13}$$

Thus, the optimal constant $k(M)$ is equal to the smallest eigenvalue of A!

Algebraic inequality

The analytical computation shows

$$k(M) = \lambda_{min} = 1 + \frac{M}{2} - \frac{1}{2}\sqrt{4M + M^2}$$

Properties:

$$\lim_{M \to \infty} k(M) = 0$$

and

$$\lim_{M \to \infty} Mk(M) = 1$$

Backward Euler scheme

In the first time step t_1, the time derivative $\dot{x}(t_1)$ is approximated by the backward Euler method as

$$\dot{X}^1 = \frac{X^1 - X^0}{k_1},$$

where $X^0 = 0$. The Hilbert space $\mathcal{H}$ is approximated by the conforming finite element (FEM) subspace

$$\S = \S^1_D(\mathcal{T}) \times \prod_{r \in I} \mathrm{dev}(\S^0(\mathcal{T})^{d \times d}_{\mathrm{sym}}),$$

which is a product space of $\mathcal{T}$- piecewise affine functions that are zero on Γ_D by

$$\S^1_D(\mathcal{T}) := \{v \in H^1_D(\Omega) : \forall T \in \mathcal{T}, v|_T \in \mathcal{P}_1(T)^d\}.$$

($\mathcal{P}_1(T)$ denotes the affine functions on T) and the space of $\mathcal{T}$- piecewise constant functions

$$\mathrm{dev}(\S^0(\mathcal{T})^{d \times d}_{\mathrm{sym}}) := \{a \in L^2(\Omega)^{d \times d} : \forall T \in \mathcal{T}, a|_T \in \mathrm{dev}\, \mathbb{R}^{d \times d}_{\mathrm{sym}}\}$$

Backward Euler scheme

The first time step problem

Find $X^1 = (U^1, (P^1_r)_{r\in I}) := (U^1, P^1) \in \S$ such that

$$\langle \ell(t_1), (Y - \frac{X^1 - X^0}{k_1})\rangle \leq a(X^1, Y - \frac{X^1 - X^0}{k_1}) + \psi(Q) - \psi(\frac{P^1 - P^0}{k_1}).$$

holds for all $Y = (V, Q) = (V, (Q_r)_{r\in I}) \in \S$.

After introducing an incremental variable $X := (U, P) = X^1 - X^0$
and a linear functional $L(Y) = \langle \ell(t_1), Y\rangle - a(X^0, Y)$
we obtain a one-time step incremental problem

$$L(Y - X) \leq a(X, Y - X) + \psi(Q) - \psi(P) \quad \text{for all } Y = (V, Q) \in \S.$$

Introducing the energy functional

Lemma (Equivalent Reformulations)

For each $X=(U,P)\in \S$ the following three conditions (a)-(c) are equivalent:

$$(a)\quad L(Y-X)\le a(X,Y-X)+\psi(Q)-\psi(P)\quad \textit{for all } Y=(V,Q)\in\S.$$

$$(b)\quad L(Y-X)= a(X,Y-X)\quad \textit{for all } Y=(V,P)\in\S \quad \textit{and}$$
$$L(Y-X)\le a(X,Y-X)+\psi(Q)-\psi(P)\quad \textit{for all } Y=(U,Q)\in\S.$$

$$(c)\quad \Phi(X)=\min_{Y\in S}\Phi(Y)\quad \textit{with } \Phi(Y)=\frac{1}{2}a(Y,Y)+\psi(Q)-L(Y).$$

Abreviations

The following matrix notation allows for a brief formulation of the discrete problem. Let

$$P := \begin{pmatrix} P_1 \\ \vdots \\ P_M \end{pmatrix}, P^0 := \begin{pmatrix} P_1^0 \\ \vdots \\ P_M^0 \end{pmatrix}, Q := \begin{pmatrix} Q_1 \\ \vdots \\ Q_M \end{pmatrix}, \hat{\Sigma} := \begin{pmatrix} \mathbb{C}\varepsilon(U) \\ \vdots \\ \mathbb{C}\varepsilon(U) \end{pmatrix},$$

$$\hat{\Sigma}^0 := \begin{pmatrix} \mathbb{C}\varepsilon(U^0) \\ \vdots \\ \mathbb{C}\varepsilon(U^0) \end{pmatrix}, \hat{\mathbb{C}} := \begin{pmatrix} \mathbb{C} & \dots & \mathbb{C} \\ \vdots & & \vdots \\ \mathbb{C} & \dots & \mathbb{C} \end{pmatrix}, \quad \hat{\mathcal{H}} := \begin{pmatrix} \mathcal{H}_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \mathcal{H}_M \end{pmatrix}.$$

Abreviations

Then there holds

$$
\begin{aligned}
-a(X, Y-X) &= \int_\Omega \left(\hat{\Sigma} - (\hat{\mathbb{C}} + \hat{\mathcal{H}})P\right) : (Q-P)\,\mathrm{d}x,\\
L(Y-X) &= \int_\Omega \left(\hat{\Sigma}^0 - (\hat{\mathbb{C}} + \hat{\mathcal{H}})P^0\right) : (Q-P)\,\mathrm{d}x,\\
\psi(Y) &= \int_\Omega |Q|_{\sigma^y}\,\mathrm{d}x.
\end{aligned}
$$

Since the plastic yield parameters $\sigma_1^y, \dots, \sigma_M^y$ are positive, the expansion

$$|(Q_1, \dots, Q_M)^T|_{\sigma^y} := \sigma_1^y |Q_1| + \cdots + \sigma_M^y |Q_M|$$

defines a norm in $\mathbb{R}^{Md\times d}$, where $|\cdot|$ denotes the Frobenius norm.

Coupled problem

Problem (Discrete problem)

Given $(U^0, P^0) \in \S$, *seek* $U^1 \in \S^1_D(\mathcal{T})$ *such that for all* $V \in S^1_D(\mathcal{T})$,

$$\int_\Omega \mathbb{C}(\varepsilon(U^1) - \sum_{r=1}^{M} P_r^1) : \varepsilon(V)\, dx - \int_\Omega f(t)V\, dx - \int_{\Gamma_N} gV\, dx = 0. \quad (14)$$

Here $P = (P_1, \dots, P_M)^T = (P_1^1, \dots, P_M^1)^T - (P_1^0, \dots, P_M^0)^T$ *satisfies*

$$(\hat{A} - (\hat{\mathbb{C}} + \hat{\mathcal{H}})P) : (Q - P) \leq |Q|_{\sigma^y} - |P|_{\sigma^y} \quad (15)$$

for all $Q = (Q_1, \dots, Q_M)^T$ *with* $Q_1, \dots, Q_M \in \operatorname{dev}(\S^0(\mathcal{T})^{d\times d}_{sym})$ *and*

$$\hat{A} := \hat{\Sigma}(U^1) + \hat{\Sigma}^0(U^0) - (\hat{\mathbb{C}} + \hat{\mathcal{H}})P^0.$$

Moreau regularization

Theorem (Moreau, 1965)

Let the function $\mathcal{F} : \mathcal{H} \times \mathcal{H} \to \overline{\mathbb{R}}$ be defined

$$\mathcal{F}(x, y) = \frac{1}{2} \|y - x\|^2_{\mathcal{H}} + \psi(x) \tag{16}$$

where ψ is a convex, proper and lower semi continuous mapping of $\mathcal{H}$ into $\overline{\mathbb{R}}$. Then

$$F(y) := \inf_{x \in \mathcal{H}} \mathcal{F}(x, y)$$

is well defined as a functional from $\mathcal{H}$ into $\mathbb{R}$ and there exists a unique mapping $\tilde{x} : \mathcal{H} \to \mathcal{H}$ such, that

$$F(y) = \mathcal{F}(\tilde{x}(y), y)$$

holds for all $y \in \mathcal{H}$. Moreover, F is strictly convex and Fréchet differentiable with the derivative

$$\mathcal{D}F(y) = \langle y - \tilde{x}(y) \, , \, \cdot \rangle_{\mathcal{H}} \in \mathcal{H}^* \qquad \forall\, y \in \mathcal{H} \, . \tag{17}$$

Moreau regularization

Theorem of Moreau implies for elastoplasticity

Theorem

There is a unique function

$$P = P(\varepsilon(U))$$

and the energy functional

$$\Phi(U) = \frac{1}{2}a(U, P(\varepsilon(U)); U, P(\varepsilon(U))) + \psi(P(\varepsilon(U))) - L(U)$$

is strictly convex and differentiable!

more details in

- Peter Gruber, Jan Valdman, Solution of one-time-step problems in elastoplasticity by a Slant Newton Method. SIAM J. Scientific Computing 31, No. 2, 1558-1580 (2009)

Analysis of single-yield model (M=1)

Localization to one element $T \in \mathcal{T}$:
One plastic strain

$$P \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}, \quad \mathrm{tr}\, P = 0,$$

the elastic matrix $\mathbb{C}$ with the (positive) Lamé coefficients μ and λ

$$\mathbb{C}P = 2\mu P + \lambda(\mathrm{tr}\,\mathcal{P})\mathbb{I} = 2\mu P,$$

the hardening matrix $\mathcal{H}$ with

$$\mathcal{H}P = hP,$$

the matrix norm

$$|P|_{\sigma^y} = \sigma^y |P|$$

and the matrix

$$A := \hat{A} := \mathbb{C}\varepsilon(U) + \mathbb{C}\varepsilon(U^0) - (\mathbb{C} + \mathcal{H})P^0.$$

Analysis of single-yield model (M=1)

Lemma (Alberty, Carstensen, Zarrabi, 1999)

Given $A \in \mathbb{R}^{d\times d}_{\text{sym}}$ *and* $\sigma^y > 0$. *There exists exactly one* $P \in \operatorname{dev} \mathbb{R}^{d\times d}_{\text{sym}}$ *that satisfies*

$$\{A - (\mathbb{C} + \mathcal{H})P\} : (Q - P) \leq \sigma^y \{|Q| - |P|\}$$

for all $Q \in \operatorname{dev} \mathbb{R}^{d\times d}_{\text{sym}}$. *This* P *is characterized as the minimiser of*

$$\frac{1}{2}(\mathbb{C} + \mathcal{H})Q : Q - Q : A + \sigma^y |Q| \tag{18}$$

(amongst trace-free symmetric $d \times d$*-matrices) and is given by*

$$P = \frac{(|\operatorname{dev} A| - \sigma^y)_+}{2\mu + h} \frac{\operatorname{dev} A}{|\operatorname{dev} A|}, \tag{19}$$

where $(\cdot)_+ := \max\{0, \cdot\}$ *denotes the non-negative part.*

Analysis of two-yield model (M=2)

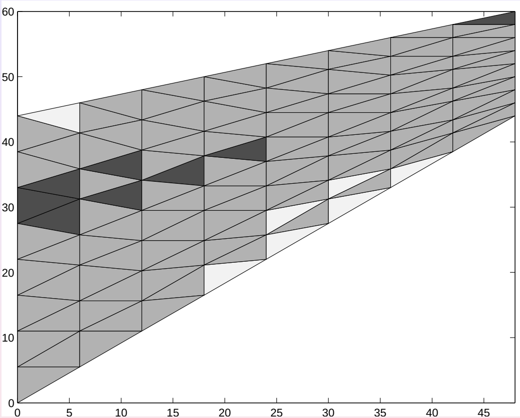


Figure: Cook's membrane problem in the first time step. The black colour shows elastic upgrade zones (where $P_1 = P_2 = 0$), brown and lighter gray colours shows the first plastic upgrade ($P_1 \neq 0, P_2 = 0$) and the both plastic upgrades ($P_1 \neq 0, P_2 \neq 0$) zones.

Analysis of two-yield model (M=2)

Two plastic strains P_1, P_2 coupled in a generalized plastic strain

$$P = (P_1, P_2)^T.$$

The generalized elasticity matrix and the generalized hardening matrices read

$$\hat{\mathbb{C}} := \begin{pmatrix} \mathbb{C} & \mathbb{C} \\ \mathbb{C} & \mathbb{C} \end{pmatrix} \quad \text{and} \quad \hat{\mathcal{H}} := \begin{pmatrix} \mathcal{H}_1 & 0 \\ 0 & \mathcal{H}_2 \end{pmatrix},$$

the generalized loading matrix reads

$$\hat{A} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} \mathbb{C}\varepsilon(U) \\ \mathbb{C}\varepsilon(U) \end{pmatrix} + \begin{pmatrix} \mathbb{C}\varepsilon(U^0) \\ \mathbb{C}\varepsilon(U^0) \end{pmatrix} - \begin{pmatrix} \mathbb{C}+\mathcal{H}_1 & \mathbb{C} \\ \mathbb{C} & \mathbb{C}+\mathcal{H}_2 \end{pmatrix} \begin{pmatrix} P_1^0 \\ P_2^0 \end{pmatrix}$$

and the matrix norm is defined by

$$|P|_{\sigma^y} = \sigma_1^y |P_1| + \sigma_2^y |P_2|.$$

Analysis of two-yield model (M=2)

Lemma

Given $\hat{A} = (A_1, A_2)^T, A_1, A_2 \in \mathbb{R}^{d\times d}_{\text{sym}}$*, there exists exactly one* $P = (P_1, P_2)^T, P_1, P_2 \in \text{dev}\,\mathbb{R}^{d\times d}_{\text{sym}}$ *that satisfies*

$$(\hat{A} - (\hat{\mathbb{C}} + \hat{\mathcal{H}})P) : (Q - P) \leq |Q|_{\sigma^y} - |P|_{\sigma^y} \tag{20}$$

for all $Q = (Q_1, Q_2)^T, Q_1, Q_2 \in \text{dev}\,\mathbb{R}^{d\times d}_{\text{sym}}$*. This* P *is characterized as the minimiser of*

$$f(Q) = \frac{1}{2}(\hat{\mathbb{C}} + \hat{\mathcal{H}})Q : Q - Q : \hat{A} + |Q|_{\sigma^y} \tag{21}$$

(amongst trace-free symmetric $d \times d$ *matrices* Q_1, Q_2*).*

Exact minimizer?

Analysis of two-yield model (M=2)

We introduce the operator

$$\mathcal{F}(M,\sigma,h) := \frac{(|M|-\sigma)_+}{2\mu+h}\frac{M}{|M|}. \tag{22}$$

Algorithm (Iterative calculation of P_1, P_2)

Input μ, h_1, h_2, σ_1^y, σ_2^y, dev A_1, dev A_2 *and* tol ≥ 0.

1. *Set* $i := 0$ *and set the initial approximation* $P_1^i = P_2^i = 0$.
2. *Update* P_2^i *via* $P_2^{i+1} = \mathcal{F}(\operatorname{dev} A_2 - 2\mu P_1^i, \sigma_2^y, h_2)$.
3. *Update* P_1^i *via* $P_1^{i+1} = \mathcal{F}(\operatorname{dev} A_1 - 2\mu P_2^{i+1}, \sigma_1^y, h_1)$.
4. *If the desired accuracy is reached, i. e., if*

$$|P_1^{i+1} - P_1^i| + |P_2^{i+1} - P_2^i| \leq \text{tol}\cdot(|P_1^{i+1}| + |P_1^i| + |P_2^{i+1}| + |P_2^i|)$$

then output solution $(P_1, P_2) = (P_1^{i+1}, P_2^{i+1})$. *Otherwise, set* $i := i+1$ *and go to step 2.*

Analysis of two-yield model (M=2)

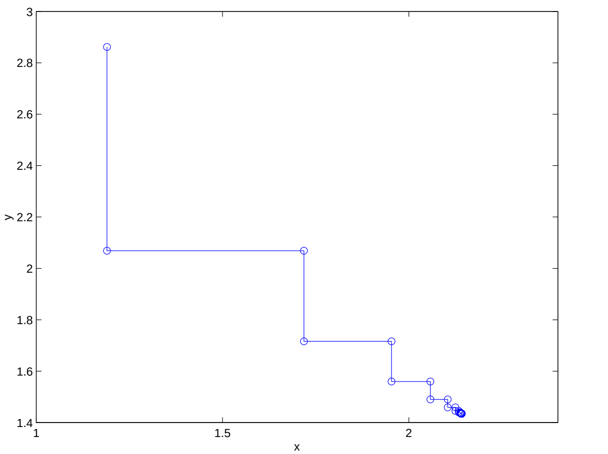


Figure: The approximations $P_1^i = (x^i, 0; 0, -x^i)$, $P_2^i = (y^i, 0; 0, -y^i)$, $i = 0, \ldots, 34$ computed by the iterative algorithm and displayed as the points (x^i, y^i) in the $x - y$ coordinate system.

Newton method

A nonlinear system of equations for $2N$ displacement unknowns $\mathbf{U^1} = (U_1^1, \dots, U_{2N}^1)^T$:

$$\mathbf{F}_i(\mathbf{U^1}) = 0 \quad \text{for all} \quad i = 1, \dots, 2N. \tag{23}$$

We use the Newton-Raphson method for the iterative solution of (23).

Algorithm (Newton-Raphson Method)

(a) Choose an initial approximation $\mathbf{U_0^1} \in \mathbb{R}^{\mathbf{2N}}$*, set* $k := 0$.
(b) Let $k := k + 1$*, solve* $\mathbf{U_k^1}$ *from*

$$D\mathbf{F}(\mathbf{U_{k-1}^1})(\mathbf{U_k^1} - \mathbf{U_{k-1}^1}) = -\mathbf{F}(\mathbf{U_{k-1}^1}).$$

(c) If $\mathbf{U_k^1} - \mathbf{U_{k-1}^1}$ *is sufficiently small then output* $\mathbf{U_k^1}$*, otherwise goto (b).*

Newton method

In order to incorporate the Dirichlet boundary conditions properly, the linear system in the step (b) is extended,

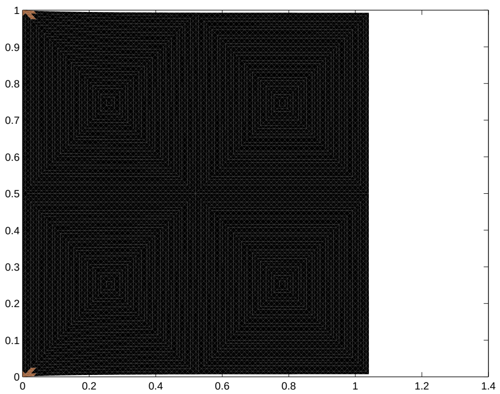
$$\begin{pmatrix} D\mathbf{F}(\mathbf{U^1_{k-1}}) & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{U^1_k} - \mathbf{U^1_{k-1}} \\ \lambda \end{pmatrix} = \begin{pmatrix} -\mathbf{F}(\mathbf{U^1_{k-1}}) \\ \mathbf{0} \end{pmatrix},$$

with some matrix B and the vector of Lagrange parameters λ. Here, $D\mathbf{F}(\mathbf{U^1_k}) \in \mathbb{R}^{2N\times 2N}$ represents a sparse tangential stiffness matrix

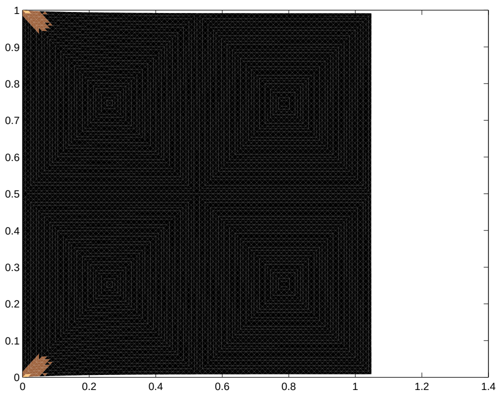
$$D\mathbf{F}(\mathbf{U})_{ij} \approx \frac{\mathbf{F}(U_1,\dots,U_j+\epsilon_j,\dots,U_{2N})_i - \mathbf{F}(U_1,\dots,U_j-\epsilon_j,\dots,U_{2N})_i}{2\epsilon_j}$$

approximated by a central difference scheme with small parameters $\epsilon_j > 0$, $j = 1,\dots,2N$.

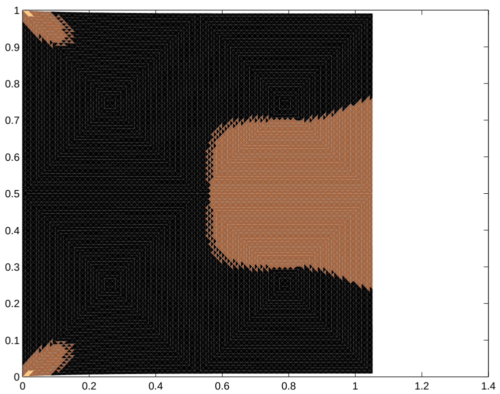
Matlab simulations: two-yield 2D beam model



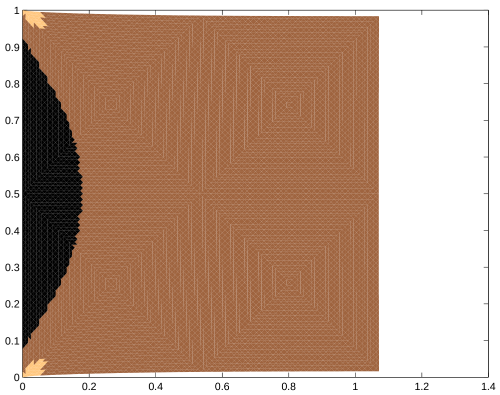
Matlab simulations: two-yield 2D beam model



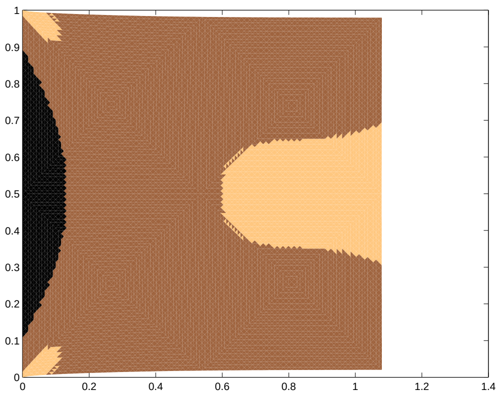
Matlab simulations: two-yield 2D beam model



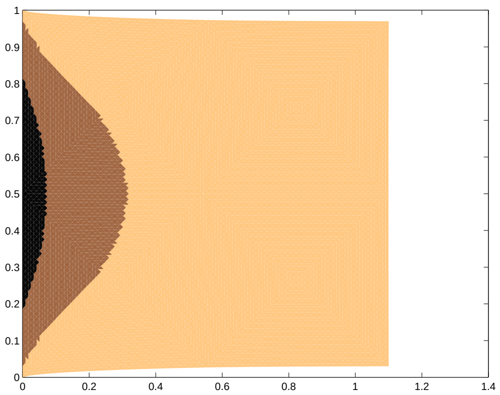
Matlab simulations: two-yield 2D beam model



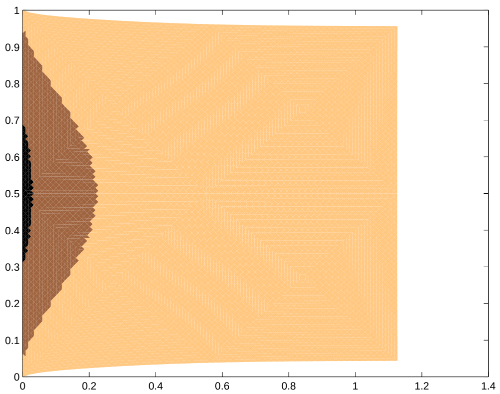
Matlab simulations: two-yield 2D beam model



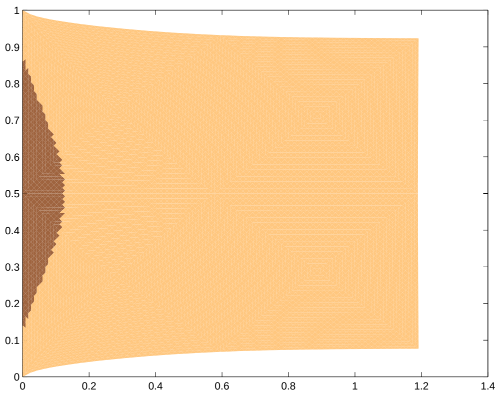
Matlab simulations: two-yield 2D beam model



Matlab simulations: two-yield 2D beam model



Matlab simulations: two-yield 2D beam model



Matlab simulations: single-yield model

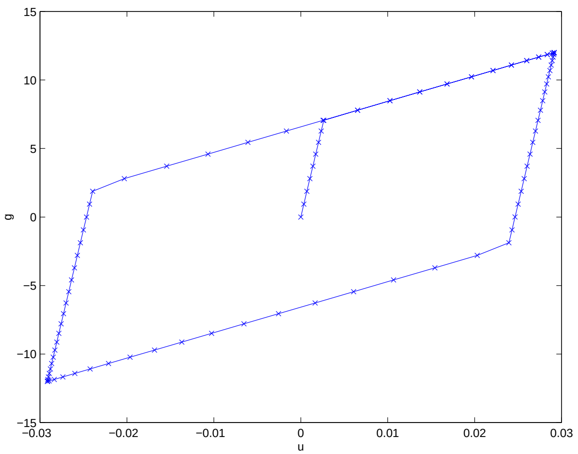


Figure: Displayed loading-deformation relation in terms of the uniform surface loading $g_x(t)$ versus the x-displacement of the point $(0,1)$ for problem of the single-yield beam with 1D effects.

Matlab simulations: two-yield model

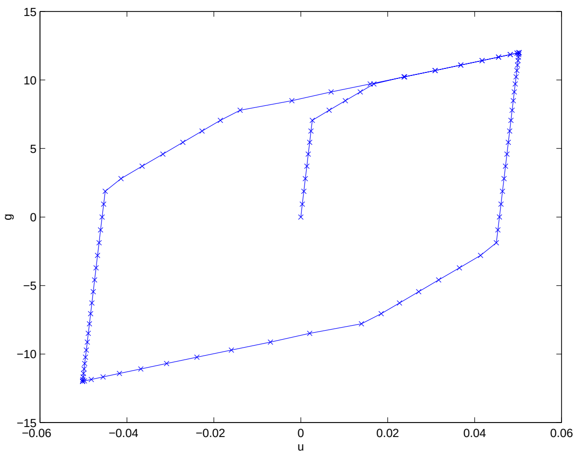


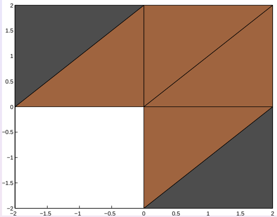
Figure: Displayed loading-deformation relation in terms of the uniform surface loading $g_x(t)$ versus the x-displacement of the point $(0, 1)$ for problem of the two-yield beam with 1D effects.

Concept of adaptivity

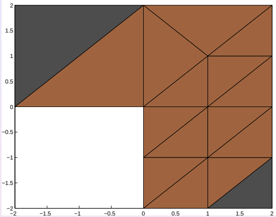
An $h-$finite element adaptive algorithm consists of successive loops of the form:

$$\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE}$$

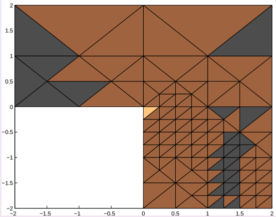
Numerical example: Adaptive meshes



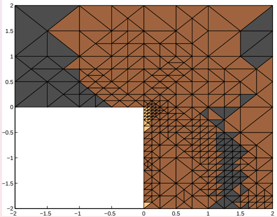
($\mathcal{T}_0$)



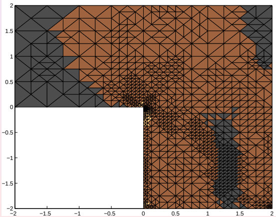
($\mathcal{T}_2$)



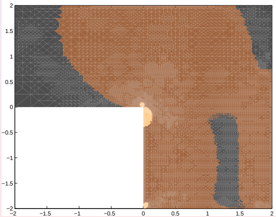
($\mathcal{T}_4$)



($\mathcal{T}_6$)



($\mathcal{T}_8$)



($\mathcal{T}_{12}$)

The SOLVE & ESTIMATE step in elastoplaticity

Reliability

$$\|\sigma - \sigma_\ell\| \leq C \left(\eta_\ell^2 + \mathrm{osc}_\ell^2 \right)^{1/2}$$

- ESTIMATE (edge-based residual):

$$\eta_\ell^2 = \sum_{E \in \mathcal{E}_\ell} \eta_E^2 \quad \text{with } \eta_E^2 = h_E \int_E |J_E|^2 \, ds, \quad J_E = [\sigma_\ell]_E \nu_E$$

- Data (node-patchwise) oscillation:

$$\mathrm{osc}_\ell^2 = \sum_{j \in \mathcal{K}_\ell} \mathrm{osc}_{j,\ell}^2 \quad \text{with } \mathrm{osc}_{j,\ell}^2 = h_{j,\ell}^2 \|f - \bar{f}_j\|_{L^2(\Omega_{j,\ell};\mathbb{R}^d)}^2$$

Main results

Oscillation reduction

$$\exists\ \rho_2 < 1 : \quad \mathrm{osc}^2_{\ell+1} \leq \rho_2 \mathrm{osc}^2_\ell$$

Energy reduction

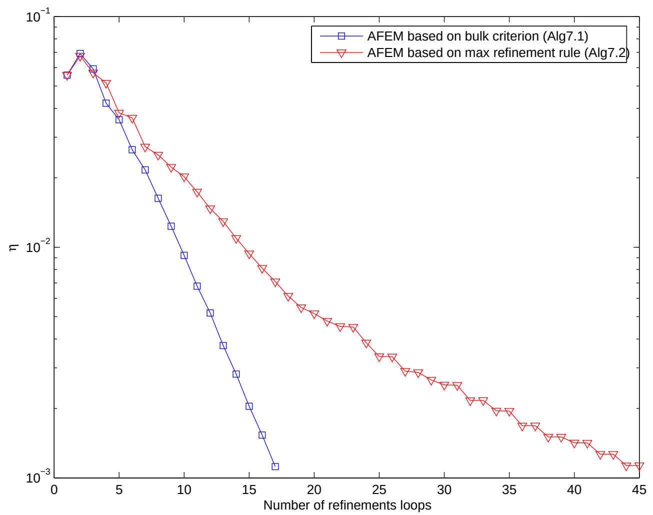
$$\exists\ \rho_1 < 1, C > 0 : \quad \delta_{\ell+1} \leq \rho_1 \delta_\ell + C \mathrm{osc}^2_\ell \quad \text{where } \delta_\ell = \mathcal{H}(w_\ell) - \mathcal{H}(w)$$

R-linear convergence of stresses

$\exists\ (\alpha_\ell)$ linearly convergent:

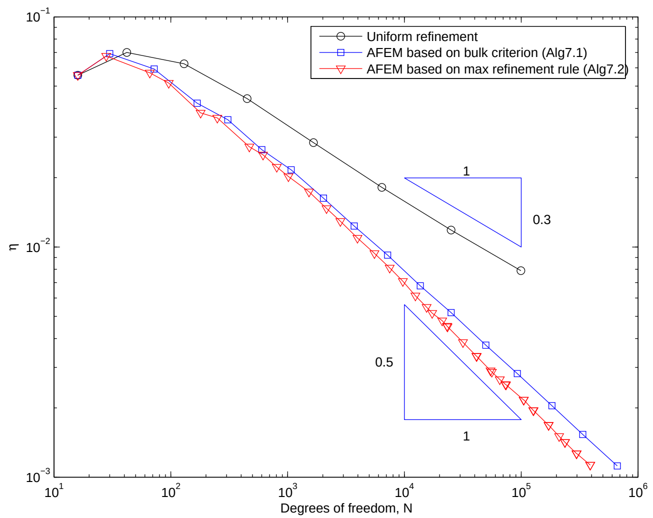
$$\|\sigma - \sigma_\ell\| \leq \alpha_\ell$$

Numerical example: Convergence rates



Bulk-criterion more efficient than max-criterion!

Numerical example: Convergence rates



Paper

1. C. Carstensen, A. Orlando, J. Valdman, A convergent adaptive finite element method for the primal problem of elastoplasticity, International Journal for Numerical Methods in Engineering 67, No. 13, 1851-1887 (2006)

Basic estimate of the deviation from exact solution

For any $w \in H$ it holds

$$\frac{1}{2}|||u - v, p - q|||^2 \leq \mathcal{H}(v, q) - \mathcal{H}(u, p),$$

where $z = (u, p)$ is an exact elastoplastic solution
and $w = (v, q)$ is a discrete approximation.

where

$$|||u - v, p - q||| := \|\mathbb{C}(\varepsilon(u - v) - (p - q))\|^2_{\mathbb{C}^{-1}} + \sigma_y^2 H^2 \|q - p\|^2 .$$

Note, $H > 0$ represents a hardening parameter (done for isotropic hardening model).

Perturbed problem

Original problem

$$\mathcal{H}(v,q) := \frac{1}{2} a(v,q;v,q) - l(v) + \int_\Omega \sigma^y |q| \, dx$$

Perturbed problem

$$\mathcal{H}_\lambda(v,q) := \frac{1}{2} a(v,q;v,q) - l(v) + \int_\Omega \sigma_y \lambda : q \, dx$$

where $\lambda \in \Lambda := \{\lambda \in L^\infty(\Omega, \mathbb{R}^{d\times d}) : |\lambda| \leq 1, tr(\lambda) = 0 \;\; a.\ e. \text{ in } \Omega\}$.

$$\sup_{\lambda\in\Lambda} \mathcal{H}_\lambda(v,q) = \mathcal{H}(v,q)$$

Lagrangian

Lagrangian

$$L_\lambda(v,q;\tau,\xi) := \int_\Omega \left(\tau : (\varepsilon(v) - q) - \frac{\mathbb{C}^{-1}\tau : \tau}{2} + \xi : q - \frac{|\xi|^2}{2\sigma_y^2 H^2} - fv\right) dx + \int_\Omega \sigma^y \lambda : q \, dx,$$

where $\tau \in Q := L^2(\Omega; \mathbb{R}^{d\times d}_{sym}), \xi \in Q_0 := \{q \in Q : tr(q) = 0 \;\; a.\ e. \text{ in } \Omega\}$.

$$\sup_{\tau\in Q, \xi \in Q_0} L_\lambda(v,q;\tau,\xi) = \mathcal{H}_\lambda(v,q)$$

First estimate

It holds for all $\lambda \in \Lambda$

$$\mathcal{H}(u,p) = \inf_{v,q} \mathcal{H}(v,q) \geq \inf_{v,q} \mathcal{H}_\lambda(v,q) \geq \inf_{v,q} L_\lambda(v,q;\tau,\xi)$$

which yields the estimate

$$\frac{1}{2}|||(u-v),(p-q)|||^2 \leq \mathcal{H}(v,q) - \inf_{v,q} L_\lambda(v,q;\tau,\xi)$$

How to compute $\inf_{v,q} L_\lambda(v,q;\tau,\xi)$?

Majorant estimate for equilibrated fields

$$\frac{1}{2}|||(u-v),(p-q)|||^2 \leq \inf_{(\tau,\xi)\in Q_{f_\lambda}} \mathcal{M}(v,q,\tau,\xi,\lambda),$$

where

$$\mathcal{M}(v,q,\tau,\xi,\lambda) = \frac{1}{2}\int_\Omega \mathbb{C}(\varepsilon(v)-q-\mathbb{C}^{-1}\tau) : (\varepsilon(v)-q-\mathbb{C}^{-1}\tau)\,dx$$
$$+\frac{1}{2}\int_\Omega \sigma_y^2 H^2 (q-\frac{1}{\sigma_y^2 H^2}\xi)^2\,dx + \int_\Omega (\sigma_y|q| - \sigma_y\lambda : q)\,dx$$

and

$$Q_{f_\lambda} := \{(\tau,\xi)\in Q\times Q_0 : \operatorname{div}\tau + f = 0, \tau^D = \xi + \sigma_y\lambda \;\; a.\ e.\ \text{in } \Omega\}.$$

Structure of Functional Majorant

$\mathcal{M}(v, q, \tau, \xi, \lambda) = 0$ if and only if

$$\tau = \mathbb{C}(\varepsilon(v) - q), \tag{24}$$

$$\operatorname{div} \tau + f = 0, \tag{25}$$

$$\lambda : q = |q|, \qquad \lambda \in \Lambda, \tag{26}$$

$$\tau^D = \xi + \sigma_y \lambda, \tag{27}$$

$$\xi = \sigma_y^2 H^2 q. \tag{28}$$

These are conditions for the exact solution (u, p) of the elastoplastic minimization problem! The majorant naturally reflects properties of the original problem.

Majorant estimate for nonequilibrated fields

$$\frac{1}{2}|||(u-v),(p-q)|||^2 \leq \inf_{(\tau,\xi)\in Q_{f_\lambda}} \hat{\mathcal{M}}(v,q;\hat{\tau},\lambda,\beta,\delta),$$

where

$$\begin{aligned}\hat{\mathcal{M}}(v,q;\hat{\tau},\lambda,\beta,\delta) := \frac{1}{2}(1+\beta)\int_\Omega \mathbb{C}(\varepsilon(v)-q-\mathbb{C}^{-1}\hat{\tau}):(\varepsilon(v)-q-\mathbb{C}^{-1}\hat{\tau})\,dx \\ +\frac{1}{2}(1+\delta)\int_\Omega \frac{1}{\sigma_y^2 H^2}(\hat{\tau}^D-\zeta)^2 dx + \int_\Omega (\sigma_y|q|-\sigma_y\lambda:q)\,dx \\ +\frac{1}{2}\left[(1+\frac{1}{\beta})+\frac{c_2}{\sigma_y^2H^2}(1+\frac{1}{\delta})\right] C^2\,\|\operatorname{div}\hat{\tau}+f\|^2\end{aligned}$$

and $\hat{\tau}\in Q_{\text{div}} := \{\tau\in Q : \operatorname{div}\tau\in L^2(\Omega,\mathbb{R}^d)\}$, $\quad \zeta := \sigma_y^2 H^2 q + \sigma_y\lambda.$

Papers

Sergey Repin, Jan Valdman, Functional a posteriori error estimates for incremental models in elasto-plasticity. Cent. Eur. J. Math. 7, No. 3, 506-519 (2009)

Papers on Matlab Implementation

- Jochen Alberty, Carsten Carstensen and Stefan A. Funken, Remarks around 50 lines of Matlab: short finite element implementation, Numerical Algorithms 20 (117), 117–137 (1999)
- Alberty, Carstensen, Funken, Klose, Matlab implementation of the finite element method in elasticity, Computing 69 (3), 239 – 263 (2002)
- Carstensen C., Klose R., Elastoviscoplastic Finite Element Analysis in 100 lines of Matlab, J. Numer. Math., 10 (3), 157–192 (2002)
- Rahman T., Valdman J., Fast MATLAB assembly of FEM stiffness- and mass matrices in 2D and 3D: nodal elements, Proceedings of conference PARA 2010 (submitted)

Computer exercises

- computation of triagulation areas
- uniform refinement in 2D
- generation of a stifness matrix
- generation of a right-hand side
- a posteriori computation of a plasticity strain from a given stress
- alternating directions iteration over equilibrium and plasticity inequality
- extension to time-dependent problems

Thank you for your attention!