Rules with parameters in modal logic

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Overview

The plan for this talk:

- General remarks on unification and admissibility
- Known results on unification and admissibility in transitive modal logics
- Unification and admissibility with parameters in transitive modal logics
- Unification and admissibility with parameters in intuitionistic logic

Unification and admissibility in propositional logics

Propositional logics

Propositional logic L:

Language: formulas Form_L built freely from atoms (variables) $\{x_n : n \in \omega\}$ using a fixed set of connectives of finite arity

Consequence relation $\vdash_L$: finitary structural Tarski-style consequence operator
I.e.: a relation $\Gamma \vdash_L \varphi$ between finite sets of formulas and formulas such that

- $\varphi \vdash_L \varphi$
- $\Gamma \vdash_L \varphi$ implies $\Gamma, \Gamma' \vdash_L \varphi$
- $\Gamma \vdash_L \varphi$ and $\Gamma, \varphi \vdash_L \psi$ imply $\Gamma \vdash_L \psi$
- $\Gamma \vdash_L \varphi$ implies $\sigma(\Gamma) \vdash_L \sigma(\varphi)$ for every substitution σ

Algebraization

L is finitely algebraizable wrt a class K of algebras if there is a finite set $F(u,v)$ of formulas and a finite set $E(x)$ of equations such that

- $\Gamma \vdash_L \varphi \Leftrightarrow E(\Gamma) \vDash_K E(\varphi)$
- $\Theta \vDash_K t \approx s \Leftrightarrow F(\Theta) \vdash_L F(t,s)$
- $x \dashv\vdash_L F(E(x))$
- $u \approx v \mathrel{=\!\!|\!\vDash}_K E(F(u,v))$

We may assume K is a quasivariety

In our case we will always have:
$E(x) = \{x \approx 1\}$, $F(u,v) = \{u \leftrightarrow v\}$, K is a variety

Equational unification

Let Θ be an equational theory (or a variety of algebras):

- Θ-unifier of a set Γ of equations:
  a substitution σ s.t. $\vDash_\Theta \sigma(t) \approx \sigma(s)$ for all $t \approx s \in \Gamma$
- Γ is Θ-unifiable if it has a Θ-unifier
- $\sigma \equiv_\Theta \tau$ iff $\vDash_\Theta \sigma(u) \approx \tau(u)$ for every variable u
- $\sigma \preceq_\Theta \tau$ (τ is more general than σ) if $\exists \varrho\, \sigma \equiv_\Theta \varrho \circ \tau$
- Complete set of unifiers of Γ: a set X of unifiers of Γ such that every unifier of Γ is less general than some $\tau \in X$
- Θ has finitary unification type if every finite Γ has a finite complete set of unifiers

Unification in propositional logics

If L is a logic finitely algebraizable wrt a variety K, we can express K-unification in terms of L:

An L-unifier of a formula φ is σ such that $\vdash_L \sigma(\varphi)$

Then we have:

- L-unifier of φ = K-unifier of $E(\varphi)$
- K-unifier of $t \approx s$ = L-unifier of $F(t, s)$
- $\sigma \equiv_L \tau$ iff $\vdash_L F(\sigma(x), \tau(x))$ for every x
  (in our case: $\vdash_L \sigma(x) \leftrightarrow \tau(x)$)
- ...

Admissible rules

Single-conclusion rule: $\Gamma \,/\, \varphi$ (Γ finite set of formulas)

Multiple-conclusion rule: $\Gamma \,/\, \Delta$ (Γ, Δ finite sets of formulas)

- $\Gamma \,/\, \Delta$ is L-derivable (or valid) if $\Gamma \vdash_L \delta$ for some $\delta \in \Delta$
- $\Gamma \,/\, \Delta$ is L-admissible (written as $\Gamma \mathrel{|\!\!\sim}_L \Delta$) if every L-unifier of Γ also unifies some $\delta \in \Delta$

$$E(\Gamma \,/\, \Delta) := \bigwedge_{\gamma\in\Gamma} E(\gamma) \to \bigvee_{\delta\in\Delta} E(\delta):$$

- $\Gamma \,/\, \Delta$ is derivable iff $E(\Gamma \,/\, \Delta)$ holds in all K-algebras
- $\Gamma \,/\, \Delta$ is admissible iff $E(\Gamma \,/\, \Delta)$ holds in free K-algebras

Note: Γ is unifiable iff $\Gamma \mathrel{|\!\!\not\sim}_L \varnothing$

Multiple-conclusion consequence relations

Single-conc. admissible rules form a consequence relation

Multiple-conc. admissible rules form a (finitary structural) multiple-conclusion consequence relation:

- $\varphi \mathrel{|\!\sim} \varphi$
- $\Gamma \mathrel{|\!\sim} \Delta$ implies $\Gamma, \Gamma' \mathrel{|\!\sim} \Delta, \Delta'$
- $\Gamma \mathrel{|\!\sim} \varphi, \Delta$ and $\Gamma, \varphi \mathrel{|\!\sim} \Delta$ imply $\Gamma \mathrel{|\!\sim} \Delta$
- $\Gamma \mathrel{|\!\sim} \Delta$ implies $\sigma(\Gamma) \mathrel{|\!\sim} \sigma(\Delta)$ for every substitution σ

A set B of rules is a basis of L-admissible rules if $\mathrel{|\!\sim}_L$ is the smallest m.-c. c. r. containing $\vdash_L$ and B

Admissibly saturated approximation

Γ is admissibly saturated if $\Gamma \mathrel{|\!\!\sim}_L \Delta$ implies $\Gamma \vdash_L \Delta$ for any Δ

Assume for simplicity that L has a well-behaved conjunction.

Admissibly saturated approximation of Γ:
a finite set of formulas Π_Γ such that

- each $\pi \in \Pi_\Gamma$ is admissibly saturated
- $\Gamma \mathrel{|\!\!\sim}_L \Pi_\Gamma$
- $\pi \vdash_L \varphi$ for each $\pi \in \Pi_\Gamma$ and $\varphi \in \Gamma$

Application of admissible saturation

Assuming every Γ has an a.s. approximation Π_Γ:

- Reduction of $\mathrel{|\!\!\sim}_L$ to $\vdash_L$:

$$\Gamma \mathrel{|\!\!\sim}_L \Delta \quad \text{iff} \quad \forall \pi \in \Pi_\Gamma \, \exists \psi \in \Delta \; \pi \vdash_L \psi$$

- If $\Gamma \mapsto \Pi_\Gamma$ is computable and $\vdash_L$ is decidable, then $\mathrel{|\!\!\sim}_L$ is decidable
- If $\Gamma \,/\, \Pi_\Gamma$ is derivable in $\vdash_L$ + a set of rules $B \subseteq \mathrel{|\!\!\sim}_L$, then B is a basis of admissible rules
- If each $\pi \in \Pi_\Gamma$ has an mgu σ_π, then $\{\sigma_\pi : \pi \in \Pi_\Gamma\}$ is a complete set of unifiers for Γ
  $\Rightarrow$ finitary unification

Projective formulas

π is projective if it has a unifier σ such that $\pi \vdash_L x \leftrightarrow \sigma(x)$ (in general: $\pi \vdash_L F(x, \sigma(x))$) for every variable x

- Every projective formula is admissibly saturated
- σ is an mgu of π: if τ is a unifier of π, then $\tau \equiv_L \tau \circ \sigma$

Projective approximation := admissibly saturated approximation consisting of projective formulas

If projective approximations exist:
convenient tool for analysis of unification and admissibility

Parameters

In real life, propositional atoms model both “variables” and “constants”

We don’t want to allow substitution for constants

$\Rightarrow$ Generalize the set-up to use two kinds of atoms:

- variables $\{x_n : n \in \omega\}$
- parameters $\{p_n : n \in \omega\}$
  (aka metavariables, constants, coefficients)

Unification with parameters

Substitutions only modify variables, we require $\sigma(p_n) = p_n$

Adapt accordingly the definitions of other notions:

- Unifier, $\sigma \preceq_L \tau$, admissible rule, m.-c. consequence relation, basis, a.s. formula and approximation, projective formula

Exception: "Propositional logic" is always assumed to be closed under substitution for parameters

Transitive modal logics

Transitive modal logics

Normal modal logics with a single modality $\Box$, include the transitivity axiom $\Box x \to \Box\Box x$ (i.e., $L \supseteq \mathbf{K4}$)

Common examples: various combinations of

logic	axiom (on top of $\mathbf{K4}$)	finite rooted transitive frames
$\mathbf{S4}$	$\Box x \to x$	reflexive
$\mathbf{D4}$	$\Diamond\top$	final clusters reflexive
$\mathbf{GL}$	$\Box(\Box x \to x) \to \Box x$	irreflexive
$\mathbf{K4Grz}$	$\Box(\Box(x \to \Box x) \to x) \to \Box x$	no proper clusters
$\mathbf{K4.1}$	$\boxdot\Diamond x \to \diamonddot\Box x$	no proper final clusters
$\mathbf{K4.2}$	$\Diamond\boxdot x \to \Box\diamonddot x$	unique final cluster
$\mathbf{K4.3}$	$\Box(\boxdot x \to y) \lor \Box(\Box y \to x)$	linear (chain of clusters)
$\mathbf{K4B}$	$x \to \Box\Diamond x$	lone cluster
$\mathbf{S5}$	$= \mathbf{S4} \oplus \mathbf{B}$	lone reflexive cluster

Some classes of transitive logics

Cofinal-subframe (csf) logics:

- complete wrt a class of frames closed under the removal of a subset of non-final points
- all combinations of logics from the table are csf

Extensible logics:

- If a frame F has a unique root r whose reflexivity is compatible with L, and $F \smallsetminus \{r\} \vDash L$, then $F \vDash L$
- $\mathbf{K4}$, $\mathbf{S4}$, $\mathbf{GL}$, $\mathbf{K4Grz}$, $\mathbf{S4Grz}$, $\mathbf{D4}$, $\mathbf{K4.1}$, ... (not $\mathbf{K4.2}$, ...)

Linear extensible logics:

- $\mathbf{K4.3}$, $\mathbf{S4.3}$, $\mathbf{GL.3}$, ...

Unification in transitive modal logics

A lot is known about admissibility without parameters:

- Admissibility is decidable in a large class of logics (Rybakov)
- Extensible logics have projective approximations (Ghilardi)
  - finitary unification type
  - complete sets of unifiers computable
- Bases of admissible rules for extensible logics (J.)
- Computational complexity of admissibility (J.)
  - Lower bounds for a quite general class of logics
  - Matching upper bounds for csf extensible logics
- . . . and more . . .

Projectivity in modal logics

Fix $L \supseteq \mathbf{K4}$ with the finite model property (fmp)

Extension property: if F is a finite L-model with a unique root r and $x \vDash \varphi$ for every $x \in F \setminus \{r\}$, then we can change valuation of variables in r to make $r \vDash \varphi$

Theorem [Ghilardi]: The following are equivalent:

- φ is projective
- φ has the extension property
- θ_φ is a unifier of φ

where θ_φ is an explicitly defined composition of substitutions of the form $\sigma(x) = \boxdot\varphi \wedge x$ or $\sigma(x) = \boxdot\varphi \to x$

Semantics of admissible rules

If L is an extensible logic with fmp, tfae:

- $\Gamma \mathrel{|\!\!\sim}_L \Delta$
- $\Gamma \,/\, \Delta$ holds in every L-frame W s.t. $\forall X \subseteq W$ finite:
  - If $L \not\supseteq \mathbf{S4}$, X has an irreflexive tight predecessor t:
  $$t{\uparrow} = X{\Uparrow}$$
  - If $L \not\supseteq \mathbf{GL}$, X has a reflexive tight predecessor t:
  $$t{\uparrow} = \{t\} \cup X{\Uparrow}$$

For linear extensible L, take only $|X| \le 1$

Notation: $x{\uparrow} = \{y : x\ R\ y\}$, $x{\Uparrow} = \{x\} \cup x{\uparrow}$, $X{\Uparrow} = \bigcup_{x \in X} x{\Uparrow}$

Bases of admissible rules

If L is an extensible logic, it has a basis of admissible rules consisting of

$$\frac{\Box y \to \Box x_1 \vee \cdots \vee \Box x_n}{\boxdot y \to x_1, \ldots, \boxdot y \to x_n} \qquad (n \in \omega)$$

if L admits an irreflexive point, and

$$\frac{\boxdot(y \leftrightarrow \Box y) \to \Box x_1 \vee \cdots \vee \Box x_n}{\boxdot y \to x_1, \ldots, \boxdot y \to x_n} \qquad (n \in \omega)$$

if L admits a reflexive point

For L linear extensible, take only $n = 0, 1$

Complexity of admissible rules

Lower bound:
Assume $L \supseteq \mathbf{K4}$ and every depth-3 tree is a skeleton of an L-frame with prescribed final clusters.
Then L-admissibility is $\textsc{coNEXP}$-hard.

Upper bounds: Admissibility in

- csf extensible logics is $\textsc{coNEXP}$-complete
- csf linearly extensible logics is $\textsc{coNP}$-complete

Unification with parameters in modal logic

Known results

Less is known about admissibility in transitive modal logics in the presence of parameters:

- Rybakov's results on decidability of admissibility also apply to admissibility with parameters
- Recently, he expanded the results to effectively construct complete sets of unifiers $\Rightarrow$ finite unification type

Terminology: From now on, admissibility and unification always allow parameters

New results

In this talk, we will show:

- Ghilardi-style characterization of projective formulas
- Existence of projective approximations for cluster-extensible (clx) logics [defined on the next slide]
- Semantic description of admissibility in clx logics
- Explicit bases of admissible rules for clx logics
- Computational complexity:
  - Lower bounds on unification in wide classes of transitive logics
  - Matching upper bounds for admissibility in csf clx logics
- Translation of these results to intuitionistic logic

Cluster-extensible logics

Let L be a transitive modal logic with fmp, $n \in \omega$, and C a finite cluster.

A finite rooted frame F is of type $\langle n, C\rangle$ if its root cluster $\mathrm{rcl}(F)$ is isomorphic to C and has n immediate successor clusters.

L is $\langle n, C\rangle$-extensible if:
For every type-$\langle n, C\rangle$ frame F, if $F \smallsetminus \mathrm{rcl}(F)$ is an L-frame, then so is F.

L is cluster-extensible (clx), if it is $\langle n, C\rangle$-extensible whenever there exists a type-$\langle n, C\rangle$ L-frame.

Examples: All combinations of $\mathbf{K4}$, $\mathbf{S4}$, $\mathbf{GL}$, $\mathbf{D4}$, $\mathbf{K4Grz}$, $\mathbf{K4.1}$, $\mathbf{K4.3}$, $\mathbf{K4B}$, $\mathbf{S5}$, $\pm$ bounded branching

Nonexamples: $\mathbf{K4.2}$, $\mathbf{S4.2}$, $\dots$

Projective formulas: the extension property

Fix $L \supseteq \mathbf{K4}$ with the fmp, and P and V finite sets of parameters and variables, resp.

- If F is a rooted model with valuation of $P \cup V$, its variant is any model F' which differs from F only by changing the value of some variables $x \in V$ in $\mathrm{rcl}(F)$
- A set M of finite rooted L-models evaluating $P \cup V$ has the model extension property, if:
  every L-model F whose all rooted generated proper submodels belong to M has a variant $F' \in M$
- A formula φ in atoms $P \cup V$ has the model extension property if $\mathrm{Mod}_L(\varphi) := \{F : \forall x \in F\ (x \vDash \varphi)\}$ does

Projective formulas: Löwenheim substitutions

Let φ be a formula in atoms $P \cup V$

- For every $D = \{\beta_x : x \in V\}$, where each β_x is a Boolean function of the parameters P, define the substitution

$$\theta_D(x) = (\boxdot\varphi \wedge x) \vee (\neg\boxdot\varphi \wedge \beta_x)$$

- Let θ_φ be the composition of substitutions θ_D for all the $2^{2^{|P|}|V|}$ possible D's, in arbitrary order

Projective formulas: a characterization

Theorem:
Let $L \supseteq \mathbf{K4}$ have the fmp, and φ be a formula in finitely many parameters P and variables V. Tfae:

- φ is projective
- φ has the model extension property
- θ_φ^N is a unifier of φ

where $N = (|B| + 1)(2^{|P|} + 1)$, $B = \{\psi : \Box\psi \subseteq \varphi\}$

Remark: If $P = \varnothing$, we have $N \leq 2|\varphi|$.
Ghilardi's original proof gives N nonelementary (tower of exponentials of height $\mathrm{md}(\varphi)$)

Projective approximations

Theorem:
If L is a clx logic, every formula φ has a projective approximation Π_φ.

Moreover, every $\pi \in \Pi_\varphi$ is a Boolean combination of subformulas of φ.

Corollary:

- $\{\theta^N_\pi : \pi \in \Pi_\varphi\}$ is a complete set of unifiers of φ
- Admissibility in L is decidable (if L r.e.?)
- If $n = |\varphi|$, then $|\Pi_\varphi| \leq 2^{2^n}$, and $|\pi| = O(n2^n)$ $\forall \pi \in \Pi_\varphi$
- $|\theta^N_\pi|$ is doubly exponential in $|B| + |V|$, and triply exponential in $|P|$. This is likely improvable.

Size of projective approximations

The bounds $|\Pi_\varphi| = 2^{2^{O(n)}}$ and $|\pi| = 2^{O(n)}$ for $\pi \in \Pi_\varphi$ are asymptotically optimal, even if $P = \varnothing$:

- If L is $\langle 2, \bullet \rangle$-extensible (e.g., $\mathbf{K4}$, $\mathbf{GL}$), consider

$$\varphi_n = \bigwedge_{i<n} (\Box x_i \lor \Box \neg x_i) \to \Box y \lor \Box \neg y$$

$$\Pi_{\varphi_n} = \Big\{ \bigwedge_{i<n} (\boxdot x_i \lor \boxdot \neg x_i) \to (y \leftrightarrow \beta(\vec{x})) \;\Big|\; \beta \colon \mathbf{2}^n \to \mathbf{2} \Big\}$$

- Similar examples work for $\langle 2, \circ \rangle$-extensible logics ($\mathbf{S4}$)

Irreflexive extension rules

Let $n < \omega$, and P a finite set of parameters.

$\mathrm{Ext}^{P}_{n,\bullet}$ is the set of rules

$$\frac{P^e \wedge \Box y \to \Box x_1 \vee \cdots \vee \Box x_n}{\boxdot y \to x_1, \dots, \boxdot y \to x_n}$$

for each assignment $e \colon P \to \mathbf{2}$

Notation:

$\varphi^1 = \varphi$, $\varphi^0 = \neg\varphi$, $P^e = \bigwedge_{p \in P} p^{e(p)}$, $\mathbf{2}^P = \{e \mid e \colon P \to \mathbf{2}\}$

Reflexive extension rules

Let C be a finite reflexive cluster

$\mathrm{Ext}^P_{n,C}$ is the set of the following rules:
Pick $E\colon C \to \mathbf{2}^P$ and $e_0 \in E(C)$, and consider

$$\frac{\begin{aligned}P^{e_0} \land \boxdot\Big(y \to \bigvee_{e\in E(C)} \Box(P^e \to y)\Big) \land \bigwedge_{e\in E(C)} \boxdot\big(\Box(P^e \to \Box y) \to y\big)&\\ \to \Box x_1 \lor \cdots \lor \Box x_n&\end{aligned}}{\boxdot y \to x_1, \dots, \boxdot y \to x_n}$$

Tight predecessors

P a finite set of parameters, C a finite cluster, $n < \omega$

- A P-L-frame is a (Kripke or general) L-frame W together with a fixed valuation of parameters $p \in P$
- If $X = \{w_1, \dots, w_n\} \subseteq W$ and $E\colon C \to \mathbf{2}^P$, a tight E-predecessor (E-tp) of X is $\{u_c : c \in C\} \subseteq W$ such that

$$u_c \vDash P^{E(c)}, \qquad u_c{\uparrow} = X{\updownarrow} \cup \{u_d : d \in c{\uparrow}\}$$

  (Note: $c{\uparrow} = C$ if C is reflexive, $c{\uparrow} = \varnothing$ if irreflexive)
- W is $\langle n, C\rangle$-extensible if every $\{w_1, \dots, w_n\} \subseteq W$ has an E-tp for every $E\colon C \to \mathbf{2}^P$
- If L is a clx logic, W is L-extensible if it is $\langle n, C\rangle$-extensible whenever L is

Correspondence and completeness

Theorem: If P is a finite set of parameters and W is a descriptive or Kripke P-$\mathbf{K4}$-frame, tfae:

- $W \vDash \mathrm{Ext}^P_{n,C}$
- W is $\langle n, C\rangle$-extensible

Corollary: For a logic $L \supseteq \mathbf{K4}$, tfae:

- L is $\langle n, C\rangle$-extensible
- $\mathrm{Ext}^P_{n,C}$ is L-admissible for every P

Theorem: If L has fmp and is $\langle n, C\rangle$-extensible for all $\langle n, C\rangle \in X$, then $L + \{\mathrm{Ext}^P_{n,C} : \langle n, C\rangle \in X\}$ is complete wrt locally finite (= all rooted subframes finite) P-L-frames, $\langle n, C\rangle$-extensible for each $\langle n, C\rangle \in X$

Semantics and bases of admissible rules

Theorem:
Let L be a clx logic, and $\Gamma \mathbin{/} \Delta$ a rule in a finite set of parameters P. Then tfae:

- $\Gamma \mathrel{|\!\!\sim}_L \Delta$
- $\Gamma \mathbin{/} \Delta$ holds in every [locally finite] L-extensible P-L-frame
- $\Gamma \mathbin{/} \Delta$ is derivable in $\vdash_L$ extended by the rules $\mathrm{Ext}^P_{n,C}$ such that L is $\langle n, C\rangle$-extensible

Corollary: If L is a clx logic, it has a basis of admissible rules consisting of $\mathrm{Ext}^P_{n,C}$ for all finite P and all $\langle n, C\rangle$ such that L is $\langle n, C\rangle$-extensible

Complexity: wide logics

Theorem:
If $L \supseteq \mathbf{K4}$ has width ≥ 2, then unification (and thus inadmissiblity) in L is NEXP-hard.

Theorem:
If L is a csf clx logic of width ≥ 2 and bounded cluster size, then inadmissibility (and thus unification) in L is NEXP-complete.

Examples: $\mathbf{GL}$, $\mathbf{K4Grz}$, $\mathbf{S4Grz}$, $\dots$

Complexity: fat logics

Theorem:
If $L \supseteq \mathbf{K4}$ has unbounded cluster size, then unification in L is coNEXP-hard.

Theorem:
If L is a clx logic of width ≤ 1 and unbounded cluster size, then inadmissibility in L is coNEXP-complete.

Examples: $\mathbf{S5}$, $\mathbf{K4.3}$, $\mathbf{S4.3}$, $\dots$

Complexity: wide and fat logics

L is "chubby" if for all $n > 0$ there is a finite rooted L-frame containing an n-element cluster C and an element incomparable with C

Recall: $\Sigma_2^{\mathrm{EXP}} = \mathrm{NEXP}^{\mathrm{NP}}$

Theorem:
If $L \supseteq \mathbf{K4}$ is chubby, then unification in L is Σ_2^{EXP}-hard.

Theorem:
If L is a csf clx logic of width ≥ 2 and unbounded cluster size, then inadmissibility in L is Σ_2^{EXP}-complete.

Examples: $\mathbf{K4}$, $\mathbf{S4}$, $\mathbf{S4.1}$, ...

Complexity: slim logics

Theorem:
If $L \supseteq \mathbf{K4}$, then unification in L is PSPACE-hard, unless L is a tabular logic of width 1.

Theorem:
If L is a clx logic of width 1, bounded cluster size, and depth > 1, then admissibility in L is PSPACE-complete.

Examples: $\mathbf{GL.3}$, $\mathbf{K4Grz.3}$, $\mathbf{S4Grz.3}$, $\dots$

Theorem:
If L is a tabular logic of width 1 and depth d, then unification and inadmissibility in L are Π^{P}_{2d}-complete.

Examples: $\mathbf{S5} + \mathbf{Alt}_n$, $\mathbf{K4} + \Box\bot$, $\dots$

Complexity: summary

We get the following classification for csf clx logics:

<table>
<tr><th colspan="2">logic</th><th rowspan="2">$\nvdash_L$</th><th colspan="2">param'r-free</th><th colspan="2">with param's</th><th rowspan="2">example</th></tr>
<tr><th>cluster size</th><th>bran-ching</th><th>unif'n</th><th>$\not\mathrel{|\!\sim}_L$</th><th>unif'n</th><th>$\not\mathrel{|\!\sim}_L$</th></tr>
<tr><td rowspan="2">$< \infty$</td><td>0</td><td colspan="4" rowspan="3">NP-complete</td><td colspan="2">Π^{P}_2-c.</td><td>$\mathbf{S5} + \mathbf{Alt}_n$</td></tr>
<tr><td>1</td><td colspan="2">PSPACE-c.</td><td>$\mathbf{GL.3}$</td></tr>
<tr><td>∞</td><td>≤ 1</td><td colspan="2">coNEXP-c.</td><td>$\mathbf{S5}$, $\mathbf{S4.3}$</td></tr>
<tr><td>$< \infty$</td><td rowspan="2">≥ 2
(∞)</td><td rowspan="2">PSPACE-c.</td><td rowspan="2">?</td><td colspan="3">NEXP-complete</td><td>$\mathbf{GL}$, $\mathbf{Grz}$</td></tr>
<tr><td>∞</td><td></td><td colspan="2">Σ^{EXP}_2-c.</td><td>$\mathbf{K4}$, $\mathbf{S4}$</td></tr>
</table>

Intuitionistic logic

Results for modal logics can be transferred to intermediate logics by means of the Block–Esakia isomorphism

The following result by Rybakov can be generalized to admissibility with parameters:

Theorem:
If $L \supseteq \mathbf{IPC}$ and σL is its largest modal companion, then

$$\Gamma \mathrel{|\!\!\sim}_L \Delta \Leftrightarrow \mathsf{T}(\Gamma) \mathrel{|\!\!\sim}_{\sigma L} \mathsf{T}(\Delta),$$

where T is the Gödel translation

[However, $\bigwedge_{p \in P} \Box(p \to \Box p) \to \mathsf{T}(\varphi)$ is often more convenient.]

Corollaries

Note: The only "clx" $L \supseteq \mathbf{IPC}$ are $\mathbf{IPC}$ itself and the bounded branching logics $\mathbf{T}_n$ (incl. $\mathbf{T}_1 = \mathbf{LC}$, $\mathbf{T}_0 = \mathbf{CPC}$)

The translation yields:

- Char. of projective formulas in $L \supseteq \mathbf{IPC}$ with fmp
- Existence of projective approximations and semantic description of $\mathrel|\joinrel\sim_L$ for $\mathbf{IPC}$ and $\mathbf{T}_n$
- Complexity (lower bounds need an extra argument): unification and inadmissibility is
  - NEXP-complete for $\mathbf{IPC}$
  - PSPACE-complete for $\mathbf{LC}$
  - Π^{P}_{2d}-complete for $\mathbf{G}_{d+1}$
  - NEXP-hard for any other intermediate logic

Intuitionistic extension rules

Bases of admissible rules require a separate construction:

A basis for $\mathbf{IPC}$ and $\mathbf{T}_n$ is given by the rules

$$\frac{\bigwedge P \wedge \Big(\bigvee_{i=1}^{n} x_i \vee \bigvee Q \to y\Big) \to \bigvee_{i=1}^{n} x_i \vee \bigvee Q}{\bigwedge P \wedge y \to x_1, \dots, \bigwedge P \wedge y \to x_n}$$

where P, Q are disjoint finite sets of parameters

Questions

- Is there a general reduction of admissibility to nonunifiability (with parameters)?
- What is the complexity of parameter-free unification for non-linear csf clx logics?
  - NP-hard and NEXP-easy
  - If L includes $\mathbf{D4}$ or $\mathbf{GL}$ or $\Box x \to x \vee \Box\bot$: NP-complete (the universal frame of rank 0 is very simple)
  - Otherwise?

Thank you for attention!

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