Fluids with viscosity depending on pressure - mathematical analysis and applications

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Balance laws for continuum:

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0 \tag{1}$$

$$\partial_t (\rho \mathbf{v}) + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) = \operatorname{div} \mathbf{T} + \rho \mathbf{f}, \qquad \mathbf{T} = \mathbf{T}^\top \tag{2}$$

$$\partial_t (\rho E) + \operatorname{div}(\rho E) + \operatorname{div} \mathbf{q} = \operatorname{div}(\mathbf{T} \mathbf{v}) + \rho \mathbf{f} \cdot \mathbf{v}, \;\; E := \frac{|\mathbf{v}|^2}{2} + e \tag{3}$$

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Constitutive equation for the stress tensor:

$$\mathbf{S}(p, \mathbf{D}(\mathbf{v})) = \nu(p, |\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v}) \tag{9}$$

Mathematical properties of pressure-dependent fluids

$$\operatorname{div} \mathbf{v} = 0$$
$$\partial_t \mathbf{v} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) + \nabla p = \operatorname{div} \mathbf{S}(p, \mathbf{D}(\mathbf{v})) + \mathbf{f}$$

In Navier-Stokes equations, only ∇p is present $\Rightarrow$ pressure is given up to an additive constant.

Here: p itself is in the equations $\Rightarrow$ its value has to be fixed by additional input parameter.

Ways of fixing p:
(1) mean value over (sub)domain
(2) boundary condition—prescribe p on some interface

Pressure-dependent fluids in practice

In most of real situations, the fluid viscosity can be considered independent of the pressure. However, there are certain situations in which the dependence on the pressure becomes significant, e.g. in elastohydrodynamics, where the pressure differs in several orders of magnitude.

Examples of commonly used experimental relations:

$$\nu(p) = \nu_0 \exp(\alpha p) \qquad \text{(Barus, 1893)},$$

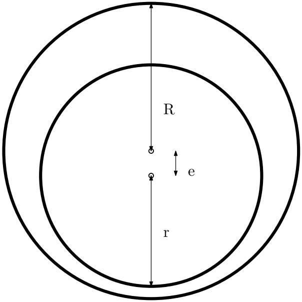
$$\nu(p) = \exp\left(-1.2 + (\log \nu_0 + 1.2)\left(1 + \frac{p}{c}\right)^Z\right) \qquad \text{(Roelands, 1966)}.$$

Particular application: journal bearing lubrication

Simple type of bearing consisting of 2 cylinders and a lubricant filling the gap in between.

R – outer ring radius
r – inner ring (shaft) radius
e – eccentricity

Wide use: e.g. steam turbines, centrifugal compressors, pumps and motors, etc.



Boundary conditions for pressure-dependent fluids

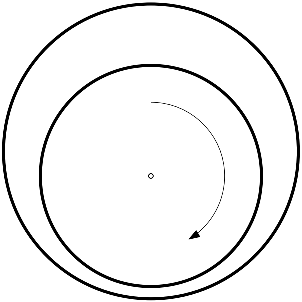
Dirichlet condition

$$\mathbf{v} = \mathbf{v}_D \text{ on } \partial\Omega$$

has to be supplemented with additional constraint fixing the level of pressure:

$$⨏_{\Omega_0} p = p_0.$$

Value of p_0 has influence on the velocity field as well.



Existence results

- Dirichlet b.c., steady-state case (Franta et al. [2005], Lanzendörfer [2009])
- Navier's b.c., unsteady case (Bulíček et al. [2007], Bulíček and Fišerová [2009])

All results consider viscosity which satisfies

(A1) $\frac{\partial\nu(p,|\mathbf{D}|^2)}{\partial|\mathbf{D}|^2} \approx (1+|\mathbf{D}|^2)^{\frac{r-4}{2}}$, $r \in (1,2)$;

(A2) $\left|\frac{\partial\nu(p,|\mathbf{D}|^2)}{\partial p}\right| \leq C(1+|\mathbf{D}|^2)^{\frac{r-4}{4}}$;

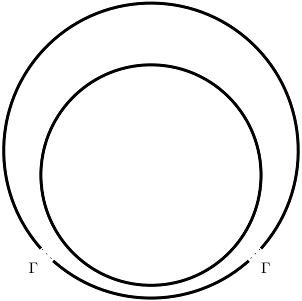
e.g.

$$\nu(p,|\mathbf{D}|^2) = (A+|\mathbf{D}|^2+(1+(\alpha p)^2)^{\frac{1}{r-2}})^{\frac{r-2}{2}}.$$

Inflow/outflow conditions

Let $\partial\Omega$ be divided into Γ_D (wall) and Γ (inflow/outflow). Prescribing boundary conditions of the type

$$\begin{aligned} \mathbf{v} &= \mathbf{v}_D && \text{on } \Gamma_D, \\ p\mathbf{n} - \mathbf{S}\mathbf{n} &= \mathbf{b}(\mathbf{v}) && \text{on } \Gamma, \end{aligned}$$



the level of pressure will be uniquely determined.

Main result

(A1) $\frac{\partial \mathbf{S}(p,|\mathbf{D}|^2)}{\partial \mathbf{D}} \approx (1+|\mathbf{D}|^2)^{\frac{r-2}{2}}$, $r \in (1,2)$;

(A2) $\left|\frac{\partial \mathbf{S}(p,|\mathbf{D}|^2)}{\partial p}\right| \leq C(1+|\mathbf{D}|^2)^{\frac{r-2}{4}}$;

(A3) for every $\boldsymbol{\varphi} \in L^\gamma(\Gamma)$:

$$\int_\Gamma \mathbf{b}(\boldsymbol{\varphi}) \cdot \boldsymbol{\varphi} \geq -\frac{1}{2} \int_\Gamma (\boldsymbol{\varphi} \cdot \mathbf{n})|\boldsymbol{\varphi}|^2.$$

Theorem (Lanzendörfer and Stebel [2008])

(i) *Let (A1)–(A3). Then there exists a weak solution* $(\mathbf{v}, p)$. *Moreover* p *is determined uniquely by* $\mathbf{v}$.

(ii) *For small data there is exactly one weak solution.*

Key arguments of the proof

1. A priori estimate of the convective term

$$\int_\Omega \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) \cdot \mathbf{v} = \int_\Omega \underbrace{\operatorname{div} \mathbf{v}}_{=0} |\mathbf{v}|^2 + \int_\Omega \mathbf{v} \cdot \nabla \left(\frac{|\mathbf{v}|^2}{2} \right)$$

$$\overset{Green}{=} \frac{1}{2} \int_\Gamma (\mathbf{v} \cdot \mathbf{n}) |\mathbf{v}|^2$$

$$\int_\Gamma \mathbf{b}(\mathbf{v}) \cdot \mathbf{v} \geq -\frac{1}{2} \int_\Gamma (\mathbf{v} \cdot \mathbf{n}) |\mathbf{v}|^2$$

Key arguments of the proof II.

2. Uniform presure estimate
   The Bogovskii operator (div^{-1})

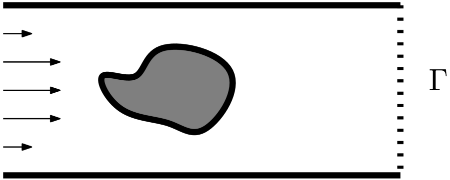
$$\mathcal{B} : L^q_0(\Omega) \to W^{1,q}_0(\Omega)^d$$

can be extended to

$$\mathcal{B}_\Gamma : L^q(\Omega) \to W^{1,q}_{\Gamma_D}(\Omega);$$

Examples of inflow/outflow b.c.

Free outflow



1. Nonreflecting conditions of the type

$$p\mathbf{n} - \mathbf{S}\mathbf{n} = \mathbf{h}(\mathbf{x}) + \frac{1}{2}(\mathbf{v}\cdot\mathbf{n})^{-}\mathbf{v}$$

2. Conditions on the Bernoulli pressure

$$\left(p + \frac{1}{2}|\mathbf{v}|^2\right)\mathbf{n} - \mathbf{S}\mathbf{n} = \mathbf{h}(\mathbf{x})$$

Examples of inflow/outflow b.c. II.

Porous wall/membrane

3. Filtration conditions of the type

$$p - \mathbf{Sn} \cdot \mathbf{n} = p_{out} + (c_1 + c_2|\mathbf{v} \cdot \mathbf{n}| + c_3|\mathbf{v} \cdot \mathbf{n}|^2)\mathbf{v} \cdot \mathbf{n},$$
$$\mathbf{v} \times \mathbf{n} = \mathbf{0}$$

p_{out} ...... given pressure at the outlet
c_1, c_2, c_3 ... coefficients from the generalized Darcy law

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