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Maximum Principles and Stability for Systems of Linear Functional Differential Equations

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1. Introduction

Consider the system

$$(M_i x)(t) \equiv x_i'(t) + \sum_{i=1}^{n} (B_{ij} x_j)(t) = f_i(t), \quad t \in [0, \omega], \ i = 1, ...n, \quad (1.1)$$

where $x = col(x_1, ..., x_n)$, $B_{ij} : C_{[0,\omega]} \to L^{\infty}_{[0,\omega]}$, $i, j = 1, ..., n$, are linear continuous operators, $C_{[0,\omega]}$ and $L^{\infty}_{[0,\omega]}$ are the spaces of continuous and measurable essentially bounded functions $y : [0, \omega] \to R^1$ respectively. The operators B_{ij} can be, for example, of the following forms

$$(B_{ij} x)(t) = \sum_{k=1}^{m} p_k(t) x(h_k(t)), \quad t \in [0, \omega],$$

$$x(\xi) = 0 \ \text{for} \ \xi \notin [0, \omega].$$

$$(B_{ij}x)(t) = \int_0^{\omega} K_{ij}(t,s)x(s)ds, \quad t \in [0,\omega],$$

and also their linear combinations or superpositions.

Let $l : C^n_{[0,\omega]} \to R^n$ be a linear bounded functional.

1) **Maximum Inequalities Principle.** Under corresponding conditions from the inequalities

$$(M_i x)(t) \equiv x_i'(t) + \sum_{i=1}^{n} (B_{ij}x_j)(t) \geq (M_i y)(t), \quad t \in [0,\omega], \ i = 1, ...n, \ lx \geq ly \tag{1.2}$$

it follows that

$$x_i(t) \geq y_i(t), \quad t \in [0,\omega], \ i = 1, ..., n. \tag{1.3}$$

1. Introduction (cont.)

S.A.Chaplygin, Foundations of new method of approximate integration of differential equations. Moscow,1919 (Collected works 1, GosTechIzdat, 1948, pp.348-368).
N.N. Luzin, About method of approximate integration of acad. S.A. Chaplygin, *Uspekhi Mathem. Nauk* **6** (1951), 3-27.
N.V. Azbelev, On an approximate solution of ordinary differential equations of the nth order based upon S. A. Čaplygin's method. (Russian) *Doklady Akad. Nauk SSSR* (N.S.) **83** (1952), 517–519.
V.Lakshmikantham and S.Leela, Differential and integral inequalities, Academic Press, 1969.
If the homogeneous BVP

$$(M_i x)(t) \equiv x_i'(t) + \sum_{i=1}^{n} (B_{ij} x_j)(t) = 0, \quad t \in [0, \omega], \; i = 1, ...n, \; lx = 0, \tag{1.4}$$

has only the trivial solution, then the BVP

$$(M_i x)(t) = f_i(t), \quad t \in [0, \omega], \; i = 1, ...n, \; lx = \alpha, \tag{1.5}$$

has for each $f_i \in L_{[0,\omega]}$, $i = 1, ..., n$, $\alpha \in R^n$ a unique solution, which has the following representation [2]

1. Introduction (cont.)

$$x(t) = \int_0^\omega G(t,s)f(s)ds + X(t)\alpha, \quad t \in [0,\omega], \tag{1.6}$$

where the $n \times n$ matrix $G(t,s)$ is called the Green's matrix of problem (1.4), $X(t)$ is a $n \times n$ fundamental matrix of the system $(M_i x)(t) = 0,\ i = 1, ..., n$, such that $lX = E$ (E is the unit $n \times n$ matrix), $f = col(f_1, ..., f_n)$.

[2] N.V. Azbelev, V.P. Maksimov and L.F. Rakhmatullina, *Introduction to the Theory of Functional Differential Equations*, Advanced Series in Math. Science and Engineering 3, Atlanta, GA: World Federation Publisher Company, 1995.

As a particular case of system (1.1) we can consider the delay system

$$x_i'(t) + \sum_{j=1}^{n} p_{ij}(t)x_j(h_{ij}(t)) = f_i(t),\ i = 1, ..., n,\ t \in [0,\omega], \tag{1.7}$$

$$x(\theta) = 0 \text{ for } \theta < 0,$$

1. Introduction (cont.)

where p_{ij} are measurable essentially bounded functions, and h_{ij} are measurable functions such that $h_{ij}(t) \leq t$ for $i, j = 1, ..., n,\ t \in [0, \omega]$. Its general solution has the representation

$$x(t) = \int_0^t C(t,s)f(s)ds + C(t,0)x(0),\ t \in [0,\omega], \qquad (1.8)$$

where $C(t,s) = \{C_{ij}(t,s)\}_{i,j=1,\dots,n}$ is called the Cauchy matrix of system (1.7). For each fixed s the matrix $C(t,s)$ is the fundamental matrix of the system

$$x_i'(t) + \sum_{j=1}^{n} p_{ij}(t)x_j(h_{ij}(t)) = 0,\ i = 1, ..., n,\ t \in [s,\omega], \qquad (1.9)$$

$$x(\theta) = 0 \text{ for } \theta < s,$$

such that $C(s,s)$ is the unit $n \times n$ matrix.
For system of ordinary differential equations

$$x_i'(t) + \sum_{j=1}^{n} p_{ij}(t)x_j(t) = f_i(t),\ i = 1, ..., n,\ t \in [0,\omega]. \qquad (1.10)$$

1. Introduction (cont.)

the classical Wazewskii's theorem claims: the condition

$$p_{ij} \leq 0 \text{ for } j \neq i, \ i,j = 1, ..., n, \tag{1.11}$$

is necessary and sufficient for nonnegativity of all elements $C_{ij}(t,s)$ of the Cauchy matrix and consequently of the property (1.2), (1.3)

T.Wazewski, Systemes des equations et des inegalites differentielled aux deuxieme membres et leurs applications, Ann. Polon. Math. 23 (1950), 112-166.
I.Kiguradze and B.Puza, *Boundary value problems for systems of linear functional differential equations.* FOLIA, Brno, Czech Republic, 2002.

Let k_i be either 1 or 2. We study the following question: when from the conditions

$$(-1)^{k_i}[(M_i x)(t) - (M_i y)(t)] \geq 0, \quad t \in [0, \omega], \quad lx = ly, \tag{1.12}$$

it does follow that for a corresponding fixed component x_r of the solution vector the inequality

$$x_r(t) \geq y_r(t), \quad t \in [0, \omega], \tag{1.13}$$

is satisfied. From the formula of solution's representation for the Cauchy problem

$$x(t) = \int_0^t C(t,s)f(s)ds + C(t,0)x(0), \ t \in [0, \omega], \tag{1.14}$$

1. Introduction (cont.)

it is clear that this property is reduced to sign-constancy of all elements $C_{ij}(t,s)$ $(j = 1, ..., n)$ standing only in the $r-$th row of the Cauchy matrix.

2) **Maximum principle as a boundedness of solutions:** *There exists a positive constant N such that* $\|x\| \leq N(\|f\| + \|\alpha\|)$.
This is a problem of continuous dependence of solutions on the right hand side f and

the boundary condition α.
Consider our equation on the semiaxis $[0, +\infty)$.
Bohl-Perron theorem for FDE with bounded memory: if for every bounded $f \implies$ solution x of the system

$$(M_i x)(t) \equiv x_i'(t) + \sum_{i=1}^{n} (B_{ij} x_j)(t) = f_i(t), \quad t \in [0, +\infty), \; i = 1, ...n, \tag{1.15}$$

is bounded, then all solutions and elements of the Cauchy matrix of

$$(M_i x)(t) \equiv x_i'(t) + \sum_{i=1}^{n} (B_{ij} x_j)(t) = 0, \quad t \in [0, +\infty), \; i = 1, ...n, \tag{1.16}$$

1. Introduction (cont.)

tend to zero exponentially, i.e. there exist N and γ,such that $\|C(t,s)\| \leq N\exp(-\gamma(t-s))$.

3) **Maximum boundaries principle:** *For the solutions of the homogeneous system (1.16) at least one of the inequalities* $x(0) \leq x(t) \leq x(\omega)$ *or* $x(\omega) \leq x(t) \leq x(0)$ *is fulfilled.* We study the case when *for a corresponding component of solution vector the inequalities* $x_r(0) \leq x_r(t) \leq x_r(\omega)$ *or* $x_r(\omega) \leq x_r(t) \leq x_r(0)$ *is fulfilled.*

2. Equation for one of the Components of Solution Vector

In this paragraph we consider the system

$$(M_i x)(t) \equiv x_i'(t) + \sum_{j=1}^{n} (B_{ij} x_j)(t) = f_i(t), \ t \in [0, \omega], \ i = 1, ..., n, \quad (2.1)$$

where $B_{ij} : C_{[0,\omega]} \to L_{[0,\omega]}$ are linear bounded Volterra operators for $i, j = 1, ..., n$.
Together with system (2.1) let us consider the following auxiliary system of the order $n - 1$

$$x_i'(t) + \sum_{i=1}^{n-1} (B_{ij} x_j)(t) = f_i(t), \quad t \in [0, \omega], \ i = 1, ...n - 1, \quad (2.2)$$

and denote by $K(t, s) = \{K_{ij}(t, s)\}_{i,j=1,...,n-1}$ its Cauchy matrix.

2. Equation for one of the Components of Solution Vector (cont.)

Lemma 2.1. *The component x_n of the solution vector of system (2.1) satisfies the following scalar functional differential equation*

$$(Mx_n)(t) \equiv x_n'(t) + (Bx_n)(t) = f^*(t), \quad t \in [0, \omega], \tag{2.3}$$

where

$$(Bx_n)(t) \equiv -\sum_{i=1}^{n-1} B_{ni}\{\int_0^t \sum_{j=1}^{n-1} K_{ij}(t,s)(B_{jn}x_n)(s)ds\}(t) + (B_{nn}x_n)(t), t \in [0, \omega] \tag{2.4}$$

and

$$f^*(t) = f_n(t) - \sum_{i=1}^{n-1} B_{ni}\{\int_0^t \sum_{j=1}^{n-1} K_{ij}(t,s)f_j(s)ds\}(t) - \sum_{i=1}^{n-1} B_{ni}\{\sum_{j=1}^{n-1} K_{ij}(t,0)x_j(0)\}(t). \tag{2.5}$$

3. Analysis of the Equation for One of the Components of Solution Vector

Consider the scalar equation

$$Mx(t) \equiv x'(t) + (Bx)(t) = f(t), \quad \text{for } t \in [0,\omega]. \tag{3.1}$$

Theorem 3.1. *Let $B : C_{[0,\omega]} \to L_{[0,\omega]}$ be a positive scalar Volterra operator, then the following assertions are equivalent*:

1) there exists a non–negative absolutely continuous function v such that

$$Mv(t) \equiv v'(t) + (Bv)(t) \leq 0, \quad \text{for } t \in [0,\omega], \quad v(\omega) > 0; \tag{3.2}$$

2) a nontrivial solution of the homogeneous equation $(Mx)(t) = 0, \ t \in [0,\omega]$ *has no zeros on* $[0,\omega]$;

3) the Cauchy function $C(t,s)$ of equation $Mx = f$ is positive for $0 \leq s \leq t \leq \omega$;

3. Analysis of the Equation for One of the Components of Solution Vector (cont.)

If in addition, the operator $B : C_{[0,\omega]} \to L_{[0,\omega]}$ is a nonzero operator, then the following assertion is included in the list of the equivalences:
4) *the problem*

$$Mx(t) = f(t), \text{ for } t \in [0,\omega], \; x(0) = x(\omega), \tag{3.3}$$

is uniquely solvable and its Green's function $P(t,s)$ is positive for $t,s \in [0,\omega]$.

S.A.Gusarenko and A.Domoshnitsky. Asymptotic and oscillation properties of first order linear scalar functional-differentional equations, Differentsial'nye uravnenija, v. 25, 12, 1989, pp. 2090-2103.

R.Hakl, A.Lomtatidze and J.Sremr. Some boundary value problems for first order scalar functional differential equations. FOLIA, Masaryk University, Brno, Czech Republic, 2002.

A.Domoshnitsky, Maximum principles and nonoscillation intervals for first order Volterra functional differential equations, Dynamics of Continuous, Discrete & Impulsive Systems. A: Mathematical Analysis, 15 (2008) 769-814.

4. Positivity of elements in the nth row of the Cauchy matrix

$$(M_i x)(t) \equiv x_i'(t) + \sum_{j=1}^{n} (B_{ij} x_j)(t) = f_i(t), \; t \in [0, \omega], \; i = 1, ..., n, \quad (4.1)$$

where $B_{ij} : C_{[0,\omega]} \to L_{[0,\omega]}$ are linear bounded Volterra operators for $i, j = 1, ..., n$.

Together with system (4.1) we consider the following auxiliary system of the order $n - 1$

$$x_i'(t) + \sum_{i=1}^{n-1} (B_{ij} x_j)(t) = f_i(t), \quad t \in [0, \omega], \; i = 1, ...n - 1, \quad (4.2)$$

4. Positivity of elements in the nth row of the Cauchy matrix (cont.)

Theorem 4.1. *Let all elements of the* $(n-1)\times(n-1)$ *Cauchy matrix* $K(t,s)$ *of system (4.2) be nonnegative, the operators* B_{nn} *and* B_{jn} *be positive and* B_{nj} *be negative for* $j = 1, ..., n-1$.

Then the following assertions are equivalent:
1) there exists an absolutely continuous vector function v *such that* $v' \in L^{\infty}_{[0,\omega]}$, $v_n(t) > 0$, $v_i(0) \leq 0$ *for* $i = 1, ..., n-1$, $(M_i v)(t) \leq 0$ *for* $i = 1, ..., n$, $t \in [0,\omega]$;
2) $C_{nn}(t,s) > 0$, $C_{nj}(t,s) \geq 0$ *for* $j = 1, ..., n-1$, $0 \leq s \leq t \leq \omega$;
3) the n-th component of the solution vector x *of the homogeneous system* $M_i x = 0$, $i = 1, ..., n$, *such that* $x_i(0) \geq 0$, $i = 1, ..., n-1$, $x_n(0) > 0$, *is positive for* $t \in [0,\omega]$.

R.Agarwal and A.Domoshnitsky, On positivity of several components of solution vector for systems of linear functional differential equations, Glasgow Mathematical Journal 52 (2010) 115-136.

4. Positivity of elements in the nth row of the Cauchy matrix (cont.)

About the proof of the implication 1)$\Longrightarrow$ 2). By virtue of Main Lemma the component x_n of the solution vector of system (4.1) satisfies equation

$$(Mx_n)(t) \equiv x_n'(t) + (Bx_n)(t) = f^*(t), \quad t \in [0, \omega], \tag{4.3}$$

It follows from the condition of positivity of the operator $-B_{nj}B_{jn}$ that B is a positive operator. The condition 1) implies that $(Mv_n)(t) \leq 0$ for $t \in [0, \omega]$. By virtue of Theorem 3.1 the Cauchy function $R(t, s)$ of the equation $Mv_n = 0$ is positive for $0 \leq s \leq t \leq \omega$.
From the formula of solution's representation and Main Lemma it follows that

$$x_n(t) = \int_0^t \sum_{j=1}^n C_{nj}(t, s) f_j(s) ds = \int_0^t R(t, s) f^*(s) ds, \quad t \in [0, \omega]. \tag{4.4}$$

If B_{nj} is a negative operator for each $j = 1, ..., n-1$, and $f_i \geq 0$ for $i = 1, ..., n$, then $f^* \geq 0$. The positivity of $R(t, s)$ implies that x_n is nonnegative and consequently $C_{nj}(t, s) \geq 0$ for $0 \leq s \leq t \leq \omega$ and $j = 1, ..., n$.

4. Positivity of elements in the nth row of the Cauchy matrix (cont.)

Let us consider the system

$$x_i'(t) + \sum_{j=1}^{n} p_{ij}(t) x_j(t - \tau_{ij}(t)) = f_i(t), \; i = 1, ..., n, \; t \in [0, +\infty), \tag{4.5}$$

where the delay $\tau_{ij} \geq 0$ for $i, j = 1, ..., n$.

Theorem 4.2. *Let the following conditions be fulfilled:*

1) $p_{ij} \leq 0$ for $i \neq j, \; i, j = 1, ..., n-1,$
2) $\int_{t-\tau_{ii}(t)}^{t} p_{ii}^{+}(s) ds \leq \frac{1}{e}$, for $i = 1, ..., n-1,$
3) $p_{jn} \geq 0, \; p_{nj} \leq 0$ for $j = 1, ..., n-1, \; p_{nn} \geq 0,$
4) *there exist positive* α and β_i *such that*

$$p_{nn}(t) e^{\alpha \tau_{nn}(t)} - \sum_{j=1}^{n-1} p_{nj}(t) \beta_j e^{\alpha \tau_{nj}(t)} \leq \alpha \leq \min_{1 \leq i \leq n-1} \{ -p_{in}(t) \frac{1}{\beta_i} e^{\alpha \tau_{in}(t)} + \sum_{j=1}^{n-1} p_{ij}(t) \frac{\beta_j}{\beta_i} e^{\alpha \tau_{jn}(t)} \} \tag{4.6}$$

Then the elements of the $n-$th row of the Cauchy matrix of system (4.5) satisfy the inequalities: $C_{nn}(t,s) > 0, \; C_{nj}(t,s) \geq 0$ *for* $j = 1, ..., n-1, \; 0 \leq s \leq t < +\infty$.

4. Positivity of elements in the nth row of the Cauchy matrix (cont.)

The idea of proof. All the elements of the $(n-1)\times(n-1)$ Cauchy matrix $K(t,s)$ of system (4.2) of the order $n-1$ are nonnegative, according to
A.Domoshnitsky and M.V.Sheina. Nonnegativity of Cauchy matrix and stability of systems with delay, Differentsial'nye uravnenija, v. 25, 1989, pp. 201-208.
Then we set $v_i(t) = -\beta_i e^{-\alpha t}$ for $i = 1, ..., n-1$, and $v_n(t) = e^{-\alpha t}$ in the condition 1) of Theorem 4.1.

For the ordinary differential system

$$x_i'(t) + \sum_{j=1}^{n} p_{ij}(t)x_j(t) = f_i(t),\ \ i = 1, ..., n,\ \ t \in [0, +\infty), \tag{4.7}$$

the Theorem 4.2 implies the following assertion.
Theorem 4.3. *Let the conditions be fulfilled:*

1) $p_{ij} \leq 0$ for $i \neq j,\ \ i,j = 1, ..., n-1,$
2) $p_{jn} \geq 0,\ \ p_{nj} \leq 0$ for $j = 1, ..., n-1,\ \ p_{nn} \geq 0,$
3) *there exist positive α and β_j $(j = 1, ..., n-1)$ such that*

4. Positivity of elements in the nth row of the Cauchy matrix (cont.)

$$p_{nn}(t) - \sum_{j=1}^{n-1} p_{nj}(t)\beta_j \leq \alpha \leq \min_{1\leq i\leq n-1}\{-p_{in}(t)\frac{1}{\beta_i} + \sum_{j=1}^{n-1} p_{ij}(t)\frac{\beta_j}{\beta_i}\},\ t \in [0, +\infty). \quad (4.8)$$

Then the elements of the $n-$th row of the Cauchy matrix of system (4.7) satisfy the inequalities: $C_{nn}(t,s) > 0,\ C_{nj}(t,s) \geq 0$ for $j = 1, ..., n-1,\ 0 \leq s \leq t < +\infty$.

If to choose $\beta_j = 1$ for $j = 1, ..., n-1$, we get

$$p_{nn}(t) - \sum_{j=1}^{n-1} p_{nj}(t) \leq \alpha \leq \min_{1\leq i\leq n-1}\{-p_{in}(t) + \sum_{j=1}^{n-1} p_{ij}(t)\},\ t \in [0, +\infty). \quad (4.9)$$

4. Positivity of elements in the nth row of the Cauchy matrix (cont.)

Consider now the following ordinary differential system of the second order

$$\begin{array}{l} x_1'(t) + p_{11}(t)x_1(t) + p_{12}(t)x_2(t) = f_1(t), \\ x_2'(t) + p_{21}(t)x_1(t) + p_{22}(t)x_2(t) = f_2(t), \end{array} \quad t \in [0, +\infty). \tag{4.10}$$

Theorem 4.4. *Let the following conditions be fulfilled:*

1) $p_{11} \geq 0,\ p_{12} \geq 0,\ p_{21} \leq 0,\ p_{22} \geq 0,$
2) *there exist positive* α *and* β *such that*

$$p_{22}(t) - p_{21}(t)\beta \leq \alpha \leq p_{11}(t) - p_{12}(t)\frac{1}{\beta},\ t \in [0, +\infty). \tag{4.11}$$

Then the elements of the second row of the Cauchy matrix of system (4.10) satisfy the inequalities: $C_{21}(t,s) \geq 0,\ C_{22}(t,s) > 0$ *for* $0 \leq s \leq t < +\infty$.

Remark 4.3. If coefficients p_{ij} are constants, let us set $\beta = \sqrt{-\frac{p_{12}}{p_{21}}}$, the second condition in Theorem 4.4 is as follows:

$$\sqrt{-p_{12}p_{21}} + p_{22} \leq \alpha \leq p_{11} - \sqrt{-p_{12}p_{21}}, \tag{4.12}$$

4. Positivity of elements in the nth row of the Cauchy matrix (cont.)

$$2\sqrt{-p_{12}p_{21}} \leq p_{11} - p_{22}, \tag{4.13}$$

Remark 4.4. It is known that for each fixed s the 2×2 matrix $C(t,s)$ is a fundamental matrix $X(t)$ of system (4.10) satisfying the condition $C(s,s) = E$, where E is the unit 2×2 matrix. Theorem 4.4 claims that elements $X_{21}(t,s)$ and $X_{22}(t,s)$ in the second row of the fundamental matrices are positive. The characteristic equation of the system

$$\begin{matrix} x_1'(t) + p_{11}x_1(t) + p_{12}x_2(t) = 0, \\ x_2'(t) + p_{21}x_1(t) + p_{22}x_2(t) = 0, \end{matrix} \quad t \in [0, +\infty), \tag{4.14}$$

with constant coefficients is as follows:

$$\lambda^2 + (p_{11} + p_{22})\lambda + p_{11}p_{22} - p_{12}p_{21} = 0, \tag{4.15}$$

and its roots are real if and only if

$$-4p_{12}p_{21} \leq (p_{11} - p_{22})^2. \tag{4.16}$$

4. Positivity of elements in the nth row of the Cauchy matrix (cont.)

$$x_i'(t) + \sum_{j=1}^{n} p_{ij}(t) x_j(t - \tau_{ij}(t)) = f_i(t),\ i = 1, ..., n,\ t \in [0, +\infty), \tag{4.19}$$

Theorem 4.5. *Let the following conditions be fulfilled:*

1) $p_{ij} \leq 0$ for $i \neq j,\ i, j = 1, ..., n-1,$
2) $\int_{t-\tau_{ii}(t)}^{t} p_{ii}(s)ds \leq \frac{1}{e}$, for $i = 1, ..., n-1,$
3) $p_{jn} \leq 0,\ p_{nj} \geq 0$ for $j = 1, ..., n-1,\ p_{nn} \geq 0,$
4) *there exist positive* α and β_i *such that*

$$p_{nn}(t) e^{\alpha \tau_{nn}(t)} + \sum_{j=1}^{n-1} p_{nj}(t) \beta_j e^{\alpha \tau_{nj}(t)} \leq \alpha \leq \min_{1 \leq i \leq n-1} \{ p_{in}(t) \frac{1}{\beta_i} e^{\alpha \tau_{ii}(t)} + \sum_{j=1}^{n-1} p_{ij}(t) \frac{\beta_j}{\beta_i} e^{\alpha \tau_{nj}(t)} \}, \tag{4.20}$$

Then the elements of the $n-$th row of the Cauchy matrix of system (4.19) satisfy the inequalities: $C_{nn}(t, s) > 0,\ C_{nj}(t, s) \leq 0$ *for* $j = 1, ..., n-1,\ 0 \leq s \leq t < +\infty.$

4. Positivity of elements in the nth row of the Cauchy matrix (cont.)

Let us consider the second order scalar differential equation

$$(Ny)(t) \equiv y''(t) + p_{11}(t)y'(t - \tau_{11}(t)) + p_{12}(t)y(t - \tau_{12}(t)) = f_1(t), \quad t \in [0, +\infty), \tag{4.21}$$

where $y(\theta) = y'(\theta) = 0$ for $\theta < 0$, and the corresponding differential system of the second order

$$\begin{matrix} x_1'(t) + p_{11}(t)x_1(t - \tau_{11}(t)) + p_{12}(t)x_2(t - \tau_{12}(t)) = f_1(t), \\ x_2'(t) - x_1(t) = 0, \end{matrix} \quad t \in [0, +\infty), \tag{4.22}$$

where $x_1(\theta) = x_2(\theta) = 0$ for $\theta < 0$.

It should be noted that the element $C_{21}(t, s)$ of the Cauchy matrix of system (4.22) coincides with the Cauchy function $W(t, s)$ of the scalar second order equation (4.21) and $C_{11}(t, s) = W_t'(t, s)$. If a function $y(t)$ is the solution of the Cauchy problem $(Ny)(t) = 0,\ t \in [0, +\infty),\ y(0) = 1,\ y'(0) = 0$, then $C_{22}(t, 0) = y(t)$ and $C_{12}(t, 0) = y'(t)$.

4. Positivity of elements in the nth row of the Cauchy matrix (cont.)

Theorem 4.6. Assume that $p_{12} \geq 0$, $p_{11}^{*}\tau_{11}^{*} \leq \frac{1}{e}$ and there exists a positive number α such that $\alpha\tau_{11}^{*} \leq \frac{1}{e}$ and

$$\alpha^2 + p_{12}(t)e^{\alpha\tau_{12}(t)} \leq \alpha p_{11}(t)e^{\alpha\tau_{11}(t)}, \quad t \in [0, +\infty). \tag{4.23}$$

Then the elements of the second row of the Cauchy matrix of system (4.22) satisfy the inequalities: $C_{21}(t,s) \geq 0$, $C_{22}(t,s) > 0$ for $0 \leq s \leq t < +\infty$.

In order to prove Theorem 4.6 we set $v_1(t) = -\alpha e^{-\alpha t}$, $v_2(t) = e^{-\alpha t}$ in the assertion 1) of Theorem 4.1.

Theorem 4.7. Assume that $p_{12} \geq 0$, $p_{11}^{*}\tau_{11}^{*} \leq \frac{1}{e}$, $\tau_{11} \geq \tau_{12}$ and

$$4p_{12}(t) \leq p_{11*}^2, \quad t \in [0, +\infty). \tag{4.24}$$

Then the elements of the second row of the Cauchy matrix of system (4.22) satisfy the inequalities: $C_{21}(t,s) \geq 0$, $C_{22}(t,s) > 0$ for $0 \leq s \leq t < +\infty$.

In order to prove Theorem 4.7 we set $\alpha = \frac{p_{11*}}{2}$ in Theorem 4.6.

4. Positivity of elements in the nth row of the Cauchy matrix (cont.)

Remark 4.6. Inequality (4.24) is best possible in the following sense. The characteristic equation for the system with constant coefficients

$$\begin{array}{l} x_1'(t) + p_{11}x_1(t) + p_{12}x_2(t) = f_1(t), \\ x_2'(t) - x_1(t) = 0, \end{array} \quad t \in [0, +\infty), \tag{4.25}$$

has real roots if and only if inequality (4.24) is fulfilled.

The problem of the asymptotic stability of delay differential systems is one of the most important applications of results on positivity of the Cauchy matrix $C(t, s)$.

5. Stabilization of linear delay systems

Consider the system

$$x_i'(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t-\sigma_{ij}(t)) = 0, \quad t \in [0,+\infty), \; i = 1,...n. \tag{5.1}$$

$$x_i(\xi) = 0, \;\; \xi < 0, \; i = 1,...n,$$

and the control

$$u_i(t) = -b_{ii}(t)x_i(t-\tau_{ii}(t)), \quad t \in [0,+\infty), \; i = 1,...,n. \tag{5.2}$$

When we substitute this control into equation (5.1), the following system is obtained

$$x_i'(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t-\sigma_{ij}(t)) + b_{ii}(t)x_i(t-\tau_{ii}(t)) = 0, \quad t \in [0,+\infty), \; i = 1,...n. \tag{5.3}$$

$$x_i(\xi) = 0, \quad \xi < 0, \ i = 1, ...n,$$

Theorem 5.1. *Assume that*
1) at least one of the conditions 1a) or 1b) be fulfilled:
1a) Let $\tau_{ii}(t) \geq \sigma_{ii}(t)$, $a_{ii}(t) \geq 0$, *and*

$$\int_{t-\tau_{ii}(t)}^{t} \left\{ b_{ii}(s) - \frac{1}{e} a_{ii}(s) \right\} ds \leq \frac{1}{e}, \ t \in (0, +\infty), \tag{5.4}$$

1b) $\tau_{ii}(t) \geq \sigma_{ii}(t), \ a_{ii}(t) \leq 0$ *for* $t \in [0, +\infty)$,

$$\int_{t-\sigma_{ii}(t)}^{t} \{b_{ii}(\xi) - a_{ii}(\xi)\} d\xi \leq \frac{1}{e}, \ t \in [0, +\infty), \tag{5.5}$$

and

$$\sup_{s \in [0, \infty)} - \int_{s+\sigma_*}^{s+\tau^*} a_{ii}(\xi) d\xi \leq \frac{1}{e}, \tag{5.6}$$

2) there exist positive numbers $z_1, ..., z_n$ *and* ε *such that*

$$(b_{ii}(t) - a_{ii}(t))z_i - \sum_{j=1,\, j\neq i}^{n} |a_{ij}(t)|\, z_j \geq \varepsilon, \quad t \in [0, +\infty),\ i = 1, ..., n; \quad (5.7)$$

Then for the fundamental and Cauchy matrices of linear system (5.3) there exist positive numbers N *and* α *such that*

$$|C_{ij}(t,s)| \leq N \exp\{-\alpha(t-s)\},\ |X_{ij}(t)| \leq N \exp\{-\alpha t\},\ i,j = 1, ..., n,\ 0 \leq s \leq t < +\infty. \quad (5.8)$$

and in the case, when $\varepsilon \geq 1$, *the integral estimates*

$$\sup_{t\in[0,\infty)} \int_0^t \sum_{j=1}^{n} |C_{ij}(t,s)|\, ds \leq z_i,\ i = 1, ..., n; \quad (5.9)$$

are true.

Remark 5.1. If $a_{ii} = 0,\ \tau_{ii} = 0,\ \ z_i = 1,\ i = 1, ..., n$, this result was obtained in:

S.A.Campbell, Delay independent stability for additive neural networks, *Differential Equations Dynam. Systems,* 9:3-4 (2001), 115-138.

J.Hofbauer and J. W.-H. So, Diagonal dominance and harmless off-diagonal delays, Proc.Amer. Math. Soc., 128 (2000), 2675-2682.

In the case $a_{ii} = 0,\ i = 1, ..., n$, in the paper

A.Domoshnitsky and M.V.Sheina. Nonnegativity of Cauchy matrix and stability of systems with delay, Differentsial'nye uravnenija, v. 25, 1989, 201-208.

Remark 5.2. Note that the constants $z_1, ..., z_n$ can be found, for example, as the solution of the algebraic system

$$(b_{ii} - a_{ii})_* z_i - \sum_{j=1}^{n} |a_{ij}|^* z_j = 1, \quad t \in [0, +\infty),\ i = 1, ..., n, \tag{5.11}$$

where $q_* = ess\inf\ q(t),\ |q|^* = ess\sup|q(t)|^*$.

Remark 5.3. From the condition 2) it is clear that $b_{ii}(t) - a_{ii}(t) \geq \varepsilon > 0$. This inequality in the case of positive $K_i(t, s),\ i = 1, ..., n$, and constant coefficients such that $b_{ii}(t) \equiv const,\ a_{ij} = 0,\ i \neq j$ is necessary for the exponential stability of linear delay system (5.3).

Remark 5.4. It is clear from the condition 2 of Theorem 5.1 that $b_{ii}(t) - a_{ii}(t)$ for $i = 1, ..., n$ have to be great enough, but from conditions *1a)* and *1b)* it follows that these differences have to be small enough. This leads us to hard limitation on the ability to get stabilization by the control in form (5.2). In the next assertions we set

$$u_i(t) = -\sum_{j=1}^{n} b_{ij}(t) x_i(t - \tau_{ij}(t)), \quad t \in [0, +\infty),\ i = 1, ..., n, \tag{5.12}$$

where $b_{ij}(t) = a_{ij}(t)$ if $i \neq j,\ \tau_{ij}(t) - \sigma_{ij}(t) \geq \varepsilon_1 > 0,\ i, j = 1, ...n, \quad t \in [0, +\infty)$, and avoid this limitation.

Theorem 5.3. *Assume that*

1) *condition 1) of Theorem 5.2 is fulfilled,*

2) there exists a positive ε such that

$$b_{ii}(t) - a_{ii}(t) \geq \varepsilon, \quad t \in [0, +\infty), \tag{5.13}$$

3)

$$\max_{1\le i\le n}\left\{\sum_{j=1,j\neq i}^{n}|a_{ij}|^{*}\left[1+\frac{1}{\varepsilon}(|a_{ij}|^{*}+|b_{ij}|^{*})\right](\tau_{ij}(t)-\sigma_{ij}(t))\right\}<1, \tag{5.14}$$

where $|a_{ij}|^{*} = \text{esssup}_{t\in[0,\infty)}|a_{ij}(t)|$, $|b_{ij}|^{*} = \text{esssup}_{t\in[0,\infty)}|b_{ij}(t)|$.
Then exponential estimates (5.8) are valid for the fundamental and Cauchy matrices of linear system

$$x_i'(t)-\sum_{j=1}^{n}a_{ij}(t)x_j(t-\sigma_{ij}(t))+\sum_{j=1}^{n}b_{ij}(t)x_j(t-\tau_{ij}(t))=f_i(t),\quad t\in[0,+\infty),\ i=1,...n. \tag{5.15}$$

5. Positivity of elements in the nth row of the Cauchy matrix and exponential stability

For systems with variable coefficients the technique of a reduction to integral equations was proposed in the papers:

A.Domoshnitsky and M.V.Sheina. Nonnegativity of Cauchy matrix and stability of systems with delay, Differentsial'nye uravnenija, v. 25, 1989, 201-208,
I.Gyori, Interaction between oscillation and global asymptotic stability in delay differential equations, *Differential & Integral Equations*, 3 (1990), 181-200.
S.A.Campbell, Delay independent stability for additive neural networks, *Differential Equations Dynam. Systems*, 9:3-4 (2001), 115-138.
J.Hofbauer and J. W.-H. So, Diagonal dominance and harmless off-diagonal delays, Proc.Amer. Math. Soc., 128 (2000), 2675-2682.
I.Gyori and F.Hartung, Fundamental solution and asymptotic stability of linear delay differential equations.*Dynamics of Continuous, Discrete and Impulsive Systems, Ser. A math. Anal. 13 (2006), 261-287*.

$$x_i'(t) + \sum_{j=1}^{n} p_{ij}(t) x_j(t - \tau_{ij}(t)) = f_i(t), \; i = 1, ..., n, \; t \in [0, +\infty), \qquad (5.1)$$

5. Positivity of elements in the nth row of the Cauchy matrix and exponential stability (cont.)

Theorem 5.1. *Let the following conditions be fulfilled:*

1) $p_{ij} \leq 0$ for $i \neq j, \ i,j = 1,...,n-1$,
2) $\int_{t-\tau_{ii}(t)}^{t} p_{ii}(s)ds \leq \frac{1}{e}$ for $i = 1,...,n-1$,
3) $p_{jn} \geq 0, \ p_{nj} \leq 0$ for $j = 1,...,n-1, \ p_{nn} \geq 0$,
4) *there exist positive* α and β_i *such that*

$$p_{nn}(t)e^{\alpha\tau_{nn}(t)} - \sum_{j=1}^{n-1} p_{nj}(t)\beta_j e^{\alpha\tau_{nj}(t)} \leq \alpha \leq \min_{1\leq i\leq n-1}\{-p_{in}(t)\frac{1}{\beta_i}e^{\alpha\tau_{in}(t)} + \sum_{j=1}^{n-1} p_{ij}(t)\frac{\beta_j}{\beta_i}e^{\alpha\tau_{jn}(t)}\}, \tag{5.2}$$

Then the elements of the $n-$th row of the Cauchy matrix of system (5.1) satisfy the inequalities: $C_{nn}(t,s) > 0, \ C_{nj}(t,s) \geq 0$ for $j = 1,...,n-1, \ 0 \leq s \leq t < +\infty$.
If in addition there exist positive ε and $z_1,...,z_{n-1}$ such that

$$\sum_{j=1}^{n-1} p_{ij}(t) \geq \varepsilon > 0,$$

$$p_{in}(t) - \sum_{j=1}^{n-1} p_{ij}(t)z_j \geq \varepsilon > 0, \ i = 1,...,n-1, \ t \in [0,+\infty), \tag{5.3}$$

$$p_{nn}(t) + \sum^{n-1} |p_{nj}(t)| z_j \geq \varepsilon > 0,$$

5. Positivity of elements in the nth row of the Cauchy matrix and exponential stability (cont.)

then elements of Cauchy matrix of system (5.1) satisfy the exponential estimate i.e. there exists N and γ such that $|C_{ij}(t,s)| \leq Ne^{-\gamma(t-s)}$ for
$0 \leq s \leq t < +\infty, \quad i,j = 1, ..., n.$

Remark 5.1. A possible case is $p_{nn} = 0$ and the principle of main diagonal dominance (even in its generalized form, assuming, for example, that the matrix $\{p_{ij}\}_{i,j=1,\dots,n}$ is M-matrix), is not fulfilled.

Remark 5.2. The inequality

$$p_{nn}(t) + \sum_{j=1}^{n-1} |p_{nj}(t)| \beta_j > 0, \; t \in [0, +\infty). \tag{5.4}$$

cannot be set instead of the inequality

$$p_{nn}(t) - \sum_{j=1}^{n-1} p_{nj}(t) \beta_j \geq \varepsilon > 0, \; t \in [0, +\infty). \tag{5.5}$$

5. Positivity of elements in the nth row of the Cauchy matrix and exponential stability (cont.)

in the condition (5.3) of Theorem 5.1. Actually in the case $p_{nj}(t) \equiv 0$ for $j = 1, ..., n-1$ and $p_{nn}(t) = \frac{1}{t^2}$, the component $x_n(t)$ of the solution vector of the homogeneous system

$$(M_i x)(t) \equiv x_i'(t) + \sum_{j=1}^{n} p_{ij}(t) x_j(t - \tau_{ij}(t)) = 0, \; i = 1, ..., n, \; t \in [0, +\infty), \tag{5.6}$$

$$x(\theta) = 0 \text{ for } \theta < 0,$$

does not tend to zero when $t \to +\infty$.

Remark 5.3. Conditions 1) and 2) imply that all elements of the Cauchy matrix $K(t,s) = \{K_{ij}(t,s)\}_{i,j=1,\ldots,n-1}$ of auxiliary system

$$x_i'(t) + \sum_{j=1}^{n-1} p_{ij}(t) x_j(t - \tau_{ij}(t)) = f_i(t), \; i = 1, ..., n-1, \; t \in [0, +\infty), \tag{5.7}$$

(5.1) are nonnegative. In the case of systems of ordinary differential equations

$$x_i'(t) + \sum_{j=1}^{n} p_{ij}(t) x_j(t) = f_i(t), \; i = 1, ..., n, \; t \in [0, +\infty), \tag{5.8}$$

5. Positivity of elements in the nth row of the Cauchy matrix and exponential stability (cont.)

condition 1) is necessary for nonnegativity of this matrix [25]. Condition 2) in a general case cannot be improved since the opposite inequality $p_{ii}\tau_{ii} > \frac{1}{e}$ in the case of scalar differential delay equation

$$x'(t) + p_{ii}x(t - \tau_{ii}) = 0, \ \ t \in [0, +\infty), \tag{5.9}$$

with constant coefficients p and τ implies oscillation of all nontrivial solutions.
Remark 5.4. The inequality

$$\sum_{j=1}^{n-1} p_{ij}(t) \geq \varepsilon > 0, \ \ t \in [0, +\infty). \tag{5.10}$$

imply that the Cauchy matrix of auxiliary system (5.7) satisfies the exponential estimate and this is essential in the case of separated nth equation i.e. when $p_{nj}(t) \equiv 0, \ \ p_{jn}(t) \equiv 0$ for $j = 1, ..., n-1$.
Remark 5.5. The condition

$$p_{in}(t) - \sum_{j=1}^{n-1} p_{ij}(t) z_j > 0, \ \ i = 1, ..., n-1, \ \ t \in [0, +\infty), \tag{5.11}$$

5. Positivity of elements in the nth row of the Cauchy matrix and exponential stability (cont.)

cannot be set instead of

$$p_{in}(t) - \sum_{j=1}^{n-1} p_{ij}(t) z_j \geq \varepsilon > 0, \ i = 1, ..., n-1, \ t \in [0, +\infty). \tag{5.12}$$

$$(Mx_n)(t) \equiv x_n'(t) + (Bx_n)(t) = 0, \tag{5.13}$$

where

$$(Bx_n)(t) \equiv -\sum_{i=1}^{n-1} B_{ni}\{\int_0^t \sum_{j=1}^{n-1} K_{ij}(t,s)(B_{jn}x_n)(s)ds\}(t) + (B_{nn}x_n)(t), \tag{5.14}$$

$$(B_{ij}y)(t) = p_{ij}(t)y(t - \tau_{ij}(t)), \quad y(\xi) = 0 \ \text{ if } \xi < 0. \tag{5.15}$$

Theorem 5.2. *Let the following conditions be fulfilled:*

1) $p_{ij} \leq 0$ for $i \neq j, \ i, j = 1, ..., n-1,$

5. Positivity of elements in the nth row of the Cauchy matrix and exponential stability (cont.)

2) $\int_{t-\tau_{ii}(t)}^{t} p_{ii}(s)ds \leq \frac{1}{e}$ for $i = 1, ..., n-1$,
3) $p_{jn} \leq 0,\ p_{nj} \geq 0$ for $j = 1, ..., n-1,\ p_{nn} \geq 0$,
4) *there exist positive* α and β_i *such that*

$$p_{nn}(t)e^{\alpha\tau_{nn}(t)} - \sum_{j=1}^{n-1} p_{nj}(t)\beta_j e^{\alpha\tau_{nj}(t)} \leq \alpha \leq \min_{1\leq i\leq n-1}\{-p_{in}(t)\frac{1}{\beta_i}e^{\alpha\tau_{in}(t)} + \sum_{j=1}^{n-1} p_{ij}(t)\frac{\beta_j}{\beta_i}e^{\alpha\tau_{jn}(t)}\} \tag{5.13}$$

Then the elements of the $n-$th row of the Cauchy matrix of system (5.1) satisfy the inequalities: $C_{nn}(t,s) > 0,\ C_{nj}(t,s) \leq 0$ for $j = 1, ..., n-1,\ 0 \leq s \leq t < +\infty$.
If in addition there exist positive $z_1, ..., z_n$ and ε such that

$$\begin{aligned} &\sum_{j=1}^{n-1} p_{ij}(t) \geq \varepsilon > 0, \\ &p_{in}(t) + \sum_{j=1}^{n-1} p_{ij}(t)z_j \leq -\varepsilon,\ i = 1, ..., n-1,\ \ t \in [0, +\infty), \\ &p_{nn}(t) + \sum_{j=1}^{n-1} p_{nj}(t)z_j \geq \varepsilon, \end{aligned} \tag{5.14}$$

5. Positivity of elements in the *n*th row of the Cauchy matrix and exponential stability (cont.)

then elements of Cauchy matrix of system (5.1) satisfy the exponential estimate i.e.there exists N and γ such that $|C_{ij}(t,s)| \leq Ne^{-\gamma(t-s)}$ for $0 \leq s \leq t < +\infty, \quad i,j = 1,...,n$.

$$\begin{array}{l} x_1'(t) + p_{11}x_1(t) + p_{12}x_2(t) = 0, \\ x_2'(t) + p_{21}x_1(t) + p_{22}x_2(t) = 0, \end{array} \quad t \in [0,+\infty), \tag{5.15}$$

Theorem 5.3. *If there exist such positive $\alpha, \beta, \varepsilon$ that the following inequalities are fulfilled:*

1) $p_{11}(t) > \varepsilon > 0,\ p_{12}(t) \leq -\varepsilon < 0,\ p_{21}(t) \geq \varepsilon > 0,\ p_{22}(t) \geq 0,$

2) $p_{21}(t)\beta + p_{22}(t) \leq \alpha \leq p_{11}(t) + p_{12}(t)\frac{1}{\beta}.$

Then the elements of Cauchy matrix of system (5.15) satisfy the inequalities $C_{21}(t,s) > 0,\ C_{22}(t,s) \geq 0$ for $0 \leq s \leq t < +\infty$ and the exponential estimate.

For system with constant coefficients and $\beta = \sqrt{\frac{-p_{12}}{p_{21}}}$ we obtain::

5. Positivity of elements in the nth row of the Cauchy matrix and exponential stability (cont.)

Corollary 5.4. *If the following inequalities are fulfilled:*
1) $p_{11} > 0,\ p_{12} < 0,\ p_{21} > 0,\ p_{22} \geq 0,$
2) $2\sqrt{-p_{12}p_{21}} \leq p_{11} - p_{22}.$
Then the elements of Cauchy matrix of system (5.15) satisfy the inequalities $C_{21}(t,s) > 0,\ C_{22}(t,s) \geq 0$ *for* $0 \leq s \leq t < +\infty$ *and the exponential estimate.*

6. Periodic problem

Consider

$$\begin{array}{l} x_1'(t) + p_{11}(t)x_1(h_{11}(t)) + p_{12}(t)x_2(h_{12}(t)) = 0, \\ x_2'(t) + p_{21}(t)x_1(h_{21}(t)) + p_{22}(t)x_2(h_{22}(t)) = 0, \end{array} \quad t \in [0, \omega], \qquad (6.1)$$

where $h_{ij}(t) \in [0, \omega]$.

Theorem 6.1. *Let the following conditions be fulfilled:*

1) $0 < \int_0^\omega p_{ii}(t)dt < 1, \quad i = 1, 2,$
2) $\delta(t) = p_{11}(t)p_{22}(t) - p_{21}(t)p_{12}(t) > 0, \quad t \in [0, \omega].$

If

$$P(t) = \begin{vmatrix} + & - \\ - & + \end{vmatrix},$$

then

$$G(t, s) = \begin{vmatrix} + & + \\ + & + \end{vmatrix}.$$

Theorem 6.2. *Let the following conditions be fulfilled:*
1) $0 < \int_0^{\omega} p_{11}(t)dt < 1,$
2) $|p_{12}|^* \int_0^{\omega} |p_{21}(t)|\, dt < p_{11*},$
3) $\delta(t) = p_{11}(t)p_{22}(t) - p_{21}(t)p_{12}(t) < 0, \quad t \in [0, \omega].$

If

$$P(t) = \begin{vmatrix} + & - \\ - & + \end{vmatrix},$$

then

$$G(t,s) = \begin{vmatrix} \cdot & \cdot \\ - & - \end{vmatrix}.$$

Consider the system with constant coefficients

$$\begin{aligned} x_1'(t) + p_{11}x_1(h_{11}(t)) + p_{12}x_2(h_{12}(t)) = 0, \\ x_2'(t) + p_{21}x_1(h_{21}(t)) + p_{22}x_2(h_{22}(t)) = 0, \end{aligned} \quad t \in [0, \omega], \tag{6.2}$$

Theorem 6.3. *Let the following conditions be fulfilled:*
1) $0 < p_{ii}\omega < 1, \quad i = 1, 2,$
2) $\omega\,|p_{21}p_{12}| < \min\{|p_{11}|, |p_{22}|\}.$

3)

$$P(t) = \begin{vmatrix} + & - \\ - & + \end{vmatrix},$$

If $\delta = p_{11}p_{22} - p_{21}p_{12} > 0,$ *then*

$$G(t,s) = \begin{vmatrix} + & + \\ + & + \end{vmatrix},$$

If $\delta = p_{11}p_{22} - p_{21}p_{12} < 0,$ *then*

$$G(t,s) = \begin{vmatrix} - & - \\ - & - \end{vmatrix}.$$

Theorem 6.4. *Let the following conditions be fulfilled:*

1) $\int_0^\omega |p_{11}(t)|\,dt < 1$,

2) *at least one from two possibilities be fulfilled: either* $\int_0^\omega p_{12}(t)dt \neq 0$, $\int_0^\omega p_{21}(t)dt \neq 0$ *or* $\int_0^\omega p_{22}(t)dt \neq 0$,

3) $\int_0^\omega \{|p_{21}(t)|\,|p_{12}|^* - |p_{22}(t)|\,|p_{11}|_*\}\,dt < |p_{11}|_*$.

Then

$$a)\ G(t,s) = \begin{vmatrix} \cdot & \cdot \\ + & + \end{vmatrix} \text{ if } P(t) = \begin{vmatrix} - & + \\ + & + \end{vmatrix},$$

$$b)\ G(t,s) = \begin{vmatrix} \cdot & \cdot \\ + & - \end{vmatrix} \text{ if } P(t) = \begin{vmatrix} + & + \\ + & - \end{vmatrix},$$

$$c)\ G(t,s) = \begin{vmatrix} \cdot & \cdot \\ - & + \end{vmatrix} \text{ if } P(t) = \begin{vmatrix} + & - \\ + & + \end{vmatrix},$$

$$d)\ G(t,s) = \begin{vmatrix} \cdot & \cdot \\ - & - \end{vmatrix} \text{ if } P(t) = \begin{vmatrix} + & - \\ - & - \end{vmatrix},$$

7. About unique solvability of boundary value problems

$$x_i'(t) + \sum_{j=1}^{n} p_{ij}(t) x_j(t - \tau_{ij}(t)) = f_i(t),\ i = 1, ..., n,\ t \in [0, \omega], \qquad (7.1)$$

$$x_i(0) = c_i,\ i = 1, ..., n-1,\ \ lx = c_n, \qquad (7.2)$$

Theorem 7.1. *Let the following conditions be fulfilled:*

1) $p_{ij} \leq 0$ for $i \neq j,\ i, j = 1, ..., n-1,$
2) $\int_{t-\tau_{ii}(t)}^{t} p_{ii}(s)ds \leq \frac{1}{e}$ for $i = 1, ..., n-1,$
3) $p_{jn} \geq 0,\ p_{nj} \leq 0$ for $j = 1, ..., n-1,\ p_{nn} \geq 0,$
4) *there exist positive* α and β_i *such that*

$$p_{nn}(t)e^{\alpha\tau_{nn}(t)} - \sum_{j=1}^{n-1} p_{nj}(t)\beta_j e^{\alpha\tau_{nj}(t)} \leq \alpha \leq \min_{1\leq i\leq n-1}\{-p_{in}(t)\frac{1}{\beta_i}e^{\alpha\tau_{in}(t)} + \sum_{j=1}^{n-1} p_{ij}(t)\frac{\beta_j}{\beta_i}e^{\alpha\tau_{jn}(t)}\} \qquad (7.3)$$

7. About unique solvability of boundary value problems (cont.)

Then the following assertions are true:

a) *If* $l : C_{[0,\omega]} \to R^1$ *is a linear nonzero positive functional, then boundary value problem (7.1),(7.2) is uniquely solvable for each summable vector function* f *and* $c \in R^1$;

b) *the boundary value problem (7.1),(7.2), where*

$$lx_n \equiv x_n(t_1) - mx_n = c, \tag{7.4}$$

and the norm of the linear functional $m : C_{[t_1,\omega]} \to R^1$ *is less than one is uniquely solvable for each summable vector function* f *and* $c \in R^1$;

c) *the boundary value problem (7.1),(7.2), where*

$$lx_n \equiv \sum_{j=1}^{2k} \alpha_j x_n(t_j) = c, \; 0 \leq t_1 < t_2 < ... < t_{2k} \leq \omega, \tag{7.5}$$

7. About unique solvability of boundary value problems (cont.)

with $\alpha_{2j-1} \geq -\alpha_{2j} \geq 0,\ j = 1, ..., k$, *and there exists an index* i *such that* $\alpha_{2i-1} > -\alpha_{2i}$, *is uniquely solvable for each summable vector function* f and $c \in R^1$;

d) *the boundary value problem (7.1), (7.2), where*

$$lx_n \equiv \sum_{j=1}^{k} \int\limits_{t_{2j-2}}^{t_{2j}} \alpha(t)x_n(t)dt = c,\ 0 = t_0 \leq t_1 < t_2 < ... < t_{2k} \leq \omega, \quad (7.6)$$

in the case when $\alpha(t) \geq 0$ for $t \in [t_{2j-2}, t_{2j-1}]$, $\alpha(t) \leq 0$ for $t \in [t_{2j-1}, t_{2j}]$, $\int\limits_{t_{2j-2}}^{t_{2j}} \alpha(t)dt \geq 0$, $j = 1, ..., k$, *and there exists* j *such that* $\int\limits_{t_{2j-2}}^{t_{2j}} \alpha(t)dt > 0$, *is uniquely solvable for each summable vector function* f and $c \in R^1$.

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