Generalized half-linear differential equations

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Contents

“Classical” half-linear differential equations

The “classical” half-linear differential equations (sometimes also called the differential equation with one-dimensional p-Laplacian) is

$$(HL) \qquad (r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-2}x, \; p > 1,$$

r, c continuous functions, $r(t) > 0$.

Special case $p = 2$

$$(SL) \qquad (r(t)x')' + c(t)x = 0.$$

linear Sturm-Liouville differential equation.

Denote $r(t) \longmapsto r^{q-1}(t)$, $\frac{1}{p} + \frac{1}{q} = 1$, i.e., $(p-1)(q-1) = 1$, then the differential term in (HL) can be written as

$$\big(r^{q-1}\Phi(x')\big)' = \big(\Phi(rx')\big)' = (p-1)|rx'|^{p-2}(rx')'$$

and (HL) as

$$\big(r(t)x'\big)' + \frac{c(t)}{p-1}|r(t)x'|^{2-p}\Phi(x) = 0$$

This motivates to introduce "generalized" half-linear differential equation (Bihari 1966-1976, Elbert 1984) as follows

Generalized half-linear equation

$$(GHL) \qquad (r(t)x')' + c(t)f(x, r(t)x') = 0$$

with the assumptions on the function f which are motivated by the "classical" case

$$f(x, rx') = \Phi(x)|rx'|^{2-p} = |x|^{p-2}x|rx'|^{2-p}$$

Assumptions on the function f

(i) The function f is continuous on $\Omega = \mathbb{R} \times \mathbb{R}_0$, where $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$;

(ii) It holds $xf(x, y) > 0$ if $xy \neq 0$;

(iii) The function f is homogeneous, i.e., $f(\lambda x, \lambda y) = \lambda f(x, y)$ for $\lambda \in \mathbb{R}$ and $(x, y) \in \Omega$;

(iv) The function f is sufficiently smooth in order to ensure the continuous dependence and the uniqueness of solutions of the initial value problem $x(t_1) = x_0$, $x'(t_1) = x_1$ at some $(x_0, x_1) \in \Omega$;

(v) Let $F(t) := tf(t,1)$, then

$$\int_{-\infty}^{\infty} \frac{dt}{1+F(t)} < \infty \quad \text{and} \quad \lim_{|t|\to\infty} F(t) = \infty.$$

(vi) The function H in appearing in Riccati type equation is strictly convex (will be defined later).

Generalized Riccati substitution

- Riccati equation associated with (HL): $w = \frac{r\Phi(x')}{\Phi(x)}$

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0.$$

- Generalized Riccati substitution. Let $g : \mathbb{R} \to \mathbb{R}$ be an increasing differentiable function (determined later) $u = rx'/x$, $v = g(u)$, then

$$\begin{aligned} v' &= g'(u)\left[-c(t)\frac{f(x, rx')}{x} - \frac{rx'^2}{x^2}\right] \\ &= -c(t)g'(u)\frac{rx'}{x}f\big(\frac{x}{rx'}, 1\big) - g'(u)\frac{u^2}{r} \end{aligned}$$

$$g'(u)uf(1/u,1)=1 \quad \Longrightarrow \quad g'(u)u^2F(u)=1.$$

We have

$$g(u)=\begin{cases} \int_{1/u}^{\infty}\frac{ds}{F(s)} & \text{if } u>0,\\ -\int_{-\infty}^{1/u}\frac{ds}{F(s)} & \text{if } u<0, \end{cases}$$

and $g(0)=0$. Then g is strictly increasing and

$$\lim_{u\to\pm\infty} g(u)=\pm\infty.$$

Generalized Riccati equation

$$(RE) \qquad v' + c(t) + \frac{H(v)}{r(t)} = 0$$

with the function

$$H(v) = [g^{-1}(v)]^2 g'(g^{-1}(v))$$

Recall that the Riccati equation corresponding to “classical” (HL) equation is

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0, \quad q > 1.$$

Sturmian theory

Let x be a solution of (GHL) with consecutive zeros $t_1 < t_2$ and let $v_x(t) = g\big(r(t)x'(t)/x(t)\big)$ be the associated solution of (RE). Then

$$v_x(t_1+) = +\infty, \quad v_x(t_2-) = -\infty.$$

Suppose that there exists a solution y of (GHL) with $y(t) \neq 0$ for $t \in [t_1, t_2]$. Then the graph of $v_y(t) = G\big(r(t)y'(t)/y(t)\big)$ must intersect the graph of $v_x \implies$ uniqueness of solution of IVP for (RE) is violated, a contradiction.

Sturmian separation theorem extends to (GHL)

Half-linear Euler differential equation

The half-linear Euler equation

$$\big(\Phi(x')\big)' + \frac{\gamma}{t^p}\Phi(x) = 0$$

is oscillatory if $\gamma > \gamma_p := \left(\frac{p-1}{p}\right)^p$ and nonoscillatory in the opposite case.

The potential t^{-p} is a border line between oscillation and nonoscillation, Kneser type (non)oscillation criteria:

$$\liminf_{t\to\infty} t^p c(t) > \gamma_p, \quad \limsup_{t\to\infty} t^p c(t) < \gamma_p.$$

An example

Consider the equation

$$(E) \qquad \big(\Phi(x')\big)' + c(t)\Phi(x) = 0$$

as a petrurbation of the critical Euler equation

$$\big(\Phi(x')\big)' + \frac{\gamma_p}{t^p}\Phi(x) = 0$$

i.e., we write in the form

$$(PE) \qquad \big(\Phi(x')\big)' + \frac{\gamma_p}{t^p}\Phi(x) + \underbrace{\left[c(t) - \frac{\gamma_p}{t^p}\right]}_{d(t)}\Phi(x) = 0$$

Consider the substitution

$$v = t^{p-1} w - \Gamma_p, \quad \Gamma_p = \left(\frac{p-1}{p}\right)^{p-1},$$

where w is a solution of the Riccati equation associated with (E). Then

$$v' + t^{p-1} d(t) + \frac{(p-1)}{t}[|v+\Gamma_p|^q - v - \gamma_p] = 0,$$

the function

$$H(v) := |v+\Gamma_p|^q - v - \gamma_p$$

satisfies all assumptions as H in the generalized Riccati equation.

Construction generalized 1/2-linear equation

Let $H(v) > 0$ for $v \neq 0$, with $H(0) = 0$, be a strictly convex such that

$$\int_{-\infty} \frac{ds}{H(s)} < \infty, \quad \int^{\infty} \frac{ds}{H(s)} < \infty,$$

define g as the solution of

$$g'(u) = \frac{1}{u^2} H(g(u)), \quad g(0) = 0,$$

and $f : \mathbb{R} \times \mathbb{R}_0 \to \mathbb{R}$ by

$$f(1, u) := \frac{1}{g'(u)}, \quad f(t, s) := \begin{cases} tf(1, t/s), & t \neq 0, \\ 0 & t = 0. \end{cases}$$

Then $(rx')' + c(t)f(x, rx') = 0$ is generalized 1/2-linear differential equation.

Conditional oscillation

Consider the generalized Riccati equation

$$v' + c(t) + H(v) = 0$$

associated with (GHL) with $r(t) \equiv 1$. Suppose that there exists $\beta > 1$ such that

$$\lim_{v \to 0+} \frac{H(v)}{v^\beta} =: L \in (0, \infty).$$

Then the associated (GHL) with $c(t) = \lambda t^{-\alpha}$, where $\alpha = \frac{\beta}{\beta - 1}$ is the conjugate exponent of β, is conditionally oscillatory with the constant of conditional oscillation

$$\lambda_0 = \left(\frac{L}{\alpha - 1} \right)^{1-\alpha} \gamma_\alpha, \quad \gamma_\alpha := \left(\frac{\alpha - 1}{\alpha} \right)^\alpha .$$

This means that the equation

$$x'' + \frac{\lambda}{t^\alpha} f(x, x') = 0$$

is oscillatory for $\lambda > \lambda_0$ and nonoscillatory for $\lambda < \lambda_0$. The limiting case $\lambda = \lambda_0$ remains generally undecided.

The perturbed Euler equation

$$\big(\Phi(x')\big)' + \left[\frac{\gamma_p}{t^p} + d(t)\right] \Phi(x) = 0$$

is conditionally conditionally oscillatory with the "limiting potential

$$d(t) = \frac{\mu_p}{t^p \log^2 t}, \quad \mu_p = \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}$$

- Picone type identity and associated energy functional. For classical 1/2-line equation it is

$$r|y'|^p - c|y|^p = [w|y|^p]' + \text{something nonnegative}$$

What instead of $|y|^p$? The term “something nonnegative” is associated with the Young inequality

$$\frac{|u|^p}{p} - uv + \frac{|v|^q}{q} \geq 0.$$

Couldn’t be this inequality replaced somehow bt the Fenchel inequality

$$H(u) - uv + H^*(v) \geq 0, \quad H^*(v) = \sup_u [uv - H(u)].$$

- Characterization of the principal solution.
- Eigenvalue problems for

$$(r(t)x')' + \lambda c(t) f(x, rx') = 0, \quad x(a) = 0 = x(b).$$

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Svaťo, vše nejlepší, hodně aktivity a zdraví v dalších letech.