Singular boundary value problems

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Overview of our common research with Svaťa

We have studied the functional differential equation

$$x''(t) = f(t, x(t), (Fx)(t), x'(t), (Hx')(t)),$$

where f satisfies the local Carathéodory conditions on $([0,1] \times \mathbb{R}^4)$ and F, H are continuous and bounded operators. We proved the existence results for various types of two-point or functional boundary conditions. Typical assumptions for f - sign conditions.

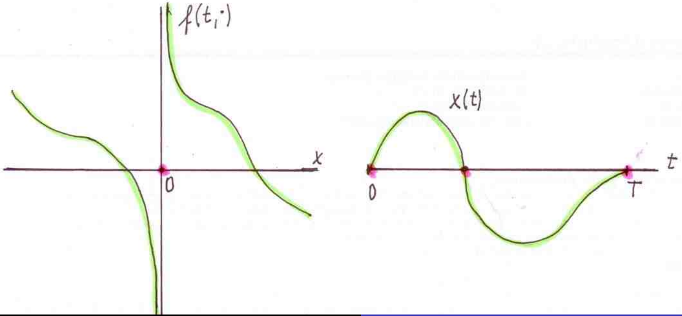
- Topological degree methods in functional boundary value problems, *Nonlinear Anal. TMA* 27 (1996), 153-166.

- Topological degree methods in functional boundary value problems at resonance, *Nonlinear Anal. TMA* 27 (1996), 271-285.

2003, Singular problems: Sign-changing solutions

- Sign-changing solutions of singular Dirichlet boundary value problems. *Archives of Inequalities and Applications* 1 (2003), 11-30.

- Connections between types of singularities in differential equations and smoothness of solutions of Dirichlet BVPs. *Dyn. Contin. Discrete Impulsive Syst.* 10 (2003), 209-222.



We have studied the singular Dirichlet boundary value problem with a positive parameter μ

$$(r(x(t))x'(t))' = \mu q(t) f(t, x(t)), \quad t \in (0, T), \tag{1}$$

$$x(0) = x(T) = 0, \quad \max_{t\in[0,T]}\{x(t)\} \cdot \min_{t\in[0,T]}\{x(t)\} < 0, \tag{2}$$

where $T \in (0, \infty)$ and $f(t, x)$ is singular at the point $x = 0$ of the phase variable x in the following sense

$$\lim_{x\to 0^-} f(t, x) = -\infty, \quad \lim_{x\to 0^+} f(t, x) = \infty \quad \text{for } t \in [0, T]. \tag{3}$$

$$(r(x(t))x'(t))' = \mu q(t) f(t, x(t))$$

The basic assumptions:

(H1) $r \in C(\mathbb{R})$, $r(x) \geq r_0 > 0$ for $x \in \mathbb{R}$,

(H2) $q \in C[0, T]$, $q < 0$ on $(0, T)$,

(H3) $f \in C([0, T] \times D)$ fulfils (3), $f(t, \cdot)$ is nonincreasing on D for $t \in [0, T]$, $D = (-\infty, 0) \cup (0, \infty)$.

Definition

A function $x \in C^1[0, T]$ is a solution of problem (1), (2) if x has precisely one zero t_0 in $(0, T)$, $r(x)x' \in C^1((0, T) \setminus \{t_0\})$, x fulfils (2) and there exists $\mu_0 > 0$ such that (1) is satisfied for $\mu = \mu_0$ and $t \in (0, T) \setminus \{t_0\}$.

Definition

We say that f has the weak singularity at $x = 0$ if f satisfies (3) and

$$0 < f(t,x)\,\mathrm{sign}\,x \le g(x) \quad \text{for } (t,x) \in [0,T] \times D, \tag{4}$$

where $g \in C(D)$ fulfils

$$\int^0 g(x)\,dx < \infty, \quad \int_0 g(x)\,dx < \infty. \tag{5}$$

For example

$$f(t,x) = \frac{\mathrm{sign}\,x}{|x|^\alpha}, \quad \alpha \in (0,1).$$

Definition

Assume that there exist a positive function $p \in C(D)$ such that

$$0 < p(x) \le f(t,x)\,\mathrm{sign}\,x \quad \text{for } (t,x) \in [0,T] \times D. \tag{6}$$

We say that f has the strong left singularity at $x = 0$ if

$$\int^0 p(x)\,dx = \infty \tag{7}$$

and we say that f has the strong right singularity at $x = 0$ if

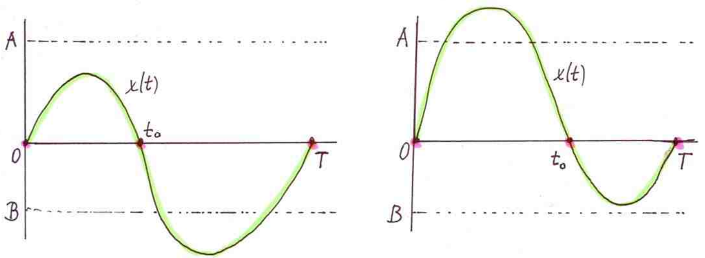
$$\int_0 p(x)\,dx = \infty. \tag{8}$$

For example

$$f(t,x) = \frac{\mathrm{sign}\,x}{|x|^\alpha}, \quad \alpha \ge 1.$$

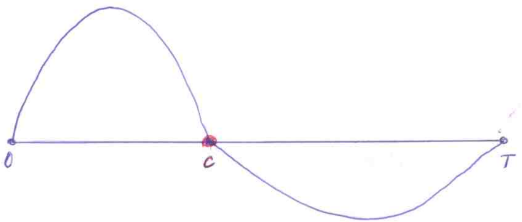
Existence and non-existence theorem - Dirichlet

1) If f has the one-sided strong singularity at $x = 0$, then problem (1), (2) has no solution;
2) If f has the week singularity at $x = 0$ then:
   (i) for each $A > 0$, there exists at least one solution having its maximum value $\leq A$ on $[0, T]$,
   (ii) for each $B < 0$ there exists at least one solution having its minimum value $\geq B$ on $[0, T]$,
   (iii) zeros of solutions on $(0, T)$ are not precisely localized.



Steps of the proof:

- The existence of positive and negative solutions of auxiliary Dirichlet problems on the intervals $[0, c]$ and $[c, T]$ for $\mu = \mu(c)$, $c \in (0, T)$.
- The construction of a sign-changing "solution" from these positive and negative solutions.
- The existence of a point $t_0 \in (0, T)$ and a parameter μ_0 such that the sign-changing solution belongs to $C^1[0, 1]$.



Steps of the proof:

- The existence of positive and negative solutions of auxiliary Dirichlet problems on the intervals $[0, c]$ and $[c, T]$ for $\mu = \mu(c)$, $c \in (0, T)$.
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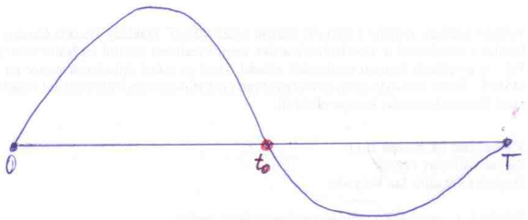


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2003-2004, Higher order singular problems

- Two-point higher order BVPs with singularities in phase variables. *Computers and Mathematics with Applications* 46 (12) (2003), 1799-1826.
- Sturm-Liouville and focal higher order BVPs with singularities in phase variables. *Georgian Mathematical Journal* 10 (2003), 165-191.
- General existence principle for singular BVPs and its applications. *Georgian Mathematical Journal* 11 (2004), 549-565.
- A singular boundary value problem for odd order differential equations. *J. Math. Anal. Appl.* 291 (2004), 741-756.

The singular Lidstone boundary value problem

$$(-1)^n x^{(2n)}(t) = f(t, x(t), \dots, x^{(2n-2)}(t)), \qquad (9)$$

$$x^{(2j)}(0) = x^{(2j)}(T) = 0, \quad 0 \le j \le n-1, \qquad (10)$$

where $n \ge 1$ and f satisfies the local Carathéodory conditions on the set $[0, T] \times \mathcal{D}$,

$$\mathcal{D} = \begin{cases} \underbrace{\mathbb{R}_+ \times \mathbb{R}_0 \times \mathbb{R}_- \times \mathbb{R}_0 \times \cdots \times \mathbb{R}_+}_{4k-3} & \text{if } n = 2k-1, \\ \underbrace{\mathbb{R}_+ \times \mathbb{R}_0 \times \mathbb{R}_- \times \mathbb{R}_0 \times \cdots \times \mathbb{R}_-}_{4k-1} & \text{if } n = 2k. \end{cases}$$

The function $f(t, x_0, \dots, x_{2n-2})$ may be singular at the points $x_i = 0$, $0 \le i \le 2n-2$, of the space variables $x_0, \dots, x_{2n-2}$.

Assumptions:

(H_1) $f \in Car([0,T] \times \mathcal{D})$ and there exists $\psi \in L_1[0,T]$ such that

$$0 < \psi(t) \le f(t, x_0, \dots, x_{2n-2})$$

for a.e. $t \in [0,T]$ and each $(x_0, \dots, x_{2n-2}) \in D$;

(H_2) For a.e. $t \in [0,T]$ and for each $(x_0, \dots, x_{2n-2}) \in \mathcal{D}$,

$$f(t, x_0, \dots, x_{2n-2}) \le \phi(t) + \sum_{j=0}^{2n-2} q_j(t)\omega_j(|x_j|) + \sum_{j=0}^{2n-2} h_j(t)|x_j|$$

where ϕ, $h_j \in L_1[0,T]$, h_j are sufficiently small and $q_j \in L_\infty[0,T]$ are nonnegative, $\omega_j : \mathbb{R}_+ \to \mathbb{R}_+$ are nonincreasing,

$$\int_0^T \omega_j(s)\, ds < \infty, \quad \omega_j(uv) \le \Lambda \omega_j(u)\omega_j(v).$$

Definition

A function $x \in AC^{2n-1}[0, T]$ is said to be a solution of BVP (9), (10) if $(-1)^j x^{(2j)}(t) > 0$ for $t \in (0, T)$ and $0 \le j \le n-1$, x satisfies the boundary conditions (10) and (9) holds a.e. on J.

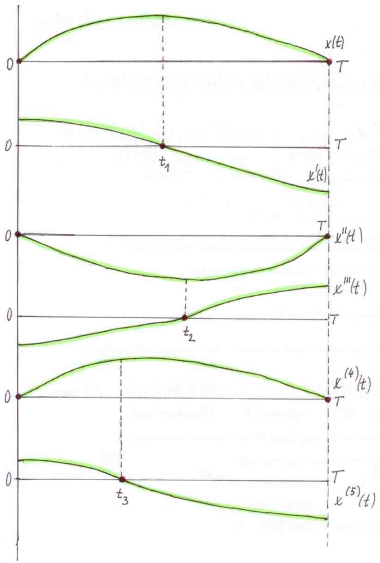
Existence theorem - Lidstone

Let assumptions (H_1) *and* (H_2) *be satisfied. Then there exists a solution of the Lidstone BVP* (9), (10).

Proofs are based on:

- a priori estimates of solutions,
- the Green's functions,
- investigation of zeros of solutions,
- the general existence principle

Singular Lidstone



Let $n \in \mathbb{N}$, $[0, T] \subset \mathbb{R}$ and $\mathcal{D} \subset \mathbb{R}^n$, $\mathcal{D} \neq \overline{\mathcal{D}}$.
We study the singular BVP

$$u^{(n)}(t) = f(t, u(t), \ldots, u^{(n-1)}(t)),$$
$$u \in \mathcal{B},$$

where f satisfies the local Carathéodory conditions on $[0, T] \times \mathcal{D}$, $f(t, x_0, \ldots, x_{n-1})$ may be singular on the boundary $\partial\mathcal{D}$ at the points $x_i = 0$, $0 \leq i \leq n-1$, of the space variables $x_0, \ldots, x_{n-1}$ and $\mathcal{B}$ is a closed subset in $C^{n-1}([0, T])$.
We look for solutions $u \in AC^{n-1}([0, T]) \cap \mathcal{B}$ such that

$$((u(t), \ldots, u^{(n-1)}(t)) \in \overline{\mathcal{D}} \subset \mathbb{R}^n \quad \text{for all } t \in [0, T].$$

Consider the singular boundary value problem

$$u^{(n)}(t) = f(t, u(t), \dots, u^{(n-1)}(t)), \quad u \in \mathcal{B}, \tag{11}$$

and the sequence of regular boundary value problems

$$u^{(n)}(t) = f_m(t, u(t), \dots, u^{(n-1)}(t)), \quad u \in \mathcal{B}, \tag{12}$$

where f_m satisfies the local Carathéodory conditions on the set $[0, T] \times \mathbb{R}^n)$, $m \in \mathbb{N}$.
Assume that for a.e. $t \in [0, T]$ and all $m \in \mathbb{N}$, $(x_0, \dots, x_{n-1}) \in \mathcal{D}$, $0 \le i \le n-1$,

$$f_m(t, x_0, \dots, x_{n-1}) = f(t, x_0, \dots, x_{n-1}) \text{ if } |x_i| \ge \frac{1}{m}.$$

Theorem - General existence principle

Assume that there is a bounded set $\Omega \subset C^{n-1}([0,T])$ such that

(*i*) $\forall m \in \mathbb{N}$, the regular BVP (12) has a solution $u_m \in \Omega$;

(*ii*) $\forall \varepsilon > 0 \; \exists \delta > 0$ such that for each $m \in \mathbb{N}$ and each system of mutually disjoint intervals $\{(\tau_j, t_j)\}_{j=1}^{\infty}$ in $[0,T]$,

$$\sum_{j=1}^{\infty}(t_j - \tau_j) < \delta \Rightarrow \sum_{j=1}^{\infty} \int_{\tau_j}^{t_j} |f_m(t, u_m(t), \dots, u_m^{(n-1)}(t))|dt < \varepsilon.$$

Then:

(*a*) there exist $u \in cl(\Omega)$ and a subsequence $\{u_k\} \subset \{u_m\}$ such that $\lim_{k\to\infty} \|u_k - u\|_{C^{n-1}} = 0$,

(*b*) if $\mu(\mathcal{V}) = 0$, where $\mathcal{V}$ is the set of all zeros of the functions $u^{(i)}$ with $0 \le i \le n-1$, then u is a solution of the singular BVP (11), (12) and $u > 0$ on $(0,T)$.

The proof of the General existence principle is based on:

- topological degree arguments,
- the combination of regularization and sequential techiques,
- the Vitali convergence theorem.

Vitali convergence theorem

Let $\varphi_m \subset L_1[0,T]$ for $m \in \mathbb{N}$ and let

$$\lim_{m\to\infty} \varphi_m(t) = \varphi(t) \quad \text{for a.e. } t \in [0,T].$$

Then the following statements are equivalent:

(i) $\varphi \in L_1[0,T]$ and $\lim_{m\to\infty} \|\varphi_m - \varphi\|_1 = 0$,

(ii) the sequence $\{\varphi_m\}$ is uniformly integrable on $[0,T]$.

The singular (n, p) boundary value problem

$$-x^{(n)}(t) = f(t, x(t), \dots, x^{(n-1)}(t)), \tag{13}$$

$$x^{(i)}(0) = 0,\ 0 \le i \le n-2,\ x^{(p)}(T) = 0, \tag{14}$$

where $n \ge 2$, $0 \le p \le n-1$, and f satisfies the local Carathéodory conditions on the set $[0, T] \times \mathcal{D}$,

$$\mathcal{D} = \mathbb{R}_+ \times \mathbb{R}_0^{n-2} \times \mathbb{R}.$$

The function $f(t, x_0, \dots, x_{n-1})$ may be singular at the points $x_i = 0$, $0 \le i \le n-2$ of the space variables $x_0, \dots, x_{n-2}$.

The singular Sturm-Liouville boundary value problem

$$
\begin{aligned}
-x^n(t) &= f(t, x(t), \dots, x^{(n-1)}(t)),\\
x^{(i)}(0) = 0, &\quad 0 \le i \le n-3,\\
\alpha x^{(n-2)}(0) &- \beta x^{(n-1)}(0) = 0,\\
\gamma x^{(n-2)}(T) &+ \delta x^{(n-1)}(T) = 0,
\end{aligned}
$$

where $n > 2$, $\alpha, \gamma > 0$, $\beta, \delta \ge 0$, and f satisfies the local Carathéodory conditions on the set $[0, T] \times \mathcal{D}$,

$$\mathcal{D} = \mathbb{R}_+^{n-1} \times \mathbb{R}_0.$$

The function $f(t, x_0, \dots, x_{n-1})$ may be singular at the points $x_i = 0$, $0 \le i \le n-1$, of all its space variables $x_0, \dots, x_{n-1}$.

Other higher order singular problems

The singular $(p, n-p)$ right focal boundary value problem

$$(-1)^{n-p} x^{(n)}(t) = f(t, x(t), \ldots, x^{(n-1)}(t)),$$
$$x^{(i)}(0) = 0,\ 0 \le i \le p-1,\ x^{(i)}(T) = 0,\ p \le i \le n-1,$$

where $n > 2$, $1 \le p \le n-1$, and f satisfies the local Carathéodory conditions on the set $[0, T] \times \mathcal{D}$,

$$\mathcal{D} = \begin{cases} \underbrace{\mathbb{R}_+^{p+1} \times \mathbb{R}_- \times \mathbb{R}_+ \times \mathbb{R}_- \times \cdots \times \mathbb{R}_+}_{n} & \text{if } n-p \text{ is odd} \\ \underbrace{\mathbb{R}_+^{p+1} \times \mathbb{R}_- \times \mathbb{R}_+ \times \mathbb{R}_- \times \cdots \times \mathbb{R}_-}_{n} & \text{if } n-p \text{ is even.} \end{cases}$$

The function $f(t, x_0, \ldots, x_{n-1})$ may be singular at the points $x_i = 0$, $0 \le i \le n-1$, of all its space variables $x_0, \ldots, x_{n-1}$.

Other higher order singular problems

The singular $(p, n-p)$ conjugate boundary value problem.

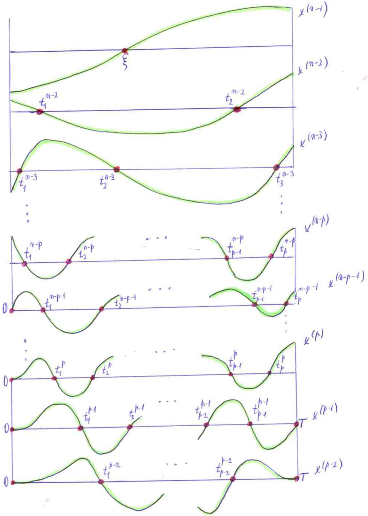
$$(-1)^p x^{(n)}(t) = f(t, x(t), \ldots, x^{(n-1)}(t)),$$
$$x^{(i)}(0) = 0, \quad x^{(j)}(T) = 0, \quad 0 \le i \le n-p-1, \; 0 \le j \le p-1,$$

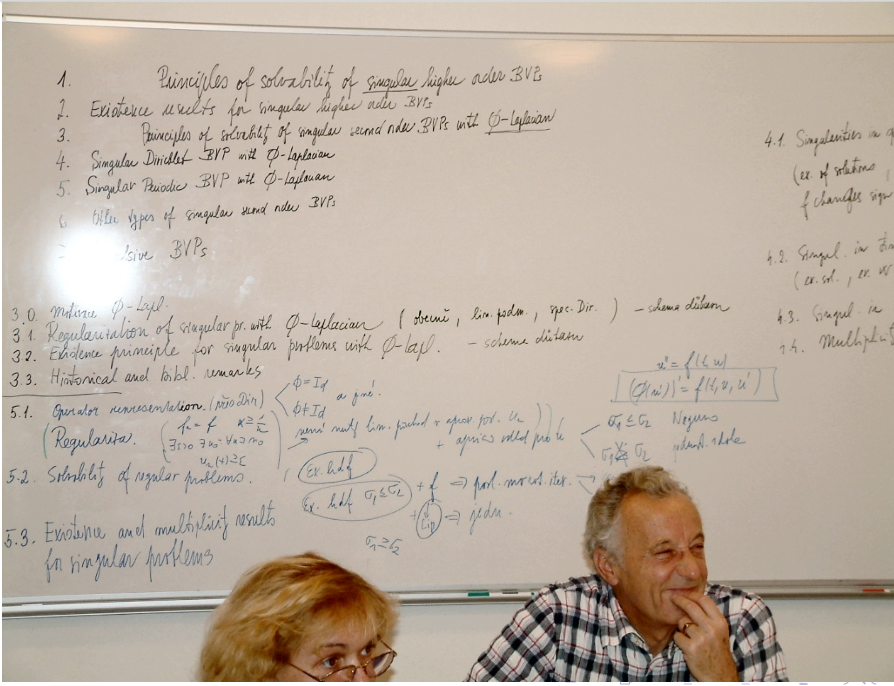
where $n > 2$, $p \le n-1$, and f satisfies the local Carathéodory conditions on the set $[0, T] \times \mathcal{D}$,

$$\mathcal{D} = ((0, \infty) \times \mathbb{R}_0^{n-1}), \quad \mathbb{R}_0 = \mathbb{R} \setminus \{0\}.$$

The function $f(t, x_0, \ldots, x_{n-1})$ may be singular at the points $x_i = 0$, $0 \le i \le n-1$, of all its space variables $x_0, \ldots, x_{n-1}$.

- Zeros of derivatives of solutions to singular (p,n-p) conjugate BVPs. *Acta Univ. Palacki. Olomuc.*, Fac. rer. nat., Mathematica 43 (2004), 137-141.




1. Principles of solvability of singular higher order BVPs
2. Existence results for singular higher order BVPs
3. Principles of solvability of singular second order BVPs with φ-Laplacian
4. Singular Dirichlet BVP with φ-Laplacian
5. Singular Periodic BVP with φ-Laplacian
6. Other types of singular second order BVPs
BVPs
3.1. Regularization of singular pr. with φ-Laplacian
3.2. Existence principle for singular problems with φ-lapl.
3.3. Historical and bibl. remarks
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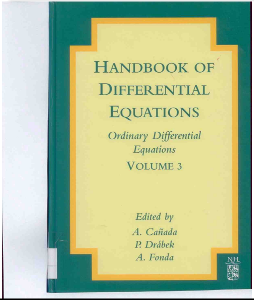
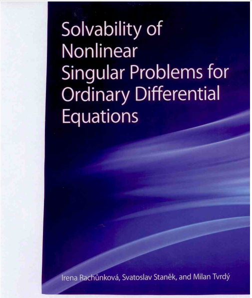
Regularization
5.2. Solvability of regular problems
5.3. Existence and multiplicity results for singular problems
4.1. Singularities in
4.2. Singul. in
4.3. Singul. in

- Singularities and Laplacians in Boundary Value Problems for Nonlinear Ordinary Differential Equations. Handbook of Differential Equations. Ordinary Differential Equations, vol.3, pp. 607-723. Ed. by A. Caňada, P. Drábek, A. Fonda. Elsevier 2006.
- Solvability of Nonlinear Singular Problems for Ordinary Differential Equations. Hindawi Publishing Corporation, New York, USA, 2009, 268 pages.





We investigate the singular second order ordinary differential equation

$$u'' - \frac{a}{t} u' = \lambda f(t, u, u'),$$

where $a \in \mathbb{R} \setminus \{0\}$, $\lambda > 0$ and $f(t, x, y)$ satisfies the local Carathéodory conditions on $[0, T] \times \mathbb{R}^2$.

- Limit properties of solutions of singular second-order differential equations. Boundary Value Problems, Vol. 2009, Article ID 905769, 28 pages.
- Neumann problems with time singularities. Computers and Mathematics with Applications 60 (2010), 722-733.

$$u'' - \frac{a}{t}u' = \lambda f(t, u, u'), \tag{15}$$

$$u(0) = u(T), \quad u'(0) = u'(T). \tag{16}$$

A function u is called a solution of equation (15) if $u \in AC^1[0, T]$ fulfilfs the equation for a.e. $t \in [0, T]$.
If $a \neq 0$, then the solution u satisfies

$$\lim_{t \to 0+} u'(t) = 0.$$

- Periodic BVPs in ODEs with time singularities. Computers and Mathematics with Applications 62 (2011), 2058-2070.

Oposite-ordered lower and upper functions.
Auxiliary boundary conditions: $u(0) = u(T)$, $u'(T) = 0$.
Leray-Schauder degree method.

Existence theorem - periodic problem

Let $a > 0$. Suppose that there exist $A, B \in \mathbb{R}$, such that $A < B$ and

$$f(t,x,y) > 0 \ \text{ for a.e. } \ t \in [0,T] \text{ and all } x \leq A,\ y \in \mathbb{R} \tag{17}$$

$$f(t,x,y) < 0 \ \text{ for a.e. } \ t \in [0,T] \text{ and all } x \geq B,\ y \in \mathbb{R}. \tag{18}$$

Further, assume that

$$|f(t,x,y)| \leq g(t)\omega(|y|) \text{ for a.e.} t \in [0,T],\ \text{all } (x,y) \in \mathbb{R}^2, \tag{19}$$

where $g \in L_1[0,T]$, $\omega(y) \in C[0,\infty)$ are positive and ω is nondecreasing.

Then problem (15), (16) has a solution for each $\lambda \in (0, \lambda^*)$, where

$$\lambda^* = \int_0^\infty \frac{\mathrm{d}s}{\omega(s)} \cdot \left(\int_0^T g(t)\ \mathrm{d}t \right)^{-1}.$$

Example

The function

$$f(t,x,y) = \frac{t}{3} - \frac{1}{4\sqrt{t}}(1+y^2)\arctan x \tag{20}$$

satisfies assumptions (17)-(19) on $[0,1]\times\mathbb{R}^2$ with $g(t) = \frac{t}{3} + \frac{\pi}{8\sqrt{t}}$ and $\omega(s) = 1+s^2$.
We have $A = 0$ and $B = \tan\left(\frac{4}{3}\right)$ and

$$\lambda^* = \int_0^\infty \frac{\mathrm{d}s}{\omega(s)} \cdot \left(\int_0^1 g(t)\,\mathrm{d}t\right)^{-1} = \frac{6\pi}{2+3\pi}.$$

For each $\lambda \in \left(0, \frac{6\pi}{2+3\pi}\right)$ there exists a solution.

Example

Let $\mu \in (0,1)$. Consider the function

$$f(t,x,y) = h(t,x,y) - \frac{1}{\sqrt[\beta]{t}} \frac{x}{\sqrt{1+x^2}} (1+|y|^\alpha) \tag{21}$$

depending on the parameters $\beta \in (1,\infty)$ and $\alpha \in (0,\infty)$. Here $h \in C([0,1]\times\mathbb{R}^2)$ and $|h(t,x,y)| \le \mu$ on $[0,1]\times\mathbb{R}^2$. Then f satisfies assumptions (17)-(19) on $[0,1]\times\mathbb{R}^2$ with $g(t) = \mu + \frac{1}{\sqrt[\beta]{t}}$, $\omega(s) = (1+s)^\alpha$.

We have $A \le -\frac{\mu}{\sqrt{1-\mu^2}}$ and $B \ge \frac{\mu}{\sqrt{1-\mu^2}}$ and

$$\lambda^* = \begin{cases} \infty & \text{for } \alpha \le 1, \\ \dfrac{\beta - 1}{(\alpha-1)(\beta+\mu(\beta-1))} & \text{for } \alpha > 1. \end{cases}$$

For each $\lambda \in (0,\lambda^*)$ there exists a solution.

- Properties of the set of positive solutions to Dirichlet boundary value problems with time singularities. Central European Journal of Mathematics, to appear.

We consider the singular Dirichlet boundary value problem

$$u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) = f(t, u(t), u'(t)), \tag{22}$$

$$u(0) = 0, \quad u(T) = 0, \tag{23}$$

where $a \in (-\infty, -1)$. For $\mathcal{D} = [0, \infty) \times \mathbb{R}$ we assume that f satisfies the local Carathéodory conditions on $[0, T] \times \mathcal{D}$. For $c \geq 0$ we consider the additional condition

$$u'(T) = -c. \tag{24}$$

Assumptions:

(H_1) There exists $\varphi \in L^1[0, T]$ such that

$$0 < \varphi(t) \le f(t, x, y) \quad \text{for a.e. } t \in [0, T] \text{ and all } (x, y) \in \mathcal{D}.$$

(H_2) For a.e. $t \in [0, T]$ and all $(x, y) \in \mathcal{D}$ the estimate

$$f(t, x, y) \le h(t, x, |y|),$$

is fulfilled, where $h \in Car([0, T] \times [0, \infty)^2)$, $h(t, x, z)$ is nondecreasing in the variables x, z, and

$$\lim_{x \to \infty} \frac{1}{x} \int_0^T h(t, x, x)\, \mathrm{d}t = 0.$$

We denote the set of all positive solutions of problem (22), (23), (24) by $\mathcal{S}_c$.

Theorem about the set $\mathcal{S}_c$

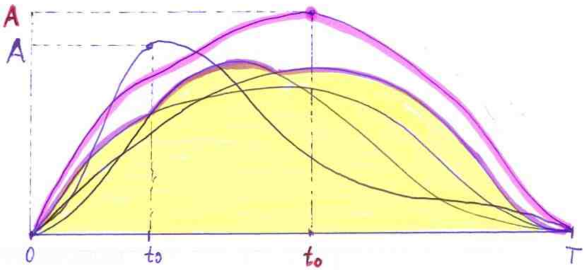
Let $(H_1), (H_2)$ hold. Then for each $c \geq 0$ the set $\mathcal{S}_c$ is nonempty and compact in $C^1[0, T]$.

$$\beta(t) = \max\{u(t) : u \in \mathcal{S}_0\} \quad \text{for } t \in [0, T].$$

$$\mathcal{S} = \cup_{c \geq 0} \mathcal{S}_c.$$

Theorem about the set $\mathcal{S} \setminus \mathcal{S}_0$

Let $(H_1), (H_2)$ hold. Then for each $t_0 \in (0, T)$ and each $A > \beta(t_0)$ there exists a positive solution u of problem (22), (23) satisfying $u(t_0) = A$.


A
A
0
t_0
t_0
T

Operators:

$$\tilde{f}(t,x,y) = \begin{cases} f(t,x,y) & \text{if } x \geq 0 \\ f(t,0,y) & \text{if } x < 0. \end{cases}$$

We define

$$(\mathcal{H}x)(t) = t\int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1}\tilde{f}(\xi, x(\xi), x'(\xi))\,\mathrm{d}\xi\right)\mathrm{d}s.$$

Further, for each $t_0 \in (0, T)$ and $A \geq 0$ we define

$$(\mathcal{K}_{t_0,A}x)(t) = \frac{t}{t_0}\frac{T^{-a-1} - t^{-a-1}}{T^{-a-1} - t_0^{-a-1}}\max\{0, A - (\mathcal{H}x)(t_0)\} + (\mathcal{H}x)(t),$$

and for each $c \geq 0$ we define

$$(\mathcal{L}_c x)(t) = t\frac{cT^{a+1}}{|a+1|}(T^{-a-1} - t^{-a-1}) + (\mathcal{H}x)(t).$$

A special case

$$u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) = f(t, u(t)). \tag{25}$$

Now we will work with the following assumptions on f:

(H_1^*) $f \in Car([0, T] \times [0, \infty))$.

(H_2^*) $0 < f(t, x)$ for a.e. $t \in [0, T]$ and all $x \in [0, \infty)$.

(H_3^*) $f(t, x)$ is increasing in x for a.e. $t \in [0, T]$ and

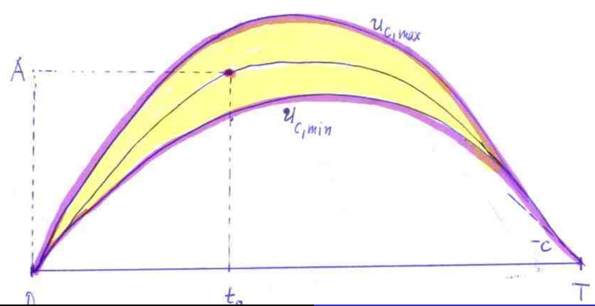
$$\lim_{x \to \infty} \frac{1}{x} \int_0^T f(t, x)\, \mathrm{d}t = 0.$$

- There exist minimal and maximal solutions $u_{c,min}, u_{c,max} \in \mathcal{S}_c$ for each $c \geq 0$.
- If the interior of the set $\{(t, x) \in \mathbb{R}^2 : 0 \leq t \leq T,\ u_{c,min}(t) \leq x \leq u_{c,max}(t)\}$ is nonempty, then this interior is covered by graphs of other solutions of $\mathcal{S}_c$ for each $c > 0$.

Theorem

Let $(H_1^*) - (H_3^*)$ hold.

- Assume that there exists $t_0 \in (0, T)$ such that $u_{c,min}(t_0) < u_{c,max}(t_0)$ for some $c > 0$. Then for each $A \in (u_{c,min}(t_0), u_{c,max}(t_0))$ there exists $u \in \mathcal{S}_c$ satisfying $u(t_0) = A$.
- The set $\mathcal{S}_c$ is one-point for each $c \in [0, \infty) \setminus \Gamma$, where $\Gamma \subset [0, \infty)$ is at most countable.



Example

Let us choose $\alpha \in [0, 1)$ and for a.e. $t \in [0, T]$ and all $x \in [0, \infty)$, define the function f by

$$f(t, x) = h_1(t) + h_2(t, x)x^{\alpha},$$

or

$$f(t, x) = h_1(t) + h_2(t, x)\frac{x}{\ln(x + 2)},$$

where $h_1 \in L^1[0, T]$, $h_1 > 0$ a.e. on $[0, T]$, h_2 is nonnegative, bounded and continuous on $[0, T] \times [0, \infty)$ and increasing in x. Then $(H_1^*) - (H_3^*)$ hold.

Happy birthday, Svaťa !!!