On Vallée-Poussin type conditions for positivity of solutions to the Darboux problem for hyperbolic functional-differential equations

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On the rectangle $\mathbb{D} = [a, b] \times [c, d]$ we consider the equation

$$\boxed{u''_{tx}(t, x) = p(t, x)u\big(\tau(t, x), \mu(t, x)\big) + q(t, x)} \tag{1}$$

- $p, q \colon \mathbb{D} \to \mathbb{R}$ integrable
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We cannot pass between equation (1) and the wave equation

$$u''_{tt}(t,x) - u''_{xx}(t,x) = \tilde{p}(t,x)u\big(\tilde{\tau}(t,x),\tilde{\mu}(t,x)\big) + \tilde{q}(t,x).$$

Consider now, on the interval $[a, b]$, the n-dimensional system of linear ordinary differential equations

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It is well known that every solution v to system (2) admits the representation

$$v(t) = C(t, t_0)v(t_0) + \int_{t_0}^{t} C(t, s)q(s)\mathrm{d}s \quad \text{for } t \in [a, b],$$

where $t_0 \in [a, b]$ and C is the Cauchy matrix of the system $v' = P(t)v$, i. e., for every $t_0 \in [a, b]$, $C(\cdot, t_0)$ is a fundamental matrix of the system $v' = P(t)v$, which satisfies $C(t_0, t_0) = E_n$.

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Remark. If $n = 1$ then, for every $t_0 \in [a, b]$, $C(\cdot, t_0)$ is a solution to the problem

$$v' = P(t)v, \qquad v(t_0) = 1.$$

Indeed, we have

$$C(t, s) = \mathrm{e}^{\int_s^t P(\xi)\mathrm{d}\xi} \quad \text{for } t, s \in [a, b]$$

in this case.

$$\boxed{u''_{tx} = p(t,x)u + q(t,x)} \tag{1'}$$

Every solution u to the equation $(1')$ admits the representation

$$u(t,x) = Z_{t,x}(a,c)u(a,c) + \int_a^t Z_{t,x}(s,c)u'_s(s,c)\mathrm{d}s +$$
$$+ \int_c^x Z_{t,x}(a,\eta)u'_\eta(a,\eta)\mathrm{d}\eta + \int_a^t \int_c^x Z_{t,x}(s,\eta)q(s,\eta)\mathrm{d}\eta\mathrm{d}s$$

for $(t,x) \in \mathbb{D}$, where $Z_{t,x}$ are the Riemann functions of the equation $u''_{tx} = p(t,x)u$.

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The Riemann function Z_{t_0,x_0} is defined as a solution to the Darboux problem

$$u''_{tx} = p(t,x)u,$$
$$u(t,x_0) = 1 \quad \text{for } t \in [a,b], \qquad u(t_0,x) = 1 \quad \text{for } x \in [c,d].$$

$$u(t,x) = Z_{t,x}(a,c)u(a,c) + \int_a^t Z_{t,x}(s,c)u'_s(s,c)\mathrm{d}s +$$
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Proposition

Let

$$Z_{t,x}(s,\eta) \geq 0 \quad \text{for } a \leq s \leq t \leq b,\ c \leq \eta \leq x \leq d. \qquad (*)$$

Then the implication

$$\left.\begin{array}{l} w \in AC(\mathbb{D}), \\ w''_{tx}(t,x) \geq p(t,x)w(t,x) \text{ for a. e. } (t,x) \in \mathbb{D}, \\ w(a,c) \geq 0, \\ w'_t(t,c) \geq 0 \text{ for a. e. } t \in [a,b], \\ w'_x(a,x) \geq 0 \text{ for a. e. } x \in [c,d] \end{array}\right\} \implies w(t,x) \geq 0 \text{ for } (t,x) \in \mathbb{D}$$

holds.

Let $(t_0, x_0) \in \mathbb{D}$. On $[a, t_0] \times [c, x_0]$ we consider the Darboux problem

$$u_{tx} = -ku \qquad (k \geq 0) \tag{3}$$

$$u(t, x_0) = 1 \quad \text{for } t \in [a, b], \qquad u(t_0, x) = 1 \quad \text{for } x \in [c, d] \tag{4}$$

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If u is looked for in the form $u(t, x) = v(z)$, where $z = \sqrt{(t_0 - t)(x_0 - x)}$, then (3), (4) is reduced to

$$\boxed{v''(z) + \frac{1}{z}\, v'(z) + 4kv(z) = 0, \qquad v(0) = 1} \tag{5}$$

on the interval $\left[0, \sqrt{(t_0 - a)(x_0 - c)}\,\right]$, which has a solution

$$v(z) = J_0\left(2\sqrt{k}\, z\right).$$

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The Riemann function Z_{t_0, x_0} of the equation (3) satisfies

$$Z_{t_0, x_0}(t, x) = J_0\left(2\sqrt{k(t_0 - t)(x_0 - x)}\right) \quad \text{for } (t, x) \in [a, t_0] \times [c, x_0].$$

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Therefore, the condition $(*)$ holds, i. e.,

$$Z_{t,x}(s, \eta) \geq 0 \quad \text{for } a \leq s \leq t \leq b, \; c \leq \eta \leq x \leq d, \tag{$*$}$$

if and only if

$$2\sqrt{k(b - a)(d - c)} \leq j_0\,,$$

i. e.,

$$k \leq \frac{j_0^2}{4(b - a)(d - c)}\,.$$

$$\boxed{u''_{tx}(t,x) = p(t,x)u\big(\tau(t,x),\mu(t,x)\big) + q(t,x)} \tag{1}$$

Definition

We say that a theorem on differential inequalities (maximum principle) holds for the equation (1) if the implication

$$\left.\begin{aligned} & w \in AC(\mathbb{D}), \\ & w''_{tx}(t,x) \geq p(t,x)w\big(\tau(t,x),\mu(t,x)\big) \text{ for a. e. } (t,x) \in \mathbb{D}, \\ & w(a,c) \geq 0, \\ & w'_t(t,c) \geq 0 \text{ for a. e. } t \in [a,b], \\ & w'_x(a,x) \geq 0 \text{ for a. e. } x \in [c,d] \end{aligned}\right\} \implies \begin{aligned} & w(t,x) \geq 0 \\ & \qquad \text{for } (t,x) \in \mathbb{D} \end{aligned}$$

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holds.

In what follows we restrict our-self to the case where

$$p(t,x) \leq 0 \quad \text{for a. e. } (t,x) \in \mathbb{D}. \tag{5}$$

$$\boxed{u''_{tx}(t,x) = p(t,x)u\big(\tau(t,x),\mu(t,x)\big) + q(t,x)} \qquad (1)$$

Proposition

Let maximum principle holds for equation (1) with non-positive p then the inequalities

$$\tau(t,x) \le t, \quad \mu(t,x) \le x \quad \text{for a. e. } (t,x) \in \mathbb{D}. \qquad (6)$$

hold, i. e., equation (1) is delayed in both arguments.

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Theorem 1

Let conditions (5) and (6) hold. Then maximum principle holds for equation (1) provided that there exists a function $\gamma \in C(\mathbb{D}) \cap AC_{loc}([a,b[\times[c,d[)$ satisfying the inequalities

$$\gamma(t,x) > 0 \quad \text{for } (t,x) \in [a,b[\times[c,d[\,,$$

$$\gamma''_{tx}(t,x) \leq p(t,x)\gamma\big(\tau(t,x),\mu(t,x)\big) \quad \text{for a. e. } (t,x) \in \mathbb{D},$$

$$\gamma'_t(t,c) \leq 0 \quad \text{for a. e. } t \in [a,b],$$

$$\gamma'_x(a,x) \leq 0 \quad \text{for a. e. } x \in [c,d].$$

$$\boxed{u''_{tx}(t,x) = p(t,x)u\big(\tau(t,x), \mu(t,x)\big) + q(t,x)} \qquad (1)$$

Corollary 1

Let conditions (5) and (6) hold and

$$\iint_{\mathbb{D}} |p(t,x)| \mathrm{d}t\mathrm{d}x \leq 1.$$

Then maximum principle holds for equation (1).

Remark. The number 1 in Corollary 1 is optimal, in general. A counter-example is constructed with

$$\tau(t,x) \equiv a, \qquad \mu(t,x) \equiv c.$$

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Proposition

Let $\tau(t,x) \equiv t$, $\mu(t,x) \equiv x$, $p(t,x) \equiv k \le 0$. Then maximum principle holds for equation (1) if and only if

$$|k|(b-a)(d-c) \le \frac{j_0^2}{4} \sim 1.4458.$$

$$\boxed{u''_{tx}(t,x) = p(t,x)u\big(\tau(t,x),\mu(t,x)\big) + q(t,x)} \tag{1}$$

j_ν ... the first positive zero of the Bessel function J_ν $(\nu > -1)$

$$E_\nu(z) := z^{-\nu} J_\nu(z) \quad \text{for } z \geq 0$$

$$j_\nu^* = \frac{E_{\nu+1}(j_\nu)}{E_{\nu+1}(0)}$$

$$\boxed{u''_{tx}(t,x) = p(t,x)u\big(\tau(t,x),\mu(t,x)\big) + q(t,x)} \tag{1}$$

Corollary 2

Let conditions (5) and (6) hold and there exist numbers $\alpha \in [0,1[$, $\beta \in [0,\alpha]$ such that the inequalities

$$|p(t,x)| \leq \frac{j^2_{-\alpha}}{4(b-a)(d-c)}\,,$$

$$\Big(E_{-\alpha}\big(z[\tau(t,x),x]\big) - E_{-\alpha}\big(z[t,x]\big)\Big)|p(t,x)| \leq \frac{\beta}{2}\,\frac{j^2_{-\alpha}}{(b-a)(d-c)}\,E_{1-\alpha}\big(z[\tau(t,x),x]\big),$$

$$\Big(E_{-\alpha}\big(z[t,x]\big) - E_{-\alpha}\big(z[t,\mu(t,x)]\big)\Big)|p(t,x)| \leq \frac{\alpha-\beta}{2}\,\frac{j^2_{-\alpha}}{(b-a)(d-c)}\,E_{1-\alpha}\big(z[\tau(t,x),x]\big)$$

are satisfied a. e. in $\mathbb{D}$, where

$$z[t,x] := \frac{j_{-\alpha}}{2}\sqrt{\frac{(t-a)(x-c)}{(b-a)(d-c)}} \quad \text{for } (t,x) \in \mathbb{D}.$$

Then maximum principle holds for equation (1).

$$\boxed{u''_{tx}(t,x) = p(t,x)u\big(\tau(t,x), \mu(t,x)\big) + q(t,x)} \qquad (1)$$

Corollary 3

Let conditions (5) and (6) hold and there exist numbers $\alpha \in [0,1[$, $\beta \in [0,\alpha]$ such that the inequalities

$$|p(t,x)| \le \frac{j^2_{-\alpha}}{4(b-a)(d-c)},$$

$$(x-c)\big(t-\tau(t,x)\big)|p(t,x)| \le \beta\, j^*_{-\alpha},$$

$$(t-a)\big(x-\mu(t,x)\big)|p(t,x)| \le (\alpha-\beta)\, j^*_{-\alpha}$$

are satisfied a. e. in $\mathbb{D}$. Then maximum principle holds for equation (1).

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Corollary 3

Let conditions (5) and (6) hold and there exist numbers $\alpha \in [0,1[$, $\beta \in [0,\alpha]$ such that the inequalities

$$|p(t,x)| \leq \frac{j_{-\alpha}^2}{4(b-a)(d-c)} \,,$$

$$(x-c)\big(t-\tau(t,x)\big)|p(t,x)| \leq \beta\, j_{-\alpha}^*,$$

$$(t-a)\big(x-\mu(t,x)\big)|p(t,x)| \leq (\alpha-\beta)\, j_{-\alpha}^*$$

are satisfied a. e. in $\mathbb{D}$. Then maximum principle holds for equation (1).

Remark. The number $\frac{j_{-\alpha}^2}{4(b-a)(d-c)}$ in Corollaries 2 and 3 is optimal, in general. A counter-example is constructed with

$$\tau(t,x) \equiv t, \qquad \mu(t,x) \equiv x, \qquad p(t,x) \equiv k \leq 0,$$

where we can choose $\alpha = \beta = 0$ and we know that the inequality

$$|k| \leq \frac{j_0^2}{4(b-a)(d-c)} \quad \text{is sufficient and necessary.}$$

$$\boxed{u''_{tx}(t,x) = p(t,x)u\big(\tau(t,x),\mu(t,x)\big) + q(t,x)} \tag{1}$$

Corollary 2

Let conditions (5) and (6) hold and there exist numbers $\alpha \in [0,1[$, $\beta \in [0,\alpha]$ such that the inequalities

$$|p(t,x)| \leq \frac{j_{-\alpha}^2}{4(b-a)(d-c)}\,,$$

$$\Big(E_{-\alpha}\big(z[\tau(t,x),x]\big) - E_{-\alpha}\big(z[t,x]\big)\Big)|p(t,x)| \leq \frac{\beta}{2}\,\frac{j_{-\alpha}^2}{(b-a)(d-c)}\,E_{1-\alpha}\big(z[t,x]\big),$$

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are satisfied a. e. in $\mathbb{D}$, where

$$z[t,x] := \frac{j_{-\alpha}}{2}\sqrt{\frac{(t-a)(x-c)}{(b-a)(d-c)}} \quad \text{for } (t,x) \in \mathbb{D}.$$

Then maximum principle holds for equation (1).

Sketch of the proof of Corollary 2

We put

$$\boxed{\gamma(t,x) = E_{-\alpha}\big(z[t,x]\big) \quad \text{for } (t,x) \in \mathbb{D}}$$

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and apply

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Let conditions (5) and (6) hold. Then maximum principle holds for equation (1) provided that there exists a function $\gamma \in C(\mathbb{D}) \cap AC_{loc}([a,b[\,\times[c,d[)$ satisfying the inequalities

$$\gamma(t,x) > 0 \quad \text{for } (t,x) \in [a,b[\,\times[c,d[\,,$$

$$\gamma''_{tx}(t,x) \le p(t,x)\gamma\big(\tau(t,x),\mu(t,x)\big) \quad \text{for a. e. } (t,x) \in \mathbb{D},$$

$$\gamma'_t(t,c) \le 0 \quad \text{for a. e. } t \in [a,b],$$

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$$J''_{-\alpha}(s) + \frac{1}{s}\,J'_{-\alpha}(s) + \left(1 - \frac{\alpha^2}{s^2}\right) J_{-\alpha}(s) = 0$$

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$$\gamma''_{tx}(t,x) = -\frac{j^2_{-\alpha}}{4(b-a)(d-c)}\,\gamma(t,x) + \frac{\beta}{x-c}\,\gamma'_t(t,x) + \frac{\beta}{t-a}\,\gamma'_x(t,x)$$

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We need to show that

$$\gamma''_{tx}(t,x) \le p(t,x)\gamma\big(\tau(t,x),\mu(t,x)\big)$$

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$$\gamma''_{tx}(t,x) \le p(t,x)\gamma\big(\tau(t,x),\mu(t,x)\big)$$

$$\begin{aligned}
\gamma(t,x) > 0 \quad & \text{for every } (t,x) \in \mathbb{D}, (t,x) \neq (b,d), \\
\gamma_t(t,x) \le 0 \quad & \text{for every } (t,x) \in \,]a,b] \times [c,d], \\
\gamma_x(t,x) \le 0 \quad & \text{for every } (t,x) \in [a,b] \times \,]c,d], \\
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\end{aligned}$$

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\end{aligned}$$

$$\begin{aligned}
\gamma\big(\tau(t,x),\mu(t,x)\big) &= \gamma(t,x) - \int_{\tau(t,x)}^{t}\gamma'_s(s,x)\mathrm{d}s - \int_{\mu(t,x)}^{x}\gamma'_\eta(t,\eta)\mathrm{d}\eta + \int_{\tau(t,x)}^{t}\int_{\mu(t,x)}^{x}\gamma''_{s\eta}(s,\eta)\mathrm{d}\eta\mathrm{d}s\\
&\leq \gamma(t,x) - \int_{\tau(t,x)}^{t}\gamma'_s(s,x)\mathrm{d}s - \int_{\mu(t,x)}^{x}\gamma'_\eta(t,\eta)\mathrm{d}\eta
\end{aligned}$$

$$\gamma''_{tx}(t,x) = -\frac{j^2_{-\alpha}}{4(b-a)(d-c)}\,\gamma(t,x) + \frac{\beta}{x-c}\,\gamma'_t(t,x) + \frac{\beta}{t-a}\,\gamma'_x(t,x)$$

$$\gamma''_{tx}(t,x) \le p(t,x)\gamma\big(\tau(t,x),\mu(t,x)\big)$$

$$\begin{aligned}
\gamma(t,x) > 0 &\quad \text{for every } (t,x) \in \mathbb{D},\, (t,x) \neq (b,d),\\
\gamma_t(t,x) \le 0 &\quad \text{for every } (t,x) \in \,]a,b] \times [c,d],\\
\gamma_x(t,x) \le 0 &\quad \text{for every } (t,x) \in [a,b] \times \,]c,d],\\
\gamma''_{tx}(t,x) \le 0 &\quad \text{for every } (t,x) \in \,]a,b[\times \,]c,d[
\end{aligned}$$

$$\begin{aligned}
\gamma\big(\tau(t,x),\mu(t,x)\big) &= \gamma(t,x) - \int_{\tau(t,x)}^{t} \gamma'_s(s,x)\mathrm{d}s - \int_{\mu(t,x)}^{x} \gamma'_\eta(t,\eta)\mathrm{d}\eta + \int_{\tau(t,x)}^{t}\int_{\mu(t,x)}^{x} \gamma''_{s\eta}(s,\eta)\mathrm{d}\eta\mathrm{d}s\\
&\le \gamma(t,x) - \int_{\tau(t,x)}^{t} \gamma'_s(s,x)\mathrm{d}s - \int_{\mu(t,x)}^{x} \gamma'_\eta(t,\eta)\mathrm{d}\eta\\
&= \gamma(t,x) - \varphi(t,x)\,\gamma'_t(t,x) \int_{\tau(t,x)}^{t} \frac{\mathrm{d}s}{\varphi(s,x)} - \psi(t,x)\,\gamma'_x(t,x) \int_{\mu(t,x)}^{x} \frac{\mathrm{d}\eta}{\psi(t,\eta)}
\end{aligned}$$

$$\gamma''_{tx}(t,x) \le p(t,x)\gamma\big(\tau(t,x),\mu(t,x)\big)$$

$$\Uparrow$$

$$|p(t,x)| \le \frac{j_{-\alpha}^{2}}{4(b-a)(d-c)}$$

$$|p(t,x)|\,\varphi(t,x)\int_{\tau(t,x)}^{t}\frac{\mathrm{d}s}{\varphi(s,x)} \le \frac{\beta}{x-c}$$

$$|p(t,x)|\,\psi(t,x)\int_{\mu(t,x)}^{x}\frac{\mathrm{d}\eta}{\psi(t,\eta)} \le \frac{\alpha-\beta}{t-a}$$

$$\varphi(t,x) := \frac{(t-a)^{1-\alpha}}{\big(z[t,x]\big)^{1-\alpha}J_{1-\alpha}\big(z[t,x]\big)}, \qquad \psi(t,x) := \frac{(x-c)^{1-\alpha}}{\big(z[t,x]\big)^{1-\alpha}J_{1-\alpha}\big(z[t,x]\big)}$$

$$\gamma''_{tx}(t,x) \le p(t,x)\gamma\big(\tau(t,x),\mu(t,x)\big)$$

$$\Uparrow$$

$$|p(t,x)| \le \frac{j_{-\alpha}^2}{4(b-a)(d-c)}$$

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$$\varphi(t,x) := \frac{(t-a)^{1-\alpha}}{\big(z[t,x]\big)^{1-\alpha}J_{1-\alpha}\big(z[t,x]\big)}, \qquad \psi(t,x) := \frac{(x-c)^{1-\alpha}}{\big(z[t,x]\big)^{1-\alpha}J_{1-\alpha}\big(z[t,x]\big)}$$

$$\int_{\tau(t,x)}^{t}\frac{\mathrm{d}s}{\varphi(s,x)} = \frac{const.}{(x-c)^{\alpha}}\Big(E_{-\alpha}\big(z[\tau(t,x),x]\big) - E_{-\alpha}\big(z[t,x]\big)\Big)$$