Nonlocal problems for the generalized Bagley-Torvik fractional differential equation

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1. INTRODUCTION

In modelling the motion of a rigid plate immersing in a Newtonian fluid, Torvik and Bagley (1984) considered the fractional differential equation

$$u''(t) + AD^{\frac{3}{2}}u(t) = au(t) + \varphi(t), \quad A, a \in \mathbb{R}, \ A \neq 0, \tag{1}$$

subject to the initial homogeneous conditions

$$u(0) = 0, \quad u'(0) = 0, \tag{2}$$

where

$$D^{\frac{3}{2}}u(t) = \frac{1}{\Gamma(\frac{1}{2})}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_0^t (t-s)^{-\frac{1}{2}}u(s)\,\mathrm{d}s$$

is the Riemann-Liouville fractional derivative of order $\frac{3}{2}$.

In the literature equation (1) is called the Bagley-Torvik equation.

Numerical solution of problem (1), (2) was given by Podlubny (1999), analytical solutions by Kilbas, Srivastava, Trujillo (2006), Ray, Bera (2005).

Numerical solutions of the problem

$$u''(t) + A{}^cD^\alpha u(t) = au(t) + \varphi(t),$$

$$u(0) = y_0,\ \ u'(0) = y_1,$$

$${}^cD^\alpha u(t) = \frac{1}{\Gamma(2-\alpha)} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_0^t (t-s)^{1-\alpha} (u(s) - u(0) - u'(0)s)\,\mathrm{d}s, \quad \alpha \in (1,2),$$

(Caputo fractional derivative of order α)

were discussed for $\alpha = \frac{3}{2}$ by Cenesiz, Keskin, Kurnaz (2010) and by Diethelm, Ford (2002), and by Edwards, Ford, Simpson (2002) for $\alpha \in (1,2)$. Applying the Adomian decomposition method, analytical solutions of the above problem were obtained by Deftardar-Gejji, Jafari (2005) for $\alpha \in (1,2)$.

Analytical and numerical solutions of the boundary value problem

$$u''(t) + A{}^cD^{\frac{3}{2}} u(t) = au(t) + \varphi(t),$$

$$u(0) = y_0, \quad y(T) = y_1.$$

were discussed by Al-Mdallal, Syam, Anwar (2010).

Wang, Wang (2010) investigated general solutions of the equations $u''(t) + A{}^cD^{\frac{3}{2}} u(t) + u(t) = 0,\ u''(t) + AD^{\frac{3}{2}} u(t) + u(t) = 0.$

Existence and uniqueness results for the generalized Bagley-Torvik fractional differential equation

$$u''(t) + A{}^{c}D^{\alpha}u(t) = f(t, u(t), {}^{c}D^{\mu}u(t), u'(t)), \quad A \in \mathbb{R} \setminus \{0\},$$

subject to the boundary conditions

$$u'(0) = 0, \;\; u(T) + au'(T) = 0, \quad a \in \mathbb{R},$$

where $\alpha \in (1,2)$, $\mu \in (0,1)$, $f \in Car([0,T] \times \mathbb{R}^3)$ were given by S.S. (2012).

Note that

$${}^{c}D^{\alpha}u(t) = \frac{1}{\Gamma(2-\alpha)} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_0^t (t-s)^{1-\alpha}(u(s) - u(0) - u'(0)s)\,\mathrm{d}s, \quad \alpha \in (1,2),$$

$${}^{c}D^{\mu}u(t) = \frac{1}{\Gamma(1-\mu)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-s)^{-\mu}(u(s) - u(0))\,\mathrm{d}s, \quad \mu \in (0,1).$$

The Riemann-Liouville fractional integral $I^\gamma v$ of $v : [0, T] \to \mathbb{R}$ of order $\gamma > 0$ is defined as

$$I^\gamma v(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} v(s)\,\mathrm{d}s$$

Properties of fractional integral:

- $I^\gamma : C[0,T] \to C[0,T]$, $I^\gamma : L^1[0,T] \to L^1[0,T]$, $\gamma \in (0,1)$,
- $I^\gamma : AC[0,1] \to AC[0,1]$, $\gamma \in (0,1)$,
- $I^\gamma : L^1[0,T] \to AC[0,T]$, $\gamma \in [1,2)$,
- $I^\beta I^\gamma v(t) = I^{\beta+\gamma} v(t)$ for $t \in [0,T]$, where $v \in L^1[0,T]$, $\beta, \gamma > 0$, $\beta + \gamma \geq 1$ (semigroup property)
- $\frac{\mathrm{d}}{\mathrm{d}t} I^{\gamma+1} v(t) = I^\gamma v(t)$ for a.e. $t \in [0,T]$, where $v \in L^1[0,T]$ and $\gamma > 0$.

LEMMA 1. *Let* $w \in C[0,1]$, $b \in C^1[0,T]$, $\alpha \in (1,2)$ *and let* $\varphi_1 \in AC[0,1]$ *be such that* $\varphi_1(0) = 0$. *Suppose that*

$$w(t) = b(t) I^{2-\alpha} w(t) + \varphi_1(t) \quad \text{for } t \in [0,1].$$

Then for each $n \in \mathbb{N}$ *there exists* $\varphi_n \in AC[0,1]$ *such that* $\varphi_n(0) = 0$ *and the equality*

$$w(t) = b^n(t) I^{n(2-\alpha)} w(t) + \varphi_n(t) \quad \text{for } t \in [0,1]$$

holds.

COROLLARY. *Let the assumptions of Lemma 1 hold. Then* $w \in AC[0,1]$.

Proof. Choose $n \in \mathbb{N}$ such that $n(2-\alpha) > 2$. Then $I^{n(2-\alpha)} w = I^1 I^{n(2-\alpha)-1} w \in C^1[0,1]$. Since $w(t) = \underbrace{b^n(t) I^{n(2-\alpha)} w(t)}_{C^1[0,T]} + \underbrace{\varphi_n(t)}_{AC[0,T]}$ for $t \in [0,1]$, we have $w \in AC[0,1]$.

The following result is a generalization of the Gronwall lemma for integrals with singular kernels (Henry (1989)).

LEMMA 2. *Let* $0 < \gamma < 1$, $b \in L^1[0, T]$ *be nonnegative and let* K *be a positive constant. Suppose* $w \in L^1[0, T]$ *is nonnegative and*

$$w(t) \leq b(t) + K \int_0^t (t-s)^{\gamma-1} w(s)\,\mathrm{d}s \quad \textit{for a.e. } t \in [0, T].$$

Then

$$w(t) \leq b(t) + LK \int_0^t (t-s)^{\gamma-1} b(s)\,\mathrm{d}s \quad \textit{for a.e. } t \in [0, T],$$

where $L = L(\gamma)$ *is a positive constant.*

$L = K\Gamma(\gamma) E_{\gamma\gamma}(K\Gamma(\gamma) \max\{1, T\})$,

$E_{\beta\gamma}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n\beta + \gamma)}$ Mittag-Leffler function

Fractional derivatives

The Caputo fractional derivative ${}^cD^\beta v$ of order $\beta > 0$, $\beta \notin \mathbb{N}$, of $x : [0, T] \to \mathbb{R}$ is defined by

$$ {}^cD^\beta x(t) = \frac{1}{\Gamma(n-\beta)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t (t-s)^{n-\beta-1} \left(x(s) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} s^k \right) \mathrm{d}s, $$

where $n = [\beta] + 1$ and where $[\beta]$ means the integral part of β.

The Riemann-Liouville fractional derivative $D^\beta v$ of $v : [0, T] \to \mathbb{R}$ of order $\beta > 0$ is given by

$$ D^\beta x(t) = \frac{1}{\Gamma(n-\beta)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t (t-s)^{n-\beta-1} x(s)\, \mathrm{d}s \ \left(= \frac{\mathrm{d}^n}{\mathrm{d}t^n} I^{n-\beta} x(t) \right), $$

where $n = [\beta] + 1$.

Let $\mathcal{A}$ be the set of functionals $\phi : C[0,1] \to \mathbb{R}$ which are

(i) continuous,

(ii) $\lim_{c\in\mathbb{R},\, c\to\pm\infty} \phi(c) = \pm\infty$,

(We identify the set of constant functions on $[0,1]$ with $\mathbb{R}$)

(iii) there exists a positive constant $L = L(\phi)$ such that

$$x \in C[0,1],\ |x(t)| > L \text{ for } t \in [0,1] \Rightarrow \phi(x) \neq 0.$$

$(\phi \in \mathcal{A},\ \phi(x) = 0 \text{ for some } x \in C[0,1] \Rightarrow \exists \xi \in [0,1] : |x(\xi)| \leq L)$

EXAMPLE. Let $p, g_j \in C(\mathbb{R})$, p be bounded, $\lim_{v\to\pm\infty} g_j(v) = \pm\infty$, $j = 0,1,\ldots,n$, and let $0 \leq t_1 < t_2 < \cdots < t_n \leq 1$. Then the functionals

$$\phi_1(x) = g_0\left(\max_{t\in[0,1]} x(t)\right), \quad \phi_2(x) = g_0\left(\min_{t\in[0,1]} x(t)\right),$$

$$\phi_3(x) = p(\|x\|) + \int_0^1 g_0(x(t))\,\mathrm{d}t, \quad \phi_4(x) = \sum_{j=1}^{n} g_j(x(t_j))$$

belong to the set $\mathcal{A}$.

LEMMA 3. *Let* $\phi \in \mathcal{A}$. *Then there exists a positive constant* $L = L(\phi)$ *such that the estimate* $|c| < L$ *holds for each* $\lambda > 0$ *and each solution* $c \in \mathbb{R}$ *of the equation*

$$\lambda\phi(c) - \phi(-c) = 0.$$

We investigate the Bagley-Torvik fractional functional differential equation

$$u''(t) + a(t)\,{}^cD^\alpha u(t) = (Fu)(t) \tag{3}$$

together with the nonlocal boundary conditions

$$u'(0) = 0, \quad \phi(u) = 0, \quad (\phi \in \mathcal{A}). \tag{4}$$

Here $\alpha \in (1,2)$, ${}^cD$ is the Caputo fractional derivative, $a \in C^1[0,1]$ and $F : C^1[0,1] \to L^1[0,1]$ is continuous.

Note that

$$ {}^cD^\alpha u(t) = \frac{1}{\Gamma(2-\alpha)} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_0^t (t-s)^{1-\alpha} (u(s) - u(0) - u'(0)s)\,\mathrm{d}s, \quad \alpha \in (1,2)$$

We say that a function $u \in AC^1[0,1]$ is *a solution of problem* (3), (4) if u satisfies (4) and (3) holds for a.e. $t \in [0,1]$.

We work with the following conditions on the function a and the operator F in (3).

(H_1) $a \in C^1[0,1]$ and $a(t) \neq 0$ for $t \in [0,1]$,

(H_2) $F : C^1[0,1] \to L^1[0,1]$ is continuous and for a.e. $t \in [0,1]$ and all $x \in C^1[0,1]$, the estimate

$$|(Fx)(t)| \leq \varphi(t)\omega(\|x\| + \|x'\|)$$

holds, where $\varphi \in L^1[0,1]$ and $\omega \in C[0,\infty)$ are nonnegative, ω is nondecreasing and

$$\lim_{v\to\infty} \frac{\omega(v)}{v} = 0.$$

4. OPERATORS

In order to prove the solvability of problem (3), (4), we define an operator $\mathcal{F}$ acting on $[0,1] \times C^1[0,1] \times \mathbb{R}$ by the formula

$$\mathcal{F}(\lambda, x, c) = \left(\mathcal{F}_1(\lambda, x, c),\ \mathcal{F}_2(x, c)\right),$$

where

$$\begin{aligned} \mathcal{F}_1(\lambda, x, c)(t) &= c + \int_0^t (\mathcal{Q}x)(s)\,\mathrm{d}s + \lambda \int_0^t (t-s)(Fx)(s)\,\mathrm{d}s, \\ \mathcal{F}_2(x, c) &= c - \phi(x), \end{aligned}$$

and

$$(\mathcal{Q}x)(t) = -a(t) I^{2-\alpha} x'(t) + \int_0^t a'(s) I^{2-\alpha} x'(s)\,\mathrm{d}s. \tag{5}$$

Here the function a and the operator F are from equation (3) and $\phi \in \mathcal{A}$ is from the boundary conditions (4)

Properties of $\mathcal{Q}$, $\mathcal{F}_1$ and $\mathcal{F}_2$

- Let (H_1) hold. Then $\mathcal{Q} : C^1[0,1] \to C[0,1]$ and $\mathcal{Q}$ is completely continuous.
- Let (H_1) and (H_2) hold. Then $\mathcal{F}_1 : [0,1] \times C^1[0,1] \times \mathbb{R} \to C^1[0,1]$ and $\mathcal{F}_1$ is completely continuous.
- Let $\phi \in \mathcal{A}$. Then $\mathcal{F}_2 : C^1[0,1] \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{F}_2$ is completely continuous.

$$\begin{aligned}
\mathcal{F}_1(\lambda, x, c)(t) &= c + \int_0^t (\mathcal{Q}x)(s)\,\mathrm{d}s + \lambda \int_0^t (t-s)(Fx)(s)\,\mathrm{d}s, \\
\mathcal{F}_2(x, c) &= c - \phi(x), \\
(\mathcal{Q}x)(t) &= -a(t)I^{2-\alpha}x'(t) + \int_0^t a'(s)I^{2-\alpha}x'(s)\,\mathrm{d}s.
\end{aligned}$$

LEMMA 4. *Let* (H_1) *and* (H_2) *hold. Then*

(a) $\mathcal{F} : [0,1] \times C^1[0,1] \times \mathbb{R} \to C^1[0,1] \times \mathbb{R}$ *and* $\mathcal{F}$ *is completely continuous,*

(b) *if* (x,c) *is a fixed point of* $\mathcal{F}(1,\cdot,\cdot)$*, then* x *is a solution of problem* (3), (4) *and* $c = x(0)$.

Proof.

(b) Let (x,c) be a fixed point of $\mathcal{F}(1,\cdot,\cdot)$. Then $x \in C^1[0,1]$,

$$x(t) = c + \int_0^t (\mathcal{Q}x)(s)\,\mathrm{d}s + \int_0^t (t-s)(Fx)(s)\,\mathrm{d}s, \quad t \in [0,1], \tag{6}$$

and $\phi(x) = 0$. Differentiating (6) gives

$$x'(t) = -a(t)I^{2-\alpha}x'(t) + \int_0^t a'(s)I^{2-\alpha}x'(s)\,\mathrm{d}s + \int_0^t (Fx)(s)\,\mathrm{d}s, \quad t \in [0,1]. \tag{7}$$

Therefore, $x'(0) = 0$, and so x satisfies the boundary conditions (4). Since $\int_0^t a'(s)I^{2-\alpha}x'(s)\,\mathrm{d}s \in C^1[0,1]$ and $\int_0^t (Fx)(s)\,\mathrm{d}s \in AC[0,1]$, (7) shows that

$$x'(t) = -a(t)I^{2-\alpha}x'(t) + \psi(t), \quad t \in [0,1],$$

where $\psi \in AC[0,1]$ and $\psi(0) = 0$. Hence, by Corollary, $x' \in AC[0,1]$. It follows from the R.-L. factional integrals that $I^{2-\alpha}x' \in AC[0,1]$.

Next we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{a(t)}\left(-x'(t)+\int_0^t a'(s)I^{2-\alpha}x'(s)\,\mathrm{d}s+\int_0^t (Fx)(s)\,\mathrm{d}s\right)\right]$$
$$=\frac{(Fx)(t)-x''(t)}{a(t)}\in L^1[0,1]\ \text{ for a.e. } t\in[0,1].$$

Since, by (7), the equality

$$I^{2-\alpha}x'(t)=\frac{1}{a(t)}\left(-x'(t)+\int_0^t a'(s)I^{2-\alpha}x'(s)\,\mathrm{d}s+\int_0^t (Fx)(s)\,\mathrm{d}s\right)$$

holds for $t\in[0,1]$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}I^{2-\alpha}x'(t)=\frac{(Fx)(t)-x''(t)}{a(t)}\ \text{ for a.e. } t\in[0,1].$$

Consequently,

$$x''(t)+\underbrace{a(t)\frac{\mathrm{d}}{\mathrm{d}t}I^{2-\alpha}x'(t)}_{{}^cD^\alpha x(t)}=(Fx)(t)\ \text{ for a.e. } t\in[0,1].$$

Since $I^{2-\alpha}x'(t) = I^{3-\alpha}x''(t) = I^1 I^{2-\alpha}x''(t)$, we have $\frac{\mathrm{d}}{\mathrm{d}t} I^{2-\alpha}x'(t) = I^{2-\alpha}x''(t)$ a.e. on $[0,1]$. Since $x'' \in L^1[0,1]$, it follows that ${}^c\!D^\alpha x(t) = I^{2-\alpha}x''(t)$ for a.e. $t \in [0,1]$. Hence $\frac{\mathrm{d}}{\mathrm{d}t} I^{2-\alpha}x'(t) = {}^c\!D^\alpha x(t)$ a.e. on $[0,1]$, and therefore x is a solution of (3). As a result x is a solution of problem (3), (4), and (6) gives $c = x(0)$.

LEMMA 5. *Let* (H_1) *and* (H_2) *hold. Then there exists a positive constant* S *such that for each* $\lambda \in [0,1]$ *and each fixed point* (x, c) *of the operator* $\mathcal{F}(\lambda, \cdot, \cdot)$ *the estimate*

$$\|x\| < S, \quad \|x'\| < S, \quad |c| < S$$

holds.

We need the following result (Deimling (1985)).

LEMMA 6. *Let X be a Banach space and let $\Omega \subset X$ be open bounded and symmetric with respect to $0 \in \Omega$. Let $\mathcal{F} : \overline{\Omega} \to X$ be a compact operator and $\mathcal{G} = \mathcal{I} - \mathcal{F}$, where $\mathcal{I}$ is the identical operator on X. If $x \neq \mathcal{F}x$ for $x \in \partial\Omega$ and $\mathcal{G}(-x) \neq \lambda\mathcal{G}(x)$ on $\partial\Omega$ for all $\lambda \geq 1$, then $\deg(\mathcal{I} - \mathcal{F}, \Omega, 0) \neq 0$.*

THEOREM 1. *Let* (H_1) *and* (H_2) *hold. Then problem* (3), (4) *has at least one solution.*

Proof. We have to show that $\mathcal{F}(1,\cdot,\cdot)$ has a fixed point (x,c). Then x is a solution of problem $(3),(4)$ and $c = x(0)$. Let S be a positive constant from Lemma 5 and let $L = L(\phi)$ be from Lemma 3 (note that $|c| < L$ holds for each $\lambda > 0$ and each solution $c \in \mathbb{R}$ of $\lambda\phi(c) - \phi(-c) = 0$). Let $W = \max\{S, L\}$ and

$$\Omega = \{(x,c) \in C^1[0,1] \times \mathbb{R} : \|x\| < W,\ \|x'\| < W,\ |c| < W\}.$$

We prove by Lemma 6 that $\deg\{\mathcal{I} - \mathcal{F}(0,\cdot,\cdot), \Omega, 0\} \neq 0$, where $\mathcal{I}$ is the identical operator on $C^1[0,1] \times \mathbb{R}$. Note that

$$\mathcal{G}(x,c) = (x,c) - \mathcal{F}(0,x,c) = \left(x(t) - c - \int_0^t (\mathcal{Q}x)(s)\,\mathrm{d}s,\ \phi(x) \right).$$

Let (x,c) be a fixed point of $\mathcal{F}(\lambda,\cdot,\cdot)$ for some $\lambda \in [0,1]$. Then, by Lemma 5, $(x,c) \notin \partial\Omega$, and therefore, by the homotopy property, $\deg(\mathcal{I} - \mathcal{F}(1,\cdot,\cdot), \Omega, 0) = \deg(\mathcal{I} - \mathcal{F}(0,\cdot,\cdot), \Omega, 0)$. Hence $\deg(\mathcal{I} - \mathcal{F}(1,\cdot,\cdot), \Omega, 0) \neq 0$. The last relation implies that $\mathcal{F}(1,\cdot,\cdot), \Omega, 0)$ has a fixed point.

EXAMPLE. Let $\varphi_1, \varphi_2 \in L^1[0,1]$, $h \in C[0,\infty)$, $p \in C(\mathbb{R})$, $\lim_{v\to\infty} \frac{h(v)}{v} = 0$ and $\lim_{|v|\to\infty} \frac{p(v)}{v} = 0$. Define an operator $F : C^1[0,1] \to L^1[0,1]$ by

$$(Fx)(t) = \varphi_1(t) \left(h(\|x'\|) + \int_0^t p(x(s))\,\mathrm{d}s \right) + \varphi_2(t).$$

Then F satisfies condition (H_2). To check it we take $\varphi(t) = |\varphi_1(t)| + |\varphi_2(t)|$ and $\omega(v) = \tilde{h}(v) + \tilde{p}(v)$, where $\tilde{h}(v) = \max\{h(\nu) : 0 \le \nu \le v\}$, $\tilde{p}(v) = \max\{p(\nu) : |\nu| \le v\}$, $v \in [0,\infty)$.

The special case of (3) is the fractional differential equation

$$u''(t) + a(t)\,{}^{c}\!D^{\alpha} u(t) = f(t, u(t), {}^{c}\!D^{\gamma} u(t), u'(t)), \tag{8}$$

where $\alpha \in (1, 2)$, $\gamma \in (0, 1)$ and f satisfies the condition

(H_3) $f \in Car([0,1] \times \mathbb{R}^3)$ and for a.e. $t \in [0,1]$ and all $(x, y, z) \in \mathbb{R}^3$ the estimate

$$|f(t, x, y, z)| \leq \varphi(t)\rho(|x| + |y| + |z|)$$

holds, where $\varphi \in L^1[0,1]$ and $\rho \in C[0,\infty)$ are nonegative, ρ is nondecreasing and $\lim_{v\to\infty} \frac{\rho(v)}{v} = 0$.

The following theorem gives an existence result for problem (8), (4).

THEOREM 2. *Let* (H_1) *and* (H_3) *hold. Then problem* (8), (4) *has at least one solution.*
Proof. Let F be an operator acting on $C^1[0,1]$ and given by

$$(Fx)(t) = f(t, x(t), {}^cD^\gamma x(t), x'(t)).$$

F satisfies condition (H_2) for $\omega(v) = \rho\left(\frac{2v}{\Gamma(2-\gamma)}\right)$. The solvability of problem (8), (4) follows from Theorem 1.

Let $\mathcal{B}$ be the set all functionals $\phi : C[0,1] \to \mathbb{R}$ which are

(i) continuous,

(ii) increasing, that is,

$$x, y \in C[0,1] \quad x(t) < y(t) \text{ for } t \in [0,1] \Rightarrow \phi(x) < \phi(y).$$

EXAMPLE. Let $g_j \in C(\mathbb{R})$ be increasing ($j = 0, 1, \ldots, n$), and let $0 \leq t_0 \leq t_1 < \cdots < t_n \leq 1$. Then the functionals

$$\phi_1(x) = g_0\left(\max_{t\in[0,1]} x(t)\right), \quad \phi_2(x) = g_0\left(\min_{t\in[0,1]} x(t)\right),$$

$$\phi_3(x) = \int_0^1 g_0(x(t))\,\mathrm{d}t, \quad \phi_4(x) = \sum_{j=1}^{n} g_j(x(t_j))$$

belong to $\mathcal{B}$.

We discuss equation (8), where $f(t,x,y,z)=\varphi(t)p(t,x,y,z)$, that is, the equation

$$u''(t)+a(t)^{c}D^{\alpha}u(t)=\varphi(t)p(t,u(t),{}^{c}D^{\gamma}u(t),u'(t)), \tag{9}$$

where $\alpha\in(1,2)$, $\gamma\in(0,1)$. Together with (9) the boundary conditions

$$u'(0)=0, \quad \phi(u)=0, \quad (\phi\in\mathcal{B}) \tag{10}$$

and

$$u'(0)=0, \quad \phi(u)=0, \quad (\phi\in\mathcal{A}\cap\mathcal{B}) \tag{11}$$

equation are investigated.

THEOREM 3. *Let*

(S_1) $a \in C^1[0,1]$, $\varphi \in L^1[0,1]$ *are such that* $a < 0$, $a' \geq 0$ *on* $[0,1]$ *and* $\varphi > 0$ *a.e. on* $[0,1]$,

(S_2) $p \in C([0,1] \times \mathbb{R}^3)$ *and* $p(t,x,y,z)$ *is increasing in the variable* x *and nondecreasing in the variables* y *and* z,

(S_3) *the exists* $\kappa > 0$ *such that for each* $\rho \in \mathbb{R}$ *the estimate*

$$|p(t, \rho + x_1, y_1, z_1) - p(t, \rho + x_2, y_2, z_2)| \leq k_\rho(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|)$$

holds for $x_j, y_j, z_j \in [-\kappa, \kappa]$, *where* $k_\rho \in C[0, 6\kappa]$, k_ρ *is nondecreasing and*

$$\limsup_{v \to 0^+} \frac{k_\rho(v)}{v} < \infty,$$

hold. Then problem (9), (10) *has at most one solution.*

EXAMPLE. Let $q_1 \in C^1(\mathbb{R})$, $q_2, q_3 \in C(\mathbb{R}) \cap C^1[-1,1]$, q_1 be increasing and q_2, q_3 be nondecreasing. Let $p_j \in C([0,1] \times \mathbb{R}^2)$ $(j = 1,2,3)$ be positive and bounded. Then the function

$$p(t,x,y,z) = p_1(t,y,z)q_1(x) + p_2(t,x,z)q_2(y) + p_3(t,x,y)q_3(z)$$

satisfies conditions (S_2) and (S_3) with $\kappa = 1$.

THEOREM 4. Let $(S_1)-(S_3)$ and

(S_4) for $t \in [0,1]$ and $(x,y,z) \in \mathbb{R}^3$ the estimate

$$|p(t,x,y,z)| \le h(|x|+|y|+|z|)$$

is fulfilled, where $h \in C[0,\infty)$, h is nondecreasing and

$$\lim_{v\to\infty} \frac{h(v)}{v} = 0.$$

hold. Then problem (9), (11) has a unique solution.

EXAMPLE. Let $q_1 \in C^1(\mathbb{R})$, $q_2, q_3 \in C(\mathbb{R}) \cap C^1[-1,1]$, q_1 be increasing and q_2, q_3 be nondecreasing. Let $p_j \in C([0,1]\times\mathbb{R}^2)$ $(j=1,2,3)$ be positive and bounded. Besides, $\lim_{v\to\infty} \frac{1}{v} \max\{|q_j(-v)|, |q_j(v)|\} = 0$ for $j=1,2,3$. Then the function

$$p(t,x,y,z) = p_1(t,y,z)q_1(x) + p_2(t,x,z)q_2(y) + p_3(t,x,y)q_3(z)$$

satisfies conditions $(S_2)-(S_4)$.

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