Existence of periodic solutions to a certain boundary value problem arising in hydrodynamics

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Dedicated to Svaťa Staněk and Franta Neuman
Malá Morávka and Brno, 2012

Valveless pumping

- assists in fluid transport in various biomedical and engineering systems
- no valves are present to regulate the flow direction
- fluid pumping efficiency of a valveless system is not necessarily lower than that having valves
- many fluid-dynamical systems in nature and engineering more or less rely upon valveless pumping to transport the working fluids therein
- blood circulation in the cardiovascular system is maintained to some extent even when the heart's valves fail
- the embryonic vertebrate heart begins pumping blood long before the development of discernable chambers and valves
- in microfluidics, valveless impedance pump have been fabricated, and are expected to be particularly suitable for handling sensitive biofluids.

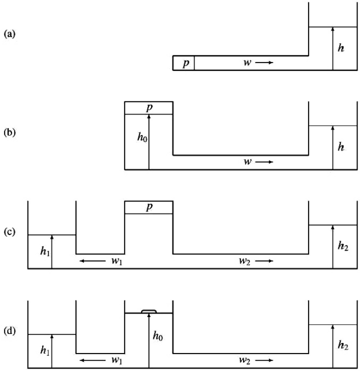
Flow configurations with rigid pipes and tanks

cross sections of the pipes are small
in comparison to cross sections of tanks

w, w_1, w_2 are flow velocities

(a)–(c): pressure p outside a massless and frictionless piston is forced

(d): level height h_0 in the middle tank is forced



Valveless pumping (1 tank - 1 pipe model)

G. Propst: Pumping effects in models of periodically forced flow configurations. *Physica D* **217** (2006), 193–201.

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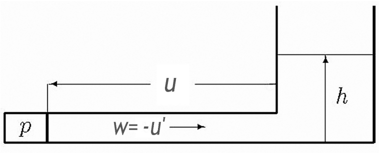
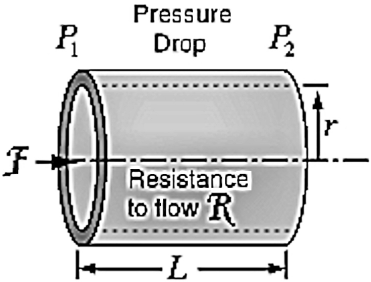


$$A_P\, u(t) + A_T\, h(t) \equiv V_0 \quad \Longrightarrow \quad h(t) \equiv \frac{1}{A_T}\big(V_0 - A_P\, u(t)\big)\,.$$

Momentum balance with Poiseuille's law and Bernoulli's equation

Poiseuille's Law



In the case of smooth flow of uniform liquids (Newtonian fluids) without turbulences,

the volume flowrate w is given by the pressure difference $P_1 - P_2$ divided by the viscous resistance R.

$$R = \frac{8\,\eta\,L}{\pi\,r^4},$$

η=viscosity, r=radius, L=length

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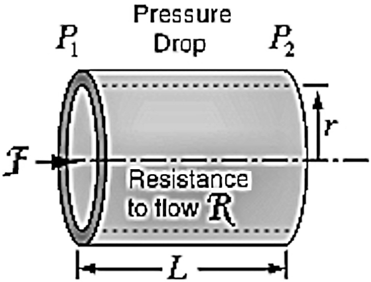

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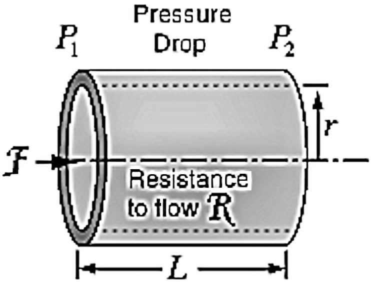

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$$w = \frac{\pi\,(P_1 - P_2)\,r^4}{8\,\eta\,L} \qquad (w = \text{volume flowrate})$$

$$\text{tube friction} = P_2 - P_1 = -r_0\,w\,L \qquad (r_0 = \text{friction coefficient})$$

Bernoulli's Equation

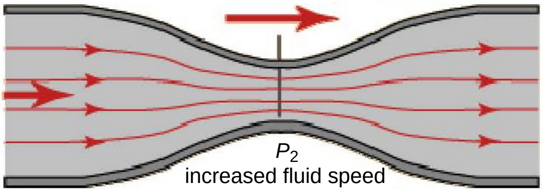
is a statement of the conservation of energy principle appropriate for flowing fluids.

The lowering of pressure in a constriction of a flow path is understandable when we consider pressure to be energy density:
kinetic energy must increase at the expense of pressure energy. (Bernoulli effect).

Energy per unit volume before $=$ Energy per unit volume after

$$\underbrace{P_1}_{\text{pressure}} + \underbrace{\frac{1}{2}\rho w_1^2}_{\text{kinetic}} + \underbrace{\rho g h_1}_{\text{potential}} = P_2 + \frac{1}{2}\rho w_2^2 + \rho g h_2$$

Flow velocity w_1 　 Flow velocity w_2



$w_2 > w_1$

$P_2 < P_1$

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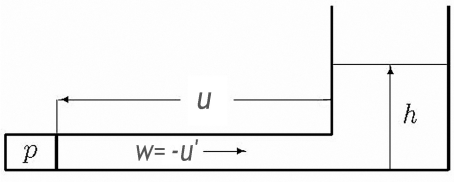
$$P_1 + \frac{1}{2}\rho w_1^2 = P_2 + \rho g h_2$$

$$P_2 - P_1 = \frac{1}{2}\rho w_1^2 - \rho g h_2$$

Flow velocity w_1 | Flow velocity 0



$h_1 = 0$
$w_2 = 0$

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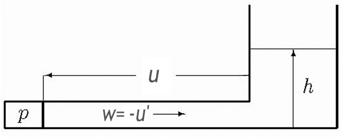

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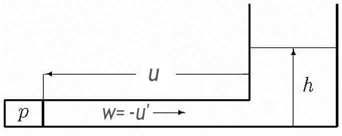

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(time derivative of the momentum of the mass of water in the pipe between piston and the tank equals the sum of the forces acting on it:
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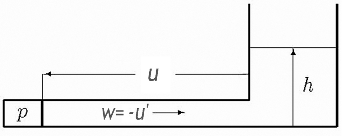

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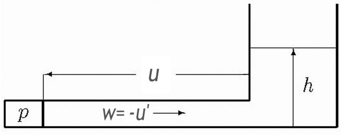

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i.e.

$$u\,u'' + a\,u\,u' + b\,(u')^2 + s\,u = e(t),$$

where

$$T > 0, \quad a = \frac{r_0}{\rho} > 0, \quad b = \big(1 + \frac{\zeta}{2}\big) \geq 3/2,$$

$$e(t) = \frac{g\,V_0}{A_T} - \frac{p(t)}{\rho} \text{ is } T-\text{periodic}, \quad 0 < s = \frac{g\,A_p}{A_T} < 1.$$

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(*) has a positive solution only if $\overline{e} \ge 0$ (i.e. $\overline{p} \le \rho\, g\, \frac{V_0}{A_T}$).

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CLAIM

Assume: $a > 0$, $e \in L_\infty[0,T]$, inf ess $e > 0$, $b \geq 1$, $0 < s < 1$.

Then: (*) has a positive solution for a sufficiently large provided that for an arbitrary interval $[r, R] \subset (0, \infty)$, the number of T-periodic solutions between r and R is finite.

Proof: Step 1

$$(*) \quad u'' + a\,u' = \frac{1}{u}\left(e(t) - b\,(u')^2\right) - s\,, \quad u(0) = u(T),\ u'(0) = u'(T),$$

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$$(\mathrm{P}) \qquad x'' + a x' + q x = f(t,x), \quad x(0) = x(T),\; x'(0) = x'(T)\,,$$

where $\quad f(t,x) = q\,x + \dfrac{1}{\mu}\left(e(t)\, x^{1-2\mu} - s\, x^{1-\mu}\right), \quad q > 0$ arbitrary.

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Lemma 1

u is a [positive solution of (*) iff $x(t)=(u(t))^{1/\mu}$ is a positive solution of (P).

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Notice: $1 - \mu > 1 - 2\,\mu \geq 0$!!!

Proof: Step 2

$$p \in \mathbb{R},\ q > 0, \quad C_T = \{y \in C^2[0,T] : y(0) = y(T),\ y'(0) = y'(T)\},$$

$$L : y \in C_T \subset C[0,T] \to y'' + p\,y' + q\,y \in C[0,T]$$

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Let $p, q \in \mathbb{R}$, $0 < q \le \left(\frac{\pi}{T}\right)^2 + \left(\frac{p}{2}\right)^2$. Then the operator L is *inverse nonnegative*,

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Let $p = a$, $0 < q \leq \left(\frac{\pi}{T}\right)^2 + \left(\frac{a}{2}\right)^2$.

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$$L^{-1} : C[0,T] \to C_T \subset C[0,T], \quad \text{i.e.} \quad L^{-1}f \ge 0 \text{ on } [0,T] \quad \text{for } f \ge 0 \text{ on } [0,T]\,.$$

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Moreover, the operator

$$F : x \in C[0,T] \to (Fx)(t) = L^{-1}(f(t, x(t))) \in C[0,T]$$

is completely continuous and x is a solution to (P) iff it is a solution to $Fx = x$.

Proof: Step 2

$$p \in \mathbb{R},\ q > 0, \quad C_T = \{y \in C^2[0,T] : y(0) = y(T),\ y'(0) = y'(T)\},$$
$$L : y \in C_T \subset C[0,T] \to y'' + py' + qy \in C[0,T]$$

LEMMA 2 [Omari & Trombetta, 1992]

Let $p, q \in \mathbb{R}$, $0 < q \leq \left(\frac{\pi}{T}\right)^2 + \left(\frac{p}{2}\right)^2$. Then the operator L is *inverse nonnegative*, i.e.

$$\left.\begin{array}{c} y'' + py' + qy \geq 0 \\ y(0) = y(T),\ y'(0) = y'(T) \end{array}\right\} \Rightarrow y \geq 0\,.$$

COROLLARY

Let $p = a$, $0 < q \leq \left(\frac{\pi}{T}\right)^2 + \left(\frac{a}{2}\right)^2$. Then the operator L has a bounded nonnegative inverse

$$L^{-1} : C[0,T] \to C_T \subset C[0,T], \quad \text{i.e.} \quad L^{-1}f \geq 0 \text{ on } [0,T] \quad \text{for } f \geq 0 \text{ on } [0,T]\,.$$

Moreover, the operator

$$F : x \in C[0,T] \to (Fx)(t) = L^{-1}(f(t, x(t))) \in C[0,T]$$

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Proof: Step 3

$$\text{(P)} \qquad F(x) = x\,,$$

where

$$(Fx)(t) = L^{-1}(f(t, x(t))),$$

$$f(t,x) = qx + \frac{1}{\mu}\left(e(t)x^{1-2\mu} - sx^{1-\mu}\right), \quad q = \left(\frac{\pi}{T}\right)^2 + \left(\frac{a}{2}\right)^2.$$

Proof: Step 3

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- $\min e > 0 \Rightarrow \exists R > r > 0$ such that $f(t,R) < 0 < f(t,r)$ for all $t \in [0,T]$,

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- $\min e > 0 \Rightarrow \exists R > r > 0$ such that $f(t,R) < 0 < f(t,r)$ for all $t \in [0,T]$, i.e. $\quad F(r) < r,\ F(R) > R$.

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- $\min e > 0 \Rightarrow \exists R > r > 0$ such that $f(t,R) < 0 < f(t,r)$ for all $t \in [0,T]$, i.e. $F(r) < r,\ F(R) > R$.
- $a > 0$ sufficiently large $\Rightarrow$ operator F is nondecreasing on $[r,R]$.
- By Torres (MJM 2004) this completes the proof.

Result

$$(\mathrm{P}) \qquad x'' + ax' + q(t)x = f(t,x), \quad x(0) = x(T),\ x'(0) = x'(T),$$

where

$$f(t,x) = q(t)x + \left(r(t)x^{\alpha} - s(t)x^{\beta}\right).$$

Result

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THEOREM 2

Assume: $q, r, s \in C([0,T],[0,\infty)),\ \bar{r} > 0,\ \bar{q} > 0,$

$$q \le \left(\tfrac{\pi}{T}\right)^2 + \left(\tfrac{a}{2}\right)^2,\ 0 < \alpha < \beta.$$

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THEOREM 2

Assume: $q, r, s \in C([0,T],[0,\infty)),\ \bar{r} > 0,\ \bar{q} > 0,$

$$q \le \left(\tfrac{\pi}{T}\right)^2 + \left(\tfrac{a}{2}\right)^2,\ 0 < \alpha < \beta.$$

Then: (P) has a positive solution for a sufficiently large provided that for an arbitrary interval $[r,R] \subset (0,\infty)$, the number of T-periodic solutions between r and R is finite..

LEMMA [Ortega & Amine, 1994]

Assume: σ_1 and σ_2 is a reversely ordered pair of strict lower and an upper functions of (P).

Then: the existence of a unique asymptotically stable T–periodic solution x such that $\sigma_2 < x < \sigma_1$ is guaranteed provided there is $\gamma \in C[0,T]$ such that $\gamma \ge 0$, $\bar{\gamma} > 0$ and $\dfrac{\partial f(t,x)}{\partial x} \ge \gamma(t)$ for $t \in [0,T]$ and $x \in [\sigma_2(t), \sigma_1(t)]$.

Rresult

$$(*) \qquad u'' + \frac{r_0}{\rho} u' = h(t, u, u'), \quad u(0) = u(T), \; u'(0) = u'(T),$$

where

$$h(t, x, y) = \frac{1}{x} \Big(\big(\frac{g V_0}{A_T} - \frac{p(t)}{\rho} \big) - \big(1 + \frac{\zeta}{2} \big) y^2 \Big) - \frac{g A_p}{A_T},$$

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COROLLARY

Assume: min $p < \rho \dfrac{g\,V_0}{A_T}$.

Then: (*) has a positive solution for $\dfrac{r_0}{\rho}$ sufficiently large.

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AŤ ŽIJE SVAŤA !!!