On the open problems connected to the results of Lazer and Solimini

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Introduction and open problem

In order to simplify the presentation we will consider the following boundary value problem with singularity at spacial variable

$$u''(t) \pm \frac{1}{u^{\lambda}(t)} = h(t) \qquad \text{for a. e. } t \in [0, \omega],$$
$$u(0) = u(\omega), \qquad u'(0) = u'(\omega),$$

where $h \in L^p([0, \omega]; \mathbb{R})$, $p \geq 1$ and $\lambda > 0$.
The pioneer paper about this topic was written by A. C. Lazer and S. Solimini and published in 1987.

Introduction and open problem

They studied the following equations

$$u''(t) - \frac{1}{u^{\lambda}(t)} = h(t), \tag{1}$$

$$u''(t) + \frac{1}{u^{\lambda}(t)} = h(t). \tag{2}$$

In their original paper they proved:

1. Assume that $h \in L([0,\omega];\mathbb{R})$ and $\lambda \geq 1$. Then (1) has periodic solutions if and only if $\overline{h} := \frac{1}{\omega}\int_0^{\omega} h(t)dt < 0$.
2. In the above theorem the assumption $\lambda \geq 1$ is essential. In otherwise they construed a continuous function h such that the equation (1) has no periodic solutions.
3. Assume that $h \in C([0,\omega];\mathbb{R})$. Then (2) has periodic solutions if and only if $\overline{h} > 0$.

$$u''(t) - \frac{1}{u^\lambda(t)} = h(t), \qquad (1)$$

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2. In the above theorem the assumption $\lambda \geq 1$ is essential. In otherwise they construed a continuous function h such that the equation (1) has no periodic solutions.
3. Assume that $h \in C([0,\omega];\mathbb{R})$. Then (2) has periodic solutions if and only if $\overline{h} > 0$.

Natural question:

Does the last result remain still valid if $h \in L([0,\omega];\mathbb{R})$?

$$u''(t) + \frac{1}{u^{\lambda}(t)} = h(t). \tag{2}$$

The framework is the following:

Lazer and Solimini, 1987

Assume that $h \in C([0,\omega];\mathbb{R})$. Then (1) has periodic solutions if and only if $\overline{h} > 0$.

Natural question:

Does the above theorem still valid if $h \in L([0,\omega];\mathbb{R})$?

This is not an innocent question. In the related literature there are many authors whom have found with this trouble.

$$u''(t) + \frac{1}{u^{\lambda}(t)} = h(t).$$

For instance, among other work we can cite

- P. Habets, L. Sanchez, *Periodic solutions of some Liénard equations with singularities*, Proc. Amer. Math. Soc. **109** (1990), 1035-1044.
- I. Rachunková, M. Tvrdý, I. Vrkoc, *Existence of nonnegative and nonpositive solutions for second order periodic boundary value problems*, J. Differential Equations **176** (2001), 445-469.
- R. Hakl, P.J. Torres, M. Zamora, *Periodic solutions of singular second order differential equations: Upper and lower functions*, Nonlinear Anal. **74** (2011), 7078-7093.
- I. Rachunková, S. Stanek, M. Tvrdý, *Solvability of nonlinear singular problems for ordinary differential equations*, Contemp. Math. Appl. **5** (2008) Hindawi Publ. Corp., 268 pp.

In addition, we can find the above natural question as an open problem formulated in:

- R. Hakl, P.J. Torres, *On periodic solutions of second-order differential equations with attractive-repulsive singularities*, J. Differential Equations **248** (2010), 111-126.

Introduction and open problem

$$u''(t) + \frac{1}{u^{\lambda}(t)} = h(t). \qquad (2)$$

In this work we show an example giving answer for the open problem

Open Problem

Prove or disprove the following conjecture: Let

$$h \in L([0,\omega];\mathbb{R}), \qquad \lambda > 0.$$

Then the equation (2) has a positive solution if and only if $\overline{h} > 0$.

In addition, we will show a optimal condition, which not affect to h, in order to have periodic solvability of (2).

Counter-Example and main results

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$$u''(t) + \frac{1}{u^{\lambda}(t)} = h(t). \tag{2}$$

Theorem:

Let $p \geq 1$, $0 < \lambda < \frac{1}{2p-1}$. Then there exists $h \in L^p([0,\omega];\mathbb{R})$ with $\overline{h} > 0$ such that (2) has no periodic solutions.

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Corollary:

Let $0 < \lambda < 1$. Then there exists $h \in L\big([0,\omega];\mathbb{R}\big)$ with $\overline{h} > 0$ such that (2) has no periodic solutions.

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Let $p \geq 1$ and $\lambda \in \left(0, \frac{1}{2p-1}\right)$. Choose $\mu \in \left(2 - \frac{1}{p\lambda}, \frac{1}{p}\right)$, $\varepsilon \in (0, \frac{\omega}{4})$, and put

$$\varphi(t) = \begin{cases} -t^{-\mu} & \text{for } t \in (0, \varepsilon], \\ 0 & \text{for } t \in \left(\varepsilon, \frac{\omega}{2} - \varepsilon\right), \\ \left(\frac{\omega}{2} - t\right)^{-\mu} & \text{for } t \in \left[\frac{\omega}{2} - \varepsilon, \frac{\omega}{2}\right), \end{cases} \qquad \varphi(t) = \varphi(\omega - t),$$

$$v''(t) = \varphi(t), \qquad v\left(\frac{\omega}{2}\right) = 0 = v'\left(\frac{\omega}{2}\right).$$

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$$v''(t) = \varphi(t), \qquad v\left(\frac{\omega}{2}\right) = 0 = v'\left(\frac{\omega}{2}\right).$$

Notice that $\varphi \in L^p([0, \omega]; \mathbb{R})$ and

$$v(t) = \int_t^{\frac{\omega}{2}} \int_s^{\frac{\omega}{2}} \varphi(\xi) d\xi ds \qquad \text{for } t \in [0, \omega].$$

PROOF:
Let $p \geq 1$ and $\lambda \in \left(0, \frac{1}{2p-1}\right)$. Choose $\mu \in \left(2 - \frac{1}{p\lambda}, \frac{1}{p}\right)$, $\varepsilon \in (0, \frac{\omega}{4})$.
Then:

$$v(t) > 0 \qquad \text{for } \; t \in [0, \omega/2) \cup (\omega/2, \omega]\,, \qquad v(\omega/2) = 0,$$
$$v(0) = v(\omega), \qquad v'(0) = 0 = v'(\omega),$$
$$v(t) = \frac{|\omega/2 - t|^{2-\mu}}{(2-\mu)(1-\mu)} \qquad \text{for } \; t \in \left(\frac{\omega}{2} - \varepsilon, \frac{\omega}{2} + \varepsilon\right).$$

Therefore

$$\frac{1}{v^{\lambda}} \in L^{p}([0, \omega]; \mathbb{R}),$$

Since $\varphi \in L^p([0,\omega];\mathbb{R})$, $\frac{1}{v^\lambda} \in L^p([0,\omega];\mathbb{R})$, we can define $h \in L^p\big([0,\omega];\mathbb{R}\big)$ such that

$$h(t) = \varphi(t) + \frac{1}{v^\lambda(t)} \qquad \text{for a. e. } t \in [0,\omega].$$

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$$h(t) = \varphi(t) + \frac{1}{v^\lambda(t)} \qquad \text{for a. e. } \ t \in [0,\omega].$$

Recalling that $v'' = \varphi(t)$, $v \in AC^1\big([0,\omega];[0,+\infty)\big)$ verifies

$$v''(t) + \frac{1}{v^\lambda(t)} = h(t) \qquad \text{for a. e. } \ t \in [0,\omega].$$

$$u''(t) + \frac{1}{u^{\lambda}(t)} = h(t). \tag{2}$$

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If w is a positive periodic solution to (2) then $w \equiv v$, which is a contradiction.

Counter-Example and main results

$$u''(t) + \frac{1}{u^{\lambda}(t)} = h(t). \qquad (2)$$

Theorem:

Let $p \geq 1$, $0 < \lambda < \frac{1}{2p-1}$. Then there exists $h \in L^p([0,\omega];\mathbb{R})$ with $\overline{h} > 0$ such that (2) has no periodic solutions.

Corollary:

Let $0 < \lambda < 1$. Then there exists $h \in L\big([0,\omega];\mathbb{R}\big)$ with $\overline{h} > 0$ such that (2) has no periodic solutions.

$$u''(t) + \frac{1}{u^\lambda(t)} = h(t). \qquad (2)$$

Corollary:

Let $0 < \lambda < 1$. Then there exists $h \in L\big([0,\omega];\mathbb{R}\big)$ with $\overline{h} > 0$ such that (2) has no periodic solutions.

What happen when $\lambda \geq 1$?

$$u''(t) + \frac{1}{u^{\lambda}(t)} = h(t). \qquad (2)$$

Corollary:

Let $0 < \lambda < 1$. Then there exists $h \in L([0, \omega]; \mathbb{R})$ with $\overline{h} > 0$ such that (2) has no periodic solutions.

Theorem:

Let $\lambda \geq 1$ and $h \in L([0, \omega]; \mathbb{R})$. Then there exists an unique periodic solution to (2) if and only if $\overline{h} > 0$.

Comments and questions

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$$u''(t) + \frac{1}{u^{\lambda}(t)} = h(t). \qquad (2)$$

Results when $h \in L\big([0,\omega];\mathbb{R}\big)$:

1. $\lambda \in (0,1) \Rightarrow \exists h \in L$ such that (2) has not periodic solutions;
2. $\lambda \geq 1 \Rightarrow$ (2) has periodic solutions.

Results when $h \in L^p\big([0,\omega];\mathbb{R}\big)$

1. $\lambda \in (0, 1/(2p-1)) \Rightarrow \exists h \in L^p$ such that (2) has not periodic solutions;
2. Conjecture: $\lambda \geq 1/(2p-1) \Rightarrow$ (2) has periodic solutions.

Comments and questions

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Corollary:

Let $0 < \lambda < 1$. Then there exists $h \in L\big([0,\omega];\mathbb{R}\big)$ with $\overline{h} > 0$ such that (2) has no periodic solutions.

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Let $0 < \lambda < 1$. Then there exists $h \in L\big([0,\omega];\mathbb{R}\big)$ with $\overline{h} > 0$ such that (2) has no periodic solutions.

We have seen that there exists $h \in L\big([0,\omega];\mathbb{R}\big)$ such that all periodic solutions to (2) have at least one zero on $[0,\omega]$.

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Corollary:

Let $0 < \lambda < 1$. Then there exists $h \in L\big([0, \omega]; \mathbb{R}\big)$ with $\overline{h} > 0$ such that (2) has no periodic solutions.

We have seen that there exists $h \in L\big([0, \omega]; \mathbb{R}\big)$ such that all periodic solutions to (2) have at least one zero on $[0, \omega]$. If we admit these type of solutions, Has always the equation (2) periodic solutions when $\overline{h} > 0$?

Thank you for your attention.