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ADAPTIVE FINITE ELEMENT ANALYSIS BASED ON PERTURBATION ARGUMENTS

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Abstract

We review some numerical analysis of an adaptive finite element method (AFEM) for a class of elliptic partial differential equations based on a perturbation argument. This argument makes use of the relationship between the general problem and a model problem, whose adaptive finite element analysis is existing, from which we get the convergence and the complexity of adaptive finite element methods for a nonsymmetric boundary value problem, an eigenvalue problem, a nonlinear boundary value problem as well as a nonlinear eigenvalue problem.

1. Introduction

In this paper, we shall apply a perturbation argument to analyze the convergence and the complexity of AFEMs for a class of elliptic partial differential equations. This perturbation argument makes use of the relationship between the general problem and a model problem, whose adaptive finite element analysis is existing. Based on the perturbation argument, we get the convergence and the complexity of AFEMs for a nonsymmetric boundary value problem, an eigenvalue problem, a nonlinear boundary value problem as well as a nonlinear eigenvalue problem.

A standard AFEM consists of successive loops of the form

$$\text{Solve} \rightarrow \text{Estimate} \rightarrow \text{Mark} \rightarrow \text{Refine}.$$

More precisely, given some finite element approximation, we generate a new mesh by refining those elements where local error estimators indicate that the errors are relatively large, and then, on this refined mesh, compute the next approximation. We repeat this procedure until a certain accuracy is obtained. In this procedure an a posteriori error estimator is crucial. For a posteriori error analysis, we refer to the books [2, 22] and the references cited therein. Since Babuška and Vogelius [3] gave an analysis of an AFEM for linear symmetric elliptic problems in one dimension, there has been much work on the numerical analysis of the convergence and the complexity of AFEM in the literature [4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21].

Let $\Omega \subset \mathbb{R}^d (d \geq 1)$ be a polytopic bounded domain. We shall use the standard notation for Sobolev spaces $W^{s,p}(\Omega)$ and their associated norms and seminorms (see, e.g., [1]). For $p=2$, we denote $H^s(\Omega)=W^{s,2}(\Omega)$ and $H_0^1(\Omega)=\{v \in H^1(\Omega) : v\mid_{\partial\Omega}=0\}$, where $v\mid_{\partial\Omega}=0$ is understood in the sense of trace, $\|\cdot\|_{s,\Omega} = \|\cdot\|_{s,2,\Omega}$. Throughout this paper, we shall use C to denote a generic positive constant which may stand for different values at its different occurrences. For convenience, the symbol $\lesssim$ will be used in this paper. The notation that $A \lesssim B$ means that $A \leq CB$ for some constant C that is independent of mesh parameters. All the constants involved are independent of mesh sizes.

This paper is organized as follows. In the next section, we review some existing results of AFEMs for a model problem. In section 3, we establish a general framework to carry out the adaptive finite element analysis for a class of elliptic problems by using the perturbation argument. Finally, we apply the general framework to four kinds of problems, including a nonsymmetric boundary value problem, an eigenvalue problem, a nonlinear boundary value problem and a nonlinear eigenvalue problem.

2. A model problem

Consider a homogeneous boundary value problem:

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1}$$

Letting $a(\cdot,\cdot) = (\nabla\cdot, \nabla\cdot)$, one sees that there exists a constant $0 < c_a < \infty$ such that

$$c_a \|v\|_{1,\Omega}^2 \leq a(v,v) \quad \forall v \in H_0^1(\Omega).$$

The energy norm $\|\cdot\|_{a,\Omega}$, which is equivalent to $\|\cdot\|_{1,\Omega}$, is defined by $\|w\|_{a,\Omega} = \sqrt{a(w,w)}$. The weak form of (1) reads as follows: find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v) \qquad \forall v \in H_0^1(\Omega). \tag{2}$$

It is well known that (2) is uniquely solvable for any $f \in H^{-1}(\Omega)$.

Let $\{\mathcal{T}_h\}$ be a shape regular family of nested conforming meshes over Ω: there exists a constant γ^* such that $\frac{h_\tau}{\rho_\tau} \leq \gamma^*$ for all $\tau \in \cup_h \mathcal{T}_h$, where h_τ is the diameter of τ, and ρ_τ is the diameter of the biggest ball contained in τ, $h=\max\{h_\tau : \tau \in \mathcal{T}_h\}$. Let $\mathcal{E}_h$ denote the set of interior faces (edges or sides) of $\mathcal{T}_h$. Let $S_0^h(\Omega) \subset H_0^1(\Omega)$ be a family of nested finite element spaces consisting of continuous piecewise polynomials over $\mathcal{T}_h$ of fixed degree $n \geq 1$, which vanish on $\partial\Omega$.

A standard finite element scheme for (2) is: find $u_h \in S_0^h(\Omega)$ satisfying

$$a(u_h,v) = (f,v) \qquad \forall v \in S_0^h(\Omega). \tag{3}$$

Let $\mathbb{T}$ denote the class of all conforming refinements by bisection of $\mathcal{T}_0$ that is the initial mesh. For $\mathcal{T}_h \in \mathbb{T}$ and $v \in S_0^h(\Omega)$ we define the element residual $\tilde{\mathcal{R}}_\tau(v)$ and the jump residual $J_e(v)$ for (3) by

$$\begin{aligned}\tilde{\mathcal{R}}_\tau(v) &= f + \Delta v \quad \text{in } \tau \in \mathcal{T}_h,\\ J_e(v) &= -\nabla v^+ \cdot \nu^+ - \nabla v^- \cdot \nu^- = [[\nabla v]]_e \cdot \nu_e \quad \text{on } e \in \mathcal{E}_h,\end{aligned}$$

where e is the common side of elements τ^+ and τ^- with unit outward normals ν^+ and ν^-, respectively, and $\nu_e = \nu^-$. For $\tau \in \mathcal{T}_h$, we define the local error indicator $\tilde{\eta}_h(v,\tau)$ and the oscillation $\widetilde{osc}_h(v,\tau)$ by

$$\tilde{\eta}_h^2(v,\tau) = h_\tau^2 \|\tilde{\mathcal{R}}_\tau(v)\|_{0,\tau}^2 + \sum_{e\in\mathcal{E}_h, e\subset\partial\tau} h_e \|J_e(v)\|_{0,e}^2, \tag{4}$$

$$\widetilde{osc}_h^2(v,\tau) = h_\tau^2 \|\tilde{\mathcal{R}}_\tau(v) - \overline{\tilde{\mathcal{R}}_\tau(v)}\|_{0,\tau}^2, \tag{5}$$

where $\overline{w}$ is the L^2-projection of $w \in L^2(\Omega)$ to polynomials of some degree on τ or e. We define the error estimator $\tilde{\eta}_h(u_h,\mathcal{T}_h)$ and the oscillation $\widetilde{osc}_h(u_h,\mathcal{T}_h)$ by

$$\tilde{\eta}_h^2(u_h,\mathcal{T}_h) = \sum_{\tau\in\mathcal{T}_h} \tilde{\eta}_h^2(u_h,\tau) \quad \text{and} \quad \widetilde{osc}_h^2(u_h,\mathcal{T}_h) = \sum_{\tau\in\mathcal{T}_h} \widetilde{osc}_h^2(u_h,\tau).$$

We recall the well-known upper and lower bounds for the energy error in terms of the residual-type estimator (see, e.g., [15, 17, 22]).

Theorem 2.1. *Let $u \in H_0^1(\Omega)$ be the solution of (2) and $u_h \in S_0^h(\Omega)$ be the solution of (3). Then there exist constants $\tilde{C}_1$, $\tilde{C}_2$ and $\tilde{C}_3 > 0$ depending only on c_a and the shape regularity γ^* such that*

$$\begin{aligned}\|u-u_h\|_{a,\Omega}^2 &\le \tilde{C}_1 \tilde{\eta}_h^2(u_h,\mathcal{T}_h),\\ \tilde{C}_2\tilde{\eta}_h^2(u_h,\mathcal{T}_h) - \tilde{C}_3 \widetilde{osc}_h^2(u_h,\mathcal{T}_h) &\le \|u-u_h\|_{a,\Omega}^2.\end{aligned}$$

Now we address the marking strategy of solving (3):

Given a parameter $0 < \theta < 1$:

1. Construct a minimal subset $\mathcal{M}_k$ of $\mathcal{T}_k$ by selecting some elements in $\mathcal{T}_k$ such that

$$\tilde{\eta}_k(u_k,\mathcal{M}_k) \ge \theta \tilde{\eta}_k(u_k,\mathcal{T}_k). \tag{6}$$

2. Mark all the elements in $\mathcal{M}_k$.

For any $\mathcal{T}_k \in \mathbb{T}$ and a subset $\mathcal{M}_k \subset \mathcal{T}_k$ of marked elements at the kth step, the "Refine" procedure outputs a conforming triangulation $\mathcal{T}_{k+1} \in \mathbb{T}$, where all elements of $\mathcal{M}_k$ are bisected at least once. We define

$$\mathcal{R}_{\mathcal{T}_k\to\mathcal{T}_{k+1}} = \mathcal{T}_k \backslash (\mathcal{T}_k \cap \mathcal{T}_{k+1})$$

as the set of refined elements, thus $\mathcal{M}_k \subset \mathcal{R}_{\mathcal{T}_k\to\mathcal{T}_{k+1}}$.

We state an adaptive finite element algorithm for solving (3) as follows:

Algorithm 2.1. *Adaptive finite element algorithm*

1. *Pick an initial mesh* $\mathcal{T}_0$ *and let* $k = 0$.
2. *Solve* (3) *on* $\mathcal{T}_k$ *and get the finite element approximation* u_k.
3. *Compute local error indicators* $\tilde{\eta}_k(u_k, \tau) \quad \forall \tau \in \mathcal{T}_k$.
4. *Construct* $\mathcal{M}_k \subset \mathcal{T}_k$ *by a marking strategy that satisfies (6).*
5. *Refine* $\mathcal{T}_k$ *to get a new conforming mesh* $\mathcal{T}_{k+1}$.
6. *Let* $k = k + 1$ *and go to 2.*

For Algorithm 2.1, we have (see [5]).

Theorem 2.2. *Let* $\{u_k\}_{k\in\mathbb{N}_0}$ *be a sequence of finite element solutions corresponding to a sequence of nested finite element spaces* $\{S_0^k(\Omega)\}_{k\in\mathbb{N}_0}$ *produced by Algorithm 2.1. Then there exist constants* $\tilde{\gamma} > 0$ *and* $\tilde{\xi} \in (0, 1)$ *depending only on the shape regularity* γ^* *and marking parameter* θ *such that for any two consecutive iterates, there holds*

$$\|u - u_{k+1}\|_{a,\Omega}^2 + \tilde{\gamma}\tilde{\eta}_{k+1}^2(u_{k+1}, \mathcal{T}_h) \;\leq\; \tilde{\xi}^2\big(\|u - u_k\|_{a,\Omega}^2 + \tilde{\gamma}\tilde{\eta}_k^2(u_k, \mathcal{T}_h)\big).$$

3. A general framework

We introduce the general framework established in [13]. Let $u \in H_0^1(\Omega)$ satisfy

$$a(u, v) + (Vu, v) = (\ell u, v) \qquad \forall v \in H_0^1(\Omega), \tag{7}$$

where $\ell : H_0^1(\Omega) \to L^2(\Omega)$ is an operator and $V : H_0^1(\Omega) \to L^2(\Omega)$ is a linear bounded operator. Some applications of ℓ and V will be shown in section 4. We assume that (7) has a unique solution $u \in H_0^1(\Omega)$.

For $h \in (0, 1)$, let $u_h \in S_0^h(\Omega)$ be a solution of the following discretization problem:

$$a(u_h, v) + (Vu_h, v) = (\ell_h u_h, v) \quad \forall v \in S_0^h(\Omega), \tag{8}$$

where $\ell_h : S_0^h(\Omega) \to L^2(\Omega)$ is some approximate operator to ℓ.

Let $K = (-\Delta)^{-1} : L^2(\Omega) \to H_0^1(\Omega)$. Then (7) and (8) can be rewritten as

$$u + KVu = K\ell u \quad \text{and} \quad u_h + P_h KVu_h = P_h K\ell_h u_h,$$

where $P_h : H_0^1(\Omega) \to S_0^h(\Omega)$ is defined by

$$a(u - P_h u, v) = 0 \quad \forall v \in S_0^h(\Omega).$$

We assume that there exists $\kappa(h) \in (0, 1)$ such that $\kappa(h) \to 0$ as $h \to 0$ and

$$\|u - w^h\|_{a,\Omega} \leq \tilde{C}\kappa(h)\|u - u_h\|_{a,\Omega}. \tag{9}$$

We have for $w^h = K\ell_h u_h - KVu_h$ that $u_h = P_h w^h$. Hence we obtain

$$\|u - u_h\|_{a,\Omega} = \|w^h - P_h w^h\|_{a,\Omega} + \mathcal{O}(\kappa(h))\|u - u_h\|_{a,\Omega}, \tag{10}$$

which implies that the error of the general problem is equivalent to that of the model problem with $\ell_h u_h - V u_h$ as a source term up to the high order term.

Following the element residual $\tilde{\mathcal{R}}_\tau(u_h)$ for (3), we define the element residual $\mathcal{R}_\tau(u_h)$ for (8) as follows:

$$\mathcal{R}_\tau(u_h) \;=\; \ell_h u_h - V u_h + \Delta u_h \quad \text{in } \tau \in \mathcal{T}_h.$$

For $\tau \in \mathcal{T}_h$, we define the local error indicator $\eta_h(u_h, \tau)$ and the oscillation $osc_h(u_h, \tau)$ from (4) and (5) by replacing $\tilde{\mathcal{R}}_\tau(u_h)$ by $\mathcal{R}_\tau(u_h)$. And we set the error estimator $\eta_h(u_h, \mathcal{T}_h)$ and oscillation $osc_h(u_h, \mathcal{T}_h)$ by

$$\eta_h^2(u_h, \mathcal{T}_h) = \sum_{\tau \in \mathcal{T}_h} \eta_h^2(u_h, \tau) \quad \text{and} \quad osc_h^2(u_h, \mathcal{T}_h) = \sum_{\tau \in \mathcal{T}_h} osc_h^2(u_h, \tau). \tag{11}$$

Let $h_0 \in (0, 1)$ be the mesh size of the initial mesh $\mathcal{T}_0$ and define

$$\tilde{\kappa}(h_0) = \sup_{h \in (0, h_0]} \max\{h, \kappa(h)\}.$$

Obviously, $\tilde{\kappa}(h_0) \ll 1$ if $h_0 \ll 1$.

Combing Theorem 2.1 with (10), we obtain the following a posteriori error estimates which will be used to analyze the convergence and the complexity [13].

Theorem 3.1. *Let $h_0 \ll 1$ and $h \in (0, h_0]$. There exist constants C_1, C_2 and C_3, which only depend on the shape regularity constant γ^* and c_a, such that*

$$\begin{aligned} \|u - u_h\|_{a,\Omega}^2 &\le C_1 \eta_h^2(u_h, \mathcal{T}_h), \\ C_2 \eta_h^2(u_h, \mathcal{T}_h) &\le \|u - u_h\|_{a,\Omega}^2 + C_3 osc_h^2(u_h, \mathcal{T}_h). \end{aligned}$$

We use $\mathcal{T}_H$ to denote a coarse mesh and $\mathcal{T}_h$ to denote a refined mesh of $\mathcal{T}_H$. Recalling that $w^h = K(\ell_h u_h - V u_h)$ and $w^H = K(\ell_H u_H - V u_H)$, we get (see [13]).

Lemma 3.1. *If $h, H \in (0, h_0]$, then*

$$\begin{aligned} \|u - u_h\|_{a,\Omega} &= \|w^H - P_h w^H\|_{a,\Omega} + \mathcal{O}(\tilde{\kappa}(h_0)) \left(\|u - u_h\|_{a,\Omega} + \|u - u_H\|_{a,\Omega}\right), \\ \eta_h(u_h, \mathcal{T}_h) &= \tilde{\eta}_h(P_h w^H, \mathcal{T}_h) + \mathcal{O}(\tilde{\kappa}(h_0)) \left(\|u - u_h\|_{a,\Omega} + \|u - u_H\|_{a,\Omega}\right), \\ osc_h(u_h, \mathcal{T}_h) &= \widetilde{osc}_h(P_h w^H, \mathcal{T}_h) + \mathcal{O}(\tilde{\kappa}(h_0)) \left(\|u - u_h\|_{a,\Omega} + \|u - u_H\|_{a,\Omega}\right). \end{aligned}$$

The adaptive algorithm of solving (8), which we call **Algorithm** D, is nothing but **Algorithm 2.1** when $\tilde{\eta}_k$ are replaced by η_k. We may obtain from Theorem 2.2 and Lemma 3.1 that **Algorithm** D of (8) is a contraction with respect to the sum of the energy error plus the scaled error estimator [13].

Theorem 3.2. *Let $\theta \in (0,1)$ and $\{u_k\}_{k\in\mathbb{N}_0}$ be a sequence of finite element solutions of (8) corresponding to a sequence of finite element spaces $\{S_0^k(\Omega)\}_{k\in\mathbb{N}_0}$ produced by* **Algorithm** *D. If $h_0 \ll 1$, then there exist constants $\gamma > 0$ and $\xi \in (0,1)$ depending only on the shape regularity constant γ^*, c_a and marking parameter θ such that*

$$\|u-u_{k+1}\|_{a,\Omega}^2 + \gamma\eta_{k+1}^2(u_{k+1},\mathcal{T}_{k+1}) \le \xi^2\big(\|u-u_k\|_{a,\Omega}^2 + \gamma\eta_k^2(u_k,\mathcal{T}_k)\big).$$

We turn to study the complexity in a class of functions defined by

$$\mathcal{A}_\gamma^s = \{v \in H_0^1(\Omega) : |v|_{s,\gamma} < \infty\},$$

where $\gamma > 0$ is some constant,

$$|v|_{s,\gamma} = \sup_{\varepsilon>0} \varepsilon \inf_{\{\mathcal{T}\subset\mathcal{T}_0 : \inf(\|v-v'\|_{a,\Omega}^2 + (\gamma+1)osc_{\mathcal{T}}^2(v',\mathcal{T}))^{1/2} \le \varepsilon : v' \in S_0^{\mathcal{T}}(\Omega)\}} \big(\#\mathcal{T} - \#\mathcal{T}_0\big)^s$$

and $\mathcal{T} \subset \mathcal{T}_0$ means $\mathcal{T}$ is a refinement of $\mathcal{T}_0$ and $S_0^{\mathcal{T}}(\Omega)$ is the associated finite element space. Since $\mathcal{A}_\gamma^s = \mathcal{A}_1^s$ for all $\gamma > 0$, we use $\mathcal{A}^s$ to stand for $\mathcal{A}_1^s$, and use $|v|_s$ to denote $|v|_{s,\gamma}$. We have the optimal complexity as follows [13].

Theorem 3.3. *Let $u \in \mathcal{A}^s$ and $\{u_k\}_{k\in\mathbb{N}_0}$ be a sequence of finite element solutions corresponding to a sequence of finite element spaces $\{S_0^k(\Omega)\}_{k\in\mathbb{N}_0}$ produced by* **Algorithm** *D. If $h_0 \ll 1$, then*

$$\|u-u_k\|_{a,\Omega}^2 + \gamma osc_k^2(u_k,\mathcal{T}_k) \lesssim (\#\mathcal{T}_k - \#\mathcal{T}_0)^{-2s}|u|_s^2,$$

where the hidden constant depends on the discrepancy between $\sqrt{\frac{C_2\gamma}{C_3(C_1+(1+2CC_1)\gamma)}}$ and θ. Here C_1, C_2, C_3 are constants appeared in Theorem 3.1 and C is some positive constant depending on the data of the problem.

4. Applications

In this section, we apply the general framework to four examples and get the convergence and the complexity of the corresponding adaptive finite element approximations.

4.1. A nonsymmetric boundary value problem

The first example is a second order nonsymmetric elliptic partial differential equation. We consider the following problem: find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) + (\mathbf{b}\cdot\nabla u, v) + (cu, v) = (f, v) \quad \forall v \in H_0^1(\Omega), \tag{12}$$

where $\Omega \subset \mathbb{R}^d (d \ge 2)$ is a ploytopic domain, $\mathbf{b} \in [L^\infty(\Omega)]^d$ is divergence free, $c \in L^\infty(\Omega)$, and $f \in L^2(\Omega)$. We assume that (12) is well-posed, namely (12) is uniquely solvable for any $f \in H^{-1}(\Omega)$.

A finite element discretization of (12) reads: find $u_h \in S_0^h(\Omega)$ such that

$$(\nabla u_h, \nabla v) + (\mathbf{b} \cdot \nabla u_h, v) + (cu_h, v) = (f, v) \qquad \forall v \in S_0^h(\Omega). \tag{13}$$

It is seen that (13) has a unique solution u_h if $h \ll 1$ (see, e.g., [23]) and (13) is a special case of (8), in which $Vw = \mathbf{b} \cdot \nabla w + cw$ and $\ell w = \ell_h w = f \;\; \forall w \in H_0^1(\Omega)$. Consequently, $w^h = K(f - Vu_h)$. The element residual becomes

$$\mathcal{R}_\tau(u_h) = f - \mathbf{b} \cdot \nabla u_h - cu_h + \Delta u_h \quad \text{ in } \tau \in \mathcal{T}_h,$$

while $\eta_h(u_h, \mathcal{T}_h)$ and $osc_h(u_h, \mathcal{T}_h)$ are defined by (11).

Note that $V : H_0^1(\Omega) \to L^2(\Omega)$ is linear bounded and KV is compact over $H_0^1(\Omega)$. Setting $\kappa(h) = \|(I + KVP_h)^{-1}\| \|KV(I - P_h)\|$, we have that (9) holds [13]. Thus Theorems 3.2 and 3.3 ensure the convergence and the complexity of AFEM for nonsymmetric problem (12) [13].

4.2. An eigenvalue problem

A number λ is called an eigenvalue of the form $a(\cdot,\cdot)$ relative to the form $(\cdot,\cdot)$ if there is a nonzero function $u \in H_0^1(\Omega)$, called an associated eigenfunction, satisfying

$$a(u, v) = \lambda(u, v) \quad \forall v \in H_0^1(\Omega). \tag{14}$$

It is known that (14) has a countable sequence of real eigenvalues $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$, and corresponding eigenfunctions $u_1, u_2, u_3, \cdots$, which can be assumed to satisfy $(u_i, u_j) = \delta_{ij},\; i, j = 1, 2, \cdots$. In the sequence $\{\lambda_j\}$, the λ_j's are repeated according to their geometric multiplicity.

A standard finite element scheme for (14) is: find a pair of (λ_h, u_h), where λ_h is a number and $0 \ne u_h \in S_0^h(\Omega)$ satisfying

$$a(u_h, v_h) = \lambda_h(u_h, v_h) \quad \forall v_h \in S_0^h(\Omega). \tag{15}$$

Let us order the eigenvalues of (15) as follows

$$\lambda_{1,h} < \lambda_{2,h} \le \cdots \le \lambda_{n_h,h}, \quad n_h = \dim S_0^h(\Omega),$$

and assume that the corresponding eigenfunctions $u_{1,h}, u_{2,h}, \cdots, u_{n_h,h}$ satisfy $(u_{i,h}, u_{j,h}) = \delta_{ij},\; i, j = 1, 2, \cdots$. (15) is a special case of (8), in which $V = 0$, $\ell u = \lambda u$ and $\ell_h u = \lambda_h u_h$. Consequently, $w^h = K\lambda_h u_h$. The element residual becomes

$$\mathcal{R}_\tau(u_h) = \lambda_h u_h + \Delta u_h \quad \text{ in } \tau \in \mathcal{T}_h,$$

while $\eta_h(u_h, \mathcal{T}_h)$ and $osc_h(u_h, \mathcal{T}_h)$ are defined by (11).

Let $\kappa(h) = \rho_\Omega(h) + \|u - u_h\|_{a,\Omega}$, where

$$\rho_\Omega(h) = \sup_{f \in L^2(\Omega), \|f\|_{0,\Omega}=1} \; \inf_{v \in S_0^h(\Omega)} \|(-\Delta)^{-1} f - v\|_{a,\Omega}.$$

We have that (9) holds for linear eigenvalue problem (14) [9]. Thus, Theorems 3.2 and 3.3 ensure the convergence and the complexity of AFEM for eigenvalue problem (14) [9].

4.3. A nonlinear boundary value problem

We consider the following nonlinear problem: find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) + (f(\cdot, u), v) = 0 \quad \forall v \in H_0^1(\Omega), \tag{16}$$

where $\Omega \subset \mathbb{R}^d (d = 1, 2, 3)$ is polytopic and $f(x, y)$ is a smooth function on $\mathbb{R}^d \times \mathbb{R}^1$.

For convenience, we shall drop the dependence of variable x in $f(x, u)$ in the following exposition and assume that (16) has a solution $u \in H_0^1(\Omega) \cap H^{1+s}(\Omega)$ $(s \in (1/2, 1])$.

A finite element discretization of (16) reads: find $u_h \in S_0^h(\Omega)$ such that

$$(\nabla u_h, \nabla v) + (f(u_h), v) = 0 \quad \forall v \in S_0^h(\Omega). \tag{17}$$

It is seen that (17) has a unique solution u_h in the neighbour of u if $h \ll 1$ (see, e.g., [23, 24]). The element residual becomes

$$\mathcal{R}_\tau(u_h) = -f(u_h) + \Delta u_h \quad \text{in } \tau \in \mathcal{T}_h,$$

while $\eta_h(u_h, \mathcal{T}_h)$ and $osc_h(u_h, \mathcal{T}_h)$ are defined by (11).

If $\|u_h\|_{0,\infty,\Omega} \lesssim 1$ and $\|u - u_h\|_{a,\Omega} \to 0$ as $h \to 0$, then (9) holds for nonlinear boundary value problem [13, 24], where $V = 0$ and $\ell_h w = -f(w)$ for any $w \in S_0^h(\Omega)$. Thus, Theorems 3.2 and 3.3 ensure the convergence and the optimal complexity of AFEM for nonlinear problem (16) [13].

4.4. A nonlinear eigenvalue problem

We turn to finite element approximations of the following nonlinear eigenvalue problem: find $\lambda \in \mathbb{R}$ and $u \in H_0^1(\Omega)$ such that $\|u\|_{0,\Omega} = 1$ and

$$(\nabla u, \nabla v) + (Vu + \mathcal{N}(u^2)u, v) \;\; = \lambda(u, v) \quad \forall v \in H_0^1(\Omega), \tag{18}$$

where $\Omega \subset \mathbb{R}^3$, $V : \Omega \to \mathbb{R}$ is a given function, and $\mathcal{N}$ has the following form:

$$\mathcal{N}(\rho) = \mathcal{N}_1(\rho) + \mathcal{N}_2(\rho),$$

where $\rho = u^2$, $\mathcal{N}_1 : [0, \infty) \to \mathbb{R}$ is a given function dominated by some polynomial, and $\mathcal{N}_2(\rho) = \int_\Omega \frac{\rho(y)}{|x-y|} dy$. This is a special case of (8), in which $\ell u = \lambda u - \mathcal{N}(u^2)u$ and $\ell_h u = \lambda_h u_h - \mathcal{N}(u_h^2)u_h$. Hence, (9) holds for this kind of nonlinear eigenvalue problems under some assumptions (see [7] for details).

Note that the element residual becomes

$$\mathcal{R}_\tau(u_h) = \lambda_h u_h - \mathcal{N}(u_h^2)u_h - Vu_h + \Delta u_h \quad \text{in } \tau \in \mathcal{T}_h,$$

while $\eta_h(u_h, \mathcal{T}_h)$ and $osc_h(u_h, \mathcal{T}_h)$ are defined by (11). Thus, Theorems 3.2 and 3.3 ensure the convergence and the complexity of AFEM for nonlinear eigenvalue problem (18) [7].

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