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AN ADAPTIVE *hp*-DISCONTINUOUS GALERKIN APPROACH FOR NONLINEAR CONVECTION-DIFFUSION PROBLEMS

Vít Dolejší
Charles University Prague, Faculty of Mathematics and Physics
Sokolovská 83, 186 75 Praha, Czech Republic
dolejsi@karlin.mff.cuni.cz

Abstract

We deal with a numerical solution of nonlinear convection-diffusion equations with the aid of the discontinuous Galerkin method (DGM). We propose a new *hp*-adaptation technique, which is based on a combination of a residuum estimator and a regularity indicator. The residuum estimator as well as the regularity indicator are easily evaluated quantities without the necessity to solve any local problem and/or any reconstruction of the approximate solution. The performance of the proposed *hp*-DGM is demonstrated

1. Introduction

Our aim is to develop a sufficiently robust, efficient and accurate numerical scheme for the simulation of viscous compressible flows. The *discontinuous Galerkin* (DG) methods have become very popular numerical techniques for the solution of the compressible Navier-Stokes equations. Recent progress of the use of the DG method for compressible flow simulations can be found in [8].

In this paper, we solve a scalar nonlinear convection-diffusion equation (which represents a model problem for the system of the compressible Navier-Stokes equations) with the aid of the DG method. We propose a *hp*-adaptive method which allows the refinement in the element size h as well as in the polynomial degree p. Similarly as the h version of the finite element methods, a posteriori error estimates can be used to determine which elements should be refined. However a single error estimate cannot simultaneously determine whether it is better to do h or p refinement. Several strategies for making this determination have been proposed over the years, see, e.g., [7] for a survey or [12]. Based on many theoretical works, e.g., monographs [10, 11] or survey paper [2], we expect that an error converges at an exponential rate in the number of degree of freedom.

There exist many theoretical works deriving a posteriori error estimates based on various approaches for linear or quasi-linear problems, e.g., [9]. On the other hand, the amount of papers dealing with a posteriori error estimates for strongly non-linear problems is significantly smaller. Some overview of a posteriori error estimates can be found in [13].

We propose a new *hp*-adaptation strategy which is based on a combination of a residuum estimator and a regularity indicator. The *residuum estimator* gives a lower estimate of the error measured in a dual norm. It is locally defined for each mesh element, it is easily evaluated and is implementation is very simple. The *regularity indicator* is based on the integration of interelement jumps of the approximate solution over the element boundary. Taking into account results from a priori error analysis (e.g., [4]), we define the regularity indicator. If this value is smaller than one then we apply a p-refinement otherwise we use a h-refinement. However, a rigorous theoretical justification of this approach is completely open. On the other hand, advantage of the proposed strategy is its simple applicability to general problems without any modification.

2. Problem description

2.1. Governing equations

We consider a stationary convection-diffusion equation

$$\nabla \cdot \boldsymbol{f}(u) = \nabla \cdot (\boldsymbol{K}(u)\nabla u) + g, \tag{1}$$

where $u : \Omega \to \mathbb{R}$ is the unknown scalar function defined in a bounded domain $\Omega \in \mathbb{R}^d$, $d = 2, 3$. Moreover, $g : \Omega \to \mathbb{R}$, $\boldsymbol{f}(u) = (f_1(u), \ldots, f_d(u)) : \mathbb{R} \to \mathbb{R}^d$ and $\boldsymbol{K}(u) = \{K_{ij}(u)\}_{i,j=1}^d : \mathbb{R} \to \mathbb{R}^{d\times d}$ are nonlinear functions of their arguments. For simplicity, we consider a homogeneous Dirichlet boundary condition over the whole boundary of Ω. However, an extension to a possible combination of nonhomogeneous Dirichlet and Neumann boundary conditions is straightforward.

2.2. Discretization of the problem

Let $\mathscr{T}_h$ $(h > 0)$ be a partition of the closure $\overline{\Omega}$ of the domain Ω into a finite number of closed d-dimensional simplicies K with mutually disjoint interiors. We call $\mathscr{T}_h = \{K\}_{K\in\mathscr{T}_h}$ a *triangulation* of Ω and do not require the conforming properties from the finite element method.

Over the triangulation $\mathscr{T}_h$ we define the so-called *broken Sobolev space*

$$H^s(\Omega, \mathscr{T}_h) := \{v; v|_K \in H^s(K)\ \forall\, K \in \mathscr{T}_h\}, \quad s \geq 0, \tag{2}$$

where $H^s(D)$ denotes the Sobolev space over domain D. Moreover, to each $K \in \mathscr{T}_h$, we assign a positive integer p_K (=local polynomial degree). Furthermore, over the triangulation $\mathscr{T}_h$ we define the finite dimensional subspace of $H^1(\Omega, \mathscr{T}_h)$

which consists of in general discontinuous piecewise polynomial functions associated with the set $\{p_K, K \in \mathscr{T}_h\}$ by

$$S_{h\mathsf{p}} = \{v;\ v \in L^2(\Omega),\ v|_K \in P_{p_K}(K)\ \forall K \in \mathscr{T}_h\}, \tag{3}$$

where $P_{p_K}(K)$ denotes the space of all polynomials on K of degree $\le p_K,\ K \in \mathscr{T}_h$.

Let the form $c_h : S_{h\mathsf{p}} \times S_{h\mathsf{p}} \to \mathbb{R}$ denote a discretization of (1) with the aid of interior penalty discontinuous Galerkin method, for its determination, see, e.g., [4, 6], particularly,

$$\begin{aligned} c_h(u,v) \ :=\ & \sum_{\Gamma\in\mathscr{F}_h} \int_\Gamma H(u|_\Gamma^{(+)}, u|_\Gamma^{(-)}, \boldsymbol{n})\, [\![v]\!]\ \mathrm{d}S - \sum_{K\in\mathscr{T}_h} \int_K \boldsymbol{f}(u)\cdot\nabla v\,\mathrm{d}x, \\ +\ & \sum_{K\in\mathscr{T}_h} \int_K \boldsymbol{K}(u)\nabla u\cdot\nabla v\,\mathrm{d}x - \int_\Omega g\,v\,\mathrm{d}x \\ -\ & \sum_{\Gamma\in\mathscr{F}_h^I} \int_\Gamma \Big(\{\!\{\boldsymbol{K}(u)\nabla u\}\!\}\cdot\boldsymbol{n}[\![v]\!] - \mathrm{g}\{\!\{\boldsymbol{K}(u)\nabla v\}\!\}\cdot\boldsymbol{n}[\![u]\!] \Big)\ \mathrm{d}S \\ -\ & \sum_{\Gamma\in\mathscr{F}_h^D} \int_\Gamma \Big(\boldsymbol{K}(u)\nabla u\cdot\boldsymbol{n}v - \mathrm{g}\boldsymbol{K}(u)\nabla v\cdot\boldsymbol{n}(u-u_D) \Big)\ \mathrm{d}S \\ +\ & \sum_{\Gamma\in\mathscr{F}_h^I} \int_\Gamma \sigma[\![u]\!]\,[\![v]\!]\ \mathrm{d}S + \sum_{\Gamma\in\mathscr{F}_h^D} \int_\Gamma \ \sigma(u-u_D)\,v\,\mathrm{d}S, \end{aligned} \tag{4}$$

where H is the numerical flux known from finite volume method, $\Gamma\in\mathscr{F}_h^I$ and $\Gamma\in\mathscr{F}_h^D$ are the sets of all interior and boundary faces, respectively, $\mathscr{F}_h = \mathscr{F}_h^I\cup\mathscr{F}_h^D$, $u|_\Gamma^{(+)}$ and $u|_\Gamma^{(-)}$ are the traces of $u \in H^s(\Omega, \mathscr{T}_h)$ on $\Gamma \in \mathscr{F}_h$, and $\{\!\{u\}\!\} = (u|_\Gamma^{(+)} + u|_\Gamma^{(-)})/2$ and $[\![u]\!] = u|_\Gamma^{(+)} - u|_\Gamma^{(-)}$ are the mean value and the jump on Γ, respectively. Moreover, u_D is the given Dirichlet boundary condition, σ is the penalty parameter and $\mathrm{g} = -1, 0, 1$ for SIPG, IIPG and NIPG variants of DGFE method, respectively.

We say that a function $u_h \in S_{h\mathsf{p}}$ is an *approximate solution* of (1), if

$$c_h(u_h, v_h) = 0 \qquad \forall v_h \in S_{h\mathsf{p}}. \tag{5}$$

Let us note that if $u \in H^2(\Omega)$ is the exact solution of (1) then the consistency of c_h gives

$$c_h(u, v) = 0 \qquad \forall v \in H^2(\Omega, \mathscr{T}_h). \tag{6}$$

3. Residuum estimates

In this section we investigate the discretization error $u - u_h$ and define estimators giving some information about this error. Based on them we propose the *hp*-adaptation strategy.

3.1. Residuum definition

In order to introduce our adaptation strategy, we proceed to a functional representation of the DG method. Let X be a linear function space such that $u \in X$ and $u_h \in X$. It is equipped with a norm $\|\cdot\|_X$. (The space X does not need to be complete with respect to $\|\cdot\|_X$.) In our case, $X := H^2(\Omega, \mathscr{T}_h)$, the norm $\|\cdot\|_X$ will be specified later. Let X' denote the dual space to X.

Moreover, let $A_h : X \to X'$ be the nonlinear operator corresponding to c_h by

$$\langle A_h u, v\rangle := c_h(u, v), \qquad u, v \in X, \tag{7}$$

where $\langle \cdot, \cdot\rangle$ denotes the duality between X' and X. We define the dual norm by

$$\|A_h u\|_{X'} := \sup_{0 \neq v \in X} \frac{\langle A_h u, v\rangle}{\|v\|_X}. \tag{8}$$

Let $u \in H^2(\Omega) \subset X$ be the solution of (1). In virtue of (6) and (7), we have $A_h u = 0$. Therefore, the value

$$\mathcal{R}(u_h) := \|A_h u_h - A_h u\|_{X'} = \|A_h u_h\|_{X'} = \sup_{0 \neq v \in X} \frac{\langle A_h u_h, v\rangle}{\|v\|_X} = \sup_{0 \neq v \in X} \frac{c_h(u_h, v)}{\|v\|_X} \tag{9}$$

defines the *residuum error in the dual norm* of the approximate solution $u_h \in S_{h\mathsf{p}} \subset X$. The right-hand side of (9) depends only on u_h and not on u. However, its is impossible to evaluate $\mathcal{R}(u_h)$, since the supremum is taken over an infinite-dimensional space. Therefore, in our approach, we seek the maximum over some sufficiently large but finite dimension subspace of X.

3.2. Global and element residuum estimators

For each $K \in \mathscr{T}_h$ and each integer $p \geq 0$, we define the space

$$S_K^p \ := \ \{\phi_h \in X,\ \phi_h|_K \in P^p(K),\ \phi_h|_{\Omega\setminus K} = 0\}. \tag{10}$$

Obviously, $S_K^p \subset S_K^{p+1} \subset S_K^{p+2} \subset \dots,\ K \in \mathscr{T}_h$. Moreover, we put

$$S_{h\mathsf{p}}^+ := \{\phi \in X; \phi = \sum_{K \in \mathscr{T}_h} c_K \phi_K,\ c_K \in \mathbb{R},\ \phi_K \in S_K^{p_K+1},\ K \in \mathscr{T}_h\}. \tag{11}$$

Finally, we observe that $S_{h\mathsf{p}} \subset S_{h\mathsf{p}}^+$.

Now, we define the *element residuum estimator*

$$\eta_K(u_h) := \sup_{0 \neq \psi_h \in S_K^{p_K+1}} \frac{c_h(u_h, \psi_h)}{\|\psi_h\|_X} = \sup_{\psi_h \in S_K^{p_K+1}, \|\psi_h\|_X = 1} c_h(u_h, \psi_h), \qquad u_h \in X, \tag{12}$$

for each $K \in \mathscr{T}_h$ and the *global residuum estimator*

$$\eta(u_h) := \sup_{0 \neq \psi_h \in S^+_{h\mathsf{p}}} \frac{c_h(u_h, \psi_h)}{\|\psi_h\|_X} = \sup_{\psi_h \in S^+_{h\mathsf{p}}, \|\psi_h\|_X = 1} c_h(u_h, \psi_h) \quad u_h \in X, \tag{13}$$

which are easily computable quantities if $\|\cdot\|_X$ is suitably chosen, see [5].

Obviously, if $u \in X$ is the exact solution of (1) then consistency (6) implies $0 = \eta(u) = \eta_K(u),\ K \in \mathscr{T}_h$. Moreover, we have immediately a lower bound

$$\eta(u_h) \leq \mathcal{R}(u_h) = \|Au_h - Au\|_{X'}. \tag{14}$$

However, it is open if there exists an upper bound, i.e., $\mathcal{R}(u_h) \leq C\eta(u_h)$, where $C > 0$. This will be the subject of a further research.

Finally, we specify the choice of the norm $\|\cdot\|_X$. This norm is generated by the scalar product $(u,v)_X := (u,v)_{L^2(\Omega)} + \varepsilon \sum_{K \in \mathscr{T}_h} (\nabla u, \nabla v)_{L^2(K)},\ u, v \in X$, where ε is a constant reflecting a ratio between “diffusion” and “convection”. For the case of the scalar equation (1) we put $\varepsilon \approx |\boldsymbol{K}(\cdot)|/|\boldsymbol{f}(\cdot)|$.

Since the spaces S_K^p and $S_{K'}^{p'}$, $K, K' \in \mathscr{T}_h,\ K \neq K'$ are orthogonal with respect to $(\cdot,\cdot)_X$, we can show ([5]) that

$$\eta(u_h)^2 = \sum_{K \in \mathscr{T}_h} \eta_K(u_h)^2. \tag{15}$$

Therefore, it is sufficient to evaluate the element residuum estimators η_K for each $K \in \mathscr{T}_h$. This is a standard task of seeking a constrained extrema over $S_K^{p_K+1}$ with the constrain $\|\psi_h\|_X = 1$. This can be done directly very fast since the dimension of $S_K^{p_K+1},\ K \in \mathscr{T}_h$ is small, namely $\dim(S_K^{p_K+1}) = (p_k+2)(p_K+3)/2$ for $d = 2$.

Our interest is to find adaptively a mesh $\mathscr{T}_h$, a set $\{p_K, K \in \mathscr{T}_h\}$ and the corresponding solution $u_h \in S_{h\mathsf{p}}$ such that the number of degree of freedom N_h $(= \dim(S_{h\mathsf{p}}))$ is small and

$$\eta(u_h) \leq \omega, \tag{16}$$

where $\omega > 0$ is a given tolerance.

In order to define an adaptive algorithm, we require that

$$\eta_K(u_h) \leq \omega (\#\mathscr{T}_h)^{-1/2} \qquad \forall K \in \mathscr{T}_h, \tag{17}$$

where $\#\mathscr{T}_h$ denotes the number of elements of $\mathscr{T}_h$. Obviously, if (17) is satisfied then, due to (15), condition (16) is valid and the adaptation process stops. Otherwise, we mark for refinement all $K \in \mathscr{T}_h$ violating (17).

Furthermore, all marked elements will be refined either by h- or by p-adaptation, namely, either we split a given mother element K into four daughter elements or we increase the degree of polynomial approximation for a given element. Thus new mesh $\mathscr{T}_{\hat{h}}$ and new set $\{\hat{p}_K, K \in \mathscr{T}_{\hat{h}}\}$ are created. We interpolate the old solution on a new mesh and perform the next adaptation step till (16) is valid.

3.3. Regularity indicator

The estimation of the regularity of the solution is an essential key of any *hp*-adaptation strategy. Our approach is based on a measure of inter-element jumps. Numerical analysis [4] carried out for scalar convection-diffusion equation gives

$$\sum_{K\in\mathscr{T}_h}\int_{\partial K}[\![u_h-u]\!]^2\,\mathrm{d}S=\sum_{K\in\mathscr{T}_h}\int_{\partial K}[\![u_h]\!]^2\,\mathrm{d}S\le C\sum_{K\in\mathscr{T}_h}h_K^{2\mu_K-1}|u|^2_{H^{s_K}(\Omega)},\tag{18}$$

where u and u_h are the exact and the approximate solutions, respectively, $C>0$ is a constant independent of h and $\mu_K=\min(p_K+1,s_K)$. Moreover, p_K is the degree of the polynomial approximation and s_K is the integer degree of local regularity of u, i.e., $u|_K\in H^{s_K}(K),\ K\in\mathscr{T}_h$. The a priori error estimates (18) imply that if the exact solution is sufficiently regular then the p-adaptation (increasing of the degree of approximation) yields to a higher decrease of the error. Otherwise, h-adaptation (element splitting) is more efficient.

Furthermore, the numerical experiments indicates that

$$\int_{\partial K}[\![u_h-u]\!]^2\,\mathrm{d}S=\int_{\partial K}[\![u_h]\!]^2\,\mathrm{d}S\approx Ch_K^{2\mu_K-1}|u|^2_{H^{s_K}(\Omega)},\ K\in\mathscr{T}_h.\tag{19}$$

Based on relation (19), we propose the *regularity indicator*

$$g_K(u_h):=\frac{\int_{\partial K\cap\Omega}[\![u_h]\!]^2\,\mathrm{d}S}{|K|h_K^{2p_K-2}},\quad K\in\mathscr{T}_h,\tag{20}$$

where $|K|$ is the area of $K\in\mathscr{T}_h$. If the exact solution is sufficiently regular, i.e., $s_K\ge p_K+1$, then $g_K(u_h)\approx O\left(h_K^{2p_K+1}/(h_K^2h_K^{2p_K-2})\right)=O(h_K)$. On the other hand, if the exact solution is not sufficiently regular, i.e., $s_K<p_K+1$ ($\Leftrightarrow s_K\le p_K$), then $g_K(u_h)\approx O\left(h_K^{2s_K-1}/(h_K^2h_K^{2p_K-2})\right)=O(h_K^{2\delta-1})$, where $\delta=s_K-p_k\le 0$. Then we use the following strategy

$$\begin{array}{lll} g_K(u_h)\le 1 & \Rightarrow \text{solution is regular} & \Rightarrow \text{p-refinement},\\ g_K(u_h)>1 & \Rightarrow \text{solution is irregular} & \Rightarrow \text{h-refinement},\end{array}\qquad K\in\mathscr{T}_h.\tag{21}$$

Finally, let us note, that on the basis of numerical experiments we use a small modification of (20), namely

$$\tilde g_K(u_h):=\frac{\int_{\partial K\cap\Omega}[\![u_h]\!]^2\,\mathrm{d}S}{|K|h_K^{2p_K-4}},\quad K\in\mathscr{T}_h,\tag{22}$$

which is more efficient than (21).

4. Numerical experiments

We present several numerical examples which demonstrate a performance of the presented *hp*-DGFE method. The DGFE discretization (5) leads to a nonlinear algebraic system which is solved iteratively with the aid of a Newton-like method.

4.1. Linear equation with boundary layers

We consider the scalar linear convection-diffusion equation (similarly as in [3])

$$-\varepsilon\triangle u - \frac{\partial u}{\partial x_1} - \frac{\partial u}{\partial x_2} = g \quad \text{in } \Omega := (0,1)^2, \tag{23}$$

where $\varepsilon > 0$ is a constant diffusion coefficient. We prescribe a Dirichlet boundary condition on the whole $\partial\Omega$. The source term g and the boundary condition are chosen so that the exact solution has the form

$$u(x_1, x_2) = (c_1 + c_2(1 - x_1) + \exp(-x_1/\varepsilon))\,(c_1 + c_2(1 - x_2) + \exp(-x_2/\varepsilon)) \tag{24}$$

with $c_1 = -\exp(-1/\varepsilon)$, $c_2 = -1 - c_1$. The solution contains two boundary layers along $x_1 = 0$ and $x_2 = 0$, whose width is proportional to ε. Here we consider $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-3}$.

The computation started on a uniform triangular grid with mesh spacing $h = 1/8$ and with piecewise linear approximation. The *hp*-DGFE method was applied with $\omega = 10^{-4}$ till the algorithm was finished. Tables 1 and 2 show the computational errors $\|e_h\|_{L^2(\Omega)}$ and $\|e_h\|_X$ for each level of the *hp*-adaptation. Moreover, the tables present the *experimental order of convergence* (EOC) with defined for each pair of successive adaptation levels l and $l+1$ by

$$\text{EOC} = \frac{\log\|e_{h_{l+1}}\| - \log\|e_{h_l}\|}{\log(1/\sqrt{N_{h_{l+1}}}) - \log(1/\sqrt{N_{h_l}})}, \qquad l = 1, 2, \ldots, \tag{25}$$

where h_l and h_{l+1} denotes the corresponding *hp*-meshes and $N_h = \dim(S_{h\mathsf{p}})$. Finally, these tables contain the value of the global residuum estimator $\eta(u_h)$ given by (13) and the *"effectivity index"* $i_{\text{eff}} := \eta(u_h)/\|e_h\|_X$. Let us not that i_{eff} is not the standard effectivity index since η is an estimation of the error in the dual norm whereas $\|e_h\|_X$ is the error in the primal norm.

lev	$\#\mathscr{T}_h$	N_h	$\|e_h\|_{L^2(\Omega)}$	EOC	$\|e_h\|_X$	EOC	$\eta(u_h)$	i_{eff}
0	128	384	6.19E-02	–	3.93E-01	–	1.04E+00	2.65
1	128	768	3.46E-02	1.68	3.91E-01	0.01	6.09E-01	1.56
2	128	1240	1.92E-02	2.46	2.52E-01	1.84	3.41E-01	1.35
3	158	1950	7.03E-03	4.44	1.21E-01	3.25	1.63E-01	1.35
4	236	3432	1.56E-03	5.33	3.72E-02	4.16	4.83E-02	1.30
5	380	6304	1.88E-04	6.95	6.93E-03	5.53	7.41E-03	1.07
6	554	10418	1.44E-05	10.24	7.86E-04	8.67	8.40E-04	1.07
7	776	17116	7.15E-07	12.09	5.76E-05	10.53	5.67E-05	0.98

Table 1: Problem (23) – (24) with $\varepsilon = 10^{-2}$: computational errors, estimator $\eta(u_h)$ and index i_{eff}.

lev	$\#\mathscr{T}_h$	N_h	$\|e_h\|_{L^2(\Omega)}$	EOC	$\|e_h\|_X$	EOC	$\eta(u_h)$	i_{eff}
0	128	384	1.89E-02	–	2.63E-02	–	6.47E-01	24.64
1	128	768	1.76E-02	0.20	5.15E-01	-8.59	5.28E-01	1.02
2	146	1172	1.82E-02	-0.17	5.27E-01	-0.11	6.20E-01	1.18
3	206	2040	1.58E-02	0.53	4.53E-01	0.55	6.61E-01	1.46
4	368	4414	1.24E-02	0.63	3.89E-01	0.39	5.46E-01	1.41
5	920	11412	7.98E-03	0.92	3.04E-01	0.52	4.19E-01	1.38
6	1982	25050	2.93E-03	2.54	1.54E-01	1.72	2.06E-01	1.34
7	4016	50528	5.78E-04	4.63	4.80E-02	3.33	6.06E-02	1.26
8	7217	91242	6.56E-05	7.36	9.32E-03	5.55	1.14E-02	1.22
9	12050	176863	6.32E-06	7.07	1.32E-03	5.92	1.69E-03	1.28
10	23684	368615	3.99E-07	7.53	8.48E-05	7.47	9.46E-05	1.11

Table 2: Problem (23) – (24) with $\varepsilon = 10^{-3}$: computational errors, estimator $\eta(u_h)$ and index i_{eff}.

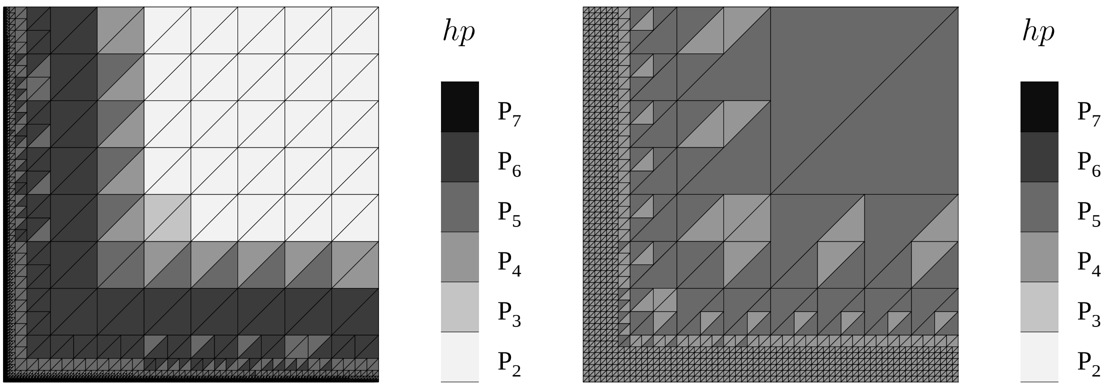

Figure 1: The final grid with the corresponding degrees of polynomial approximation, the whole domain (left) and its detail $(0, 1/16) \times (0, 1/16)$ (right) for $\varepsilon = 10^{-3}$.

We observe that the computational error e_h converge exponentially in both presented norms. Moreover, we found that the effectivity index i_{eff} is very close to one for increasing N_h. However, a theoretical justification of this favorable property is quite open and it will be a subject of the further research.

Furthermore, Figure 1 shows the final hp-grid obtained with the aid of the hp-DGFE algorithm for $\varepsilon = 10^{-3}$. We observe that the h-adaptation was carried out in regions with the boundary layers are presented. On the other hand, the p-adaptation appears in regions where the solution is regular.

Finally, let us note that the presented strategy is not too efficient for problems with boundary layers since our h-adaptation is only isotropic. More efficient is the use of an anisotropic mesh adaptation.

lev	$\#\mathscr{T}_h$	N_h	$\|e_h\|_{L^2(\Omega)}$	EOC	$\|e_h\|_X$	EOC	$\eta(u_h)$	i_{eff}
0	128	384	8.28E-03	–	9.13E-03	–	8.24E-02	9.03
1	128	768	1.83E-03	4.35	2.71E-03	3.50	1.95E-02	7.20
2	128	1272	6.92E-04	3.86	1.64E-03	2.00	7.00E-03	4.27
3	128	1522	7.18E-04	-0.41	1.42E-03	1.58	3.29E-03	2.31
4	131	1693	3.10E-04	15.81	9.93E-04	6.75	1.62E-03	1.64
5	143	2095	1.53E-04	6.60	7.59E-04	2.52	6.37E-04	0.84
6	161	2540	6.86E-05	8.35	5.39E-04	3.56	4.98E-04	0.92
7	167	2661	2.67E-05	40.15	3.74E-04	15.57	3.45E-04	0.92
8	203	3383	1.02E-05	8.02	2.63E-04	2.92	2.32E-04	0.88
9	206	3449	4.68E-06	80.24	1.87E-04	35.53	1.61E-04	0.86
10	215	3632	3.35E-06	12.94	1.32E-04	13.36	1.14E-04	0.86
11	227	3854	3.14E-06	2.17	9.37E-05	11.65	8.06E-05	0.86

Table 3: Problem (26): computational errors, estimator $\eta(u_h)$ and index i_{eff}.

4.2. Nonlinear convection-diffusion equation

We consider the scalar nonlinear convection-diffusion equation

$$-\nabla \cdot (\boldsymbol{K}(u)\nabla u) - \frac{\partial u^2}{\partial x_1} - \frac{\partial u^2}{\partial x_2} \quad = \quad g \qquad \text{in } \Omega := (0,1)^2, \tag{26}$$

where $\boldsymbol{K}(u)$ is the nonsymmetric matrix given by

$$\boldsymbol{K}(u) = \varepsilon \begin{pmatrix} 2 + \arctan(u) & (2 - \arctan(u))/4 \\ 0 & (4 + \arctan(u))/2 \end{pmatrix}. \tag{27}$$

We put $\varepsilon = 10^{-4}$ and prescribe a Dirichlet boundary condition on the whole $\partial\Omega$. The source term g and the boundary condition are chosen so that the exact solution is $u(x_1, x_2) = (x_1^2 + x_2^2)^{-3/4} x_1 x_2 (1 - x_1)(1 - x_2)$. This function has a singularity at $x_1 = x_2 = 0$ and it is possible to show (see [1]) that $u \in H^\kappa(\Omega)$, $\kappa \in (0, 3/2)$, where $H^\kappa(\Omega)$ denotes the Sobolev-Slobodetskii space of functions with "non-integer derivatives". Numerical examples presented in [6], carried out for a little different problem, show that this singularity avoids to achieve the orders of convergence better than $O(h^{3/2})$ in the L^2-norm and $O(h^{1/2})$ in the H^1-seminorm for any degree of polynomial approximation. Nevertheless, the exact solution is regular outside of the singularity.

The computation was started on a uniform triangular grid with mesh spacing $h = 1/8$ and with piecewise linear approximation. Then the *hp*-DGFE method was applied with $\omega = 10^{-4}$ till the algorithm was finished. Table 3 shows the computational errors $\|e_h\|_{L^2(\Omega)}$ and $\|e_h\|_X$ for each level of the *hp*-adaptation including EOC, the global residuum estimator $\eta(u_h)$ and the effectivity index i_{eff}. We observe that the adaptive algorithm significantly reduces the computational error e_h with a small N_h. Moreover, the effectivity index i_{eff} converges to a constant value.

5. Conclusion and outlook

We presented a new *hp*-adaptive method for the solution of convection-diffusion problems. This approach is based on a combination of the residuum estimator and the regularity indicator. Numerical experiments indicate its efficiency and a reliability. The subject of the further research will be numerical analysis of the presented method, and an extension to unsteady problems.

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