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A MULTILEVEL CORRECTION TYPE OF ADAPTIVE FINITE ELEMENT METHOD FOR STEKLOV EIGENVALUE PROBLEMS

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Abstract

Adaptive finite element method based on multilevel correction scheme is proposed to solve Steklov eigenvalue problems. In this method, each adaptive step involves solving associated boundary value problems on the adaptive partitions and small scale eigenvalue problems on the coarsest partitions. Solving eigenvalue problem in the finest partition is not required. Hence the efficiency of solving Steklov eigenvalue problems can be improved to the similar efficiency of the adaptive finite element method for the associated boundary value problems. The efficiency of the proposed method is also investigated by a numerical experiment.

1. Introduction

The main goal of this paper is to present a multilevel correction type of adaptive finite element method (AFEM) for Steklov eigenvalue problems. These type of eigenvalue problems arise in a number of applications (see, e.g., [1, 6, 7, 10, 11, 15]). The analysis of finite element methods for Steklov eigenvalue problems have been given in [2, 3, 8, 9, 14, 16, 17] and the references cited therein.

In this paper, we are concerned with the following model problem

$$\left\{\begin{array}{rcll} -\Delta u + u &=& 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &=& \lambda u & \text{on } \partial\Omega, \end{array}\right. \tag{1}$$

where $\Omega \subset \mathcal{R}^2$ is a bounded polygonal domain and $\frac{\partial}{\partial \nu}$ the outward normal derivative on $\partial\Omega$.

As we know, the AFEM is a very useful and efficient way for solving eigenvalue problems. Recently, one active topic is to use AFEM to solve the Steklov eigenvalue problems (see, e.g., [4, 13, 21]). The purpose of this paper is to propose and analyze a multilevel correction type of AFEM to solve Steklov eigenvalue problems based on the recent work on multi-level correction method (see [18, 23]). In the new scheme, the cost of solving eigenvalue problems is almost the same as solving the associated boundary value problems.

The corresponding weak form of the problem (1) is:
Find $\lambda \in \mathcal{R}$ and $u \in H^1(\Omega)$ such that $\|u\|_b = 1$ and

$$a(u,v) = \lambda b(u,v) \qquad \forall v \in H^1(\Omega), \tag{2}$$

where

$$a(u,v) = \int_\Omega \big(\nabla u \nabla v + uv\big) d\Omega, \tag{3}$$

$$b(u,v) = \int_{\partial\Omega} uv ds, \ \|u\|_b = b(u,u)^{\frac{1}{2}}. \tag{4}$$

Evidently the bilinear form $a(\cdot,\cdot)$ is symmetric, continuous and coercive over the product space $H^1(\Omega) \times H^1(\Omega)$.

From [5], we know the eigenvalue problem (2) has an eigenvalue sequence $\{\lambda_j\}$:

$$0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots, \quad \lim_{k\to\infty} \lambda_k = \infty,$$

and the associated eigenfunctions

$$u_1, u_2, \cdots, u_j, \cdots,$$

where $b(u_i, u_j) = \delta_{ij}$. In the sequence $\{\lambda_j\}$, the λ_j are repeated according to their geometric multiplicity.

An outline of the paper goes as follows. In section 2, we introduce finite element method for the Steklov eigenvalue problem and the corresponding error estimates. A multilevel correction type of AFEM for Steklov eigenvalue problems is given in section 3. In section 4, a numerical example is presented to demonstrate the efficiency of the AFEM and some concluding remarks are given in the last section.

2. Discretization by finite element method and error estimates

In this paper, the letter C (with or without subscripts) denotes a generic positive constant which may be different at different occurrences. For convenience, the symbols $\lesssim$, $\gtrsim$ and $\approx$ will be used in this paper. That $x_1 \lesssim y_1, x_2 \gtrsim y_2$ and $x_3 \approx y_3$, mean that $x_1 \le C_1 y_1$, $x_2 \ge c_2 y_2$ and $c_3 x_3 \le y_3 \le C_3 x_3$ for some constants C_1, c_2, c_3 and C_3 that are independent of mesh sizes.

Set $V := H^1(\Omega)$. Let us define the finite element approximation of (2). First we generate a shape-regular decomposition of the computing domain $\Omega \subset \mathcal{R}^d$ $(d = 2, 3)$ into triangles for $d = 2$ (tetrahedrons for $d = 3$). The diameter of a cell $T \in \mathcal{T}_h$ is denoted by h_T. The mesh diameter h describes the maximum diameter of all cells $T \in \mathcal{T}_h$. Based on the partition $\mathcal{T}_h$, we construct the linear finite element space denoted by $V_h \subset V$. Let $\mathcal{E}_h$ denote the set of interior faces (edges or sides) of $\mathcal{T}_h$ and $\mathcal{E}_{\partial\Omega}$ the faces on the boundary $\partial\Omega$.

Therefore we can define the approximation of eigenpair (λ, u) of (2) by the finite element method as:

Find $(\lambda_h, u_h) \in \mathcal{R} \times V_h$ such that $b(u_h, u_h) = 1$ and

$$a(u_h, v_h) \quad = \quad \lambda_h b(u_h, v_h) \qquad \forall v_h \in V_h. \tag{5}$$

Similarly, we know from [5] the eigenvalue problem (5) has eigenvalues

$$0 < \lambda_{1,h} \leq \lambda_{2,h} \leq \cdots \leq \lambda_{k,h} \leq \cdots \leq \lambda_{N_h,h},$$

and the corresponding eigenfunctions

$$u_{1,h}, u_{2,h}, \cdots, u_{k,h}, \cdots, u_{N_h,h},$$

where $b(u_{i,h}, u_{j,h}) = \delta_{ij}, 1 \leq i, j \leq N_h$ (N_h is the dimension of the finite element space V_h).

Let P_h be the finite element projection operator of V onto V_h defined by

$$a(w - P_h w, v) = 0 \quad \forall w \in V \text{ and } \forall v \in V_h. \tag{6}$$

Obviously

$$\|P_h w\|_1 \leq \|w\|_1 \quad \forall w \in V. \tag{7}$$

Define $\eta_a(h)$ as

$$\eta_a(h) = \sup_{f \in H^{1/2}(\partial\Omega), \|f\|_{1/2,\partial\Omega}=1} \inf_{v \in V_h} \|Kf - v\|_1, \tag{8}$$

where the operator $K : H^{-1/2}(\partial\Omega) \mapsto V$ is defined as

$$a(Kf, v) \quad = \quad b(f, v) \quad \forall f \in H^{-1/2}(\partial\Omega) \text{ and } \forall v \in V. \tag{9}$$

For the aim of convergence analysis by the finite element method, we introduce the following regularity result for the boundary value problem (9).

Lemma 2.1. *([9, (4.10)], [7, Proposition 4.4]) For the Steklov type boundary value problem (9), if $f \in L^2(\partial\Omega)$, then $Kf \in H^{1+\gamma/2}(\Omega)$ and*

$$\|Kf\|_{1+\gamma/2} \leq C\|f\|_b, \tag{10}$$

where $\gamma = 1$ if Ω is convex and $\gamma < \frac{\pi}{\omega}$ (with ω being the largest inner angle of Ω). Furthermore, if $f \in H^{\frac{1}{2}}(\partial\Omega)$, we have $Kf \in H^{1+\gamma}(\Omega)$ and

$$\|Kf\|_{1+\gamma} \leq C\|f\|_{1/2,\partial\Omega}. \tag{11}$$

In order to derive the error estimate of eigenpair approximation in the norm $\|\cdot\|_{-1/2,\partial\Omega}$, we need the following error estimate of the finite element projection operator P_h in the norm $\|\cdot\|_{-1/2,\partial\Omega}$.

Lemma 2.2. *([5, Lemma 3.3 and Lemma 3.4], [3, Proposition 3.1])*

$$\eta_a(h) \;=\; o(1) \quad \text{as } h \to 0, \tag{12}$$

and

$$\|w - P_h w\|_{-1/2,\partial\Omega} \;\lesssim\; \eta_a(h)\|w - P_h w\|_1 \quad \forall w \in V. \tag{13}$$

Proof. In order to obtain the error estimate in $\|\cdot\|_{-1/2,\partial\Omega}$, we chose a function $\varphi \in H^{1/2}(\partial\Omega)$ such that $\|\varphi\|_{1/2,\partial\Omega} = 1$ and $\|u - P_h u\|_{-1/2,\partial\Omega} = b(u - P_h u, \varphi)$. Then we have

$$\begin{aligned} \|u - P_h u\|_{-1/2,\partial\Omega} &= b(\varphi, u - P_h u) = a(K\varphi, u - P_h u) \\ &= a(K\varphi - \psi_h, u - P_h u) \quad \forall \psi_h \in V_h. \end{aligned} \tag{14}$$

This means we obtain the desired result (13) and the proof is complete. □

From the minimum-maximum principle [5], the following upper bound result holds

$$\lambda_i \le \lambda_{i,h}, \quad i = 1, 2, \cdots, N_h.$$

Let $M(\lambda_i)$ denote the eigenspace corresponding to the eigenvalue λ_i which is defined by

$$\begin{aligned} M(\lambda_i) \;=\; \big\{ & w \in V : w \text{ is an eigenfunction of (2) corresponding to } \lambda_i \\ & \text{and } \|w\|_b = 1 \big\}. \end{aligned} \tag{15}$$

Then we define

$$\delta_h(\lambda_i) = \sup_{w \in M(\lambda_i)} \inf_{v \in V_h} \|w - v\|_1. \tag{16}$$

For the eigenpair approximations by finite element method, there exist the following error estimates.

Proposition 2.1. *([5, P. 699], [3] and [7])*
(i) For any eigenfunction approximation $u_{i,h}$ of (5) ($i = 1, 2, \cdots, N_h$), there is an eigenfunction u_i of (2) corresponding to λ_i such that $\|u_i\|_b = 1$ and

$$\|u_i - u_{i,h}\|_1 \;\le\; C_i \delta_h(\lambda_i). \tag{17}$$

Furthermore,

$$\|u_i - u_{i,h}\|_{-1/2,\partial\Omega} \le C_i \eta_a(h) \|u_i - u_{i,h}\|_1. \tag{18}$$

(ii) For each eigenvalue, we have

$$\lambda_i \le \lambda_{i,h} \le \lambda_i + C_i \delta_h^2(\lambda_i). \tag{19}$$

3. Adaptive multilevel correction algorithm for eigenvalue problem

In this section, we present a residual type of a posteriori error estimate and give the multilevel correction type of AFEM for Steklov eigenvalue problems.

We follow the classic routine to define an a posteriori error estimator for (5) (see [4, 13]). Let us define the element residual $\mathcal{R}_K(u_h)$

$$\mathcal{R}_K(u_h) := u_h \quad \text{in} \ \ T \in \mathcal{T}_h, \tag{20}$$

and the jump residual $\mathcal{J}_E(u_h)$ by

$$\mathcal{J}_E(u_h) := \begin{cases} \frac{1}{2}\big(\nabla u_h^+ \cdot \nu^+ + \nabla u_h^- \cdot \nu^-\big) := \frac{1}{2}[[\nabla u_h]]_E \cdot \nu_E & \text{for} \ \ E \in \mathcal{E}_h, \\ \nabla u_h \cdot \nu - \lambda_h u_h & \text{for} \ \ E \in \mathcal{E}_{\partial\Omega}, \end{cases}$$

where E is the common side of elements T^+ and T^- with outward normals ν^+ and ν^-, $\nu_E = \nu^-$, and $\omega_E := T^+ \cap T^-$ that share the same edge E.

For the element $T \in \mathcal{T}_h$, we define the local error indicator $\eta_h(u_h, T)$ by

$$\eta_h(u_h, T) := \left(h_T^2 \|\mathcal{R}_T(u_h)\|_{0,T}^2 + \sum_{E \in \mathcal{E}_h, E \subset \partial T} h_E \|\mathcal{J}_E(u_h)\|_{0,E}^2 \right)^{1/2}, \tag{21}$$

and the error indicator for a subdomain $\omega \subset \Omega$ by

$$\eta_h(u_h, \omega) := \left(\sum_{T \in \mathcal{T}_h, T \subset w} \eta_h^2(u_h, T) \right)^{1/2}. \tag{22}$$

Thus $\eta_h(u_h, \Omega)$ denotes the error estimator of u_h with respect to $\mathcal{T}_h$.

Now we summarize the reliability and the efficiency of the a posterior error estimator (see, e.g., [4, 13]):

Lemma 3.1. *([4, 13]) The error estimator (22) has the reliability*

$$\|u - u_h\|_1 \lesssim \Big\{ \eta_h(u_h, \Omega) + \frac{\lambda + \lambda_h}{2} \|u - u_h\|_{0,\partial\Omega} \Big\}. \tag{23}$$

Furthermore, the error estimator has the efficiency

a) For $T \in \mathcal{T}_h$, if $\mathcal{E}_T \cap \partial\Omega = \emptyset$

$$\eta_h(u_h, T) \lesssim \|u - u_h\|_{1,\omega_T}^2, \tag{24}$$

where ω_T contains all the elements that share at least a side with T.

b) For $T \in \mathcal{T}_h$, if $\mathcal{E}_T \cap \partial\Omega \neq \emptyset$

$$\eta_h(u_h, T) \lesssim \Big\{ \|u - u_h\|_{1,\omega_T} + \sum_{E \in \mathcal{E}_T \cap \partial\Omega} h_E \|\lambda u - \lambda_h u_h\|_{0,E} \Big\}. \tag{25}$$

The adaptive procedure consists of loops of the form

Solve → Estimate → Mark → Refine.

Now we state our adaptive finite element method to compute the Steklov eigenvalue problem (5) in the multilevel correction framework.

Adaptive Algorithm C

1. Pick up an initial mesh $\mathcal{T}_{h_0}$ with mesh size h_0.
2. Construct the finite element space V_{h_0} and solve the following eigenvalue problem to get the discrete solution $(\lambda_{h_0}, u_{h_0}) \in \mathcal{R} \times V_{h_0}$ such that $\|u_{h_0}\|_b = 1$ and

$$a(u_{h_0}, v_{h_0}) \;=\; \lambda_{h_0} b(u_{h_0}, v_{h_0}) \qquad \forall v_{h_0} \in V_{h_0}. \tag{26}$$

3. Let $k = 0$.
4. Compute the local error indicators $\eta_{h_k}(u_{h_k}, T)$.
5. Construct $\widehat{\mathcal{T}}_{h_k} \subset \mathcal{T}_{h_k}$ by **Marking Strategy E** and parameter θ.
6. Refine $\mathcal{T}_{h_k}$ to get a new conforming mesh $\mathcal{T}_{h_{k+1}}$ by procedure **Refine**.
7. Solve the following source problem on $\mathcal{T}_{h_{k+1}}$ for the discrete solution $\widetilde{u}_{h_{k+1}} \in V_{h_{k+1}}$:

$$a(\widetilde{u}_{h_{k+1}}, v_{h_{k+1}}) \;=\; \lambda_{h_k} b(u_{h_k}, v_{h_{k+1}}) \qquad \forall v_{h_k} \in V_{h_k}. \tag{27}$$

8. Construct the new finite element space $V_{h_0,h_{k+1}} = V_{h_0} + \mathrm{span}\{\widetilde{u}_{h_{k+1}}\}$ and solve the eigenvalue problem to get the solution $(\lambda_{h_{k+1}}, u_{h_{k+1}}) \in \mathcal{R} \times V_{h_0,h_{k+1}}$ such that $\|u_{h_{k+1}}\|_b = 1$ and

$$a(u_{h_{k+1}}, v_{h_{h_0,h_{k+1}}}) = \lambda_{h_{k+1}} b(u_{h_{k+1}}, v_{h_0,h_{k+1}}) \quad \forall v_{h_0,h_{k+1}} \in V_{h_0,h_{k+1}}. \tag{28}$$

9. Let $k = k + 1$ and go to Step 4.

Here we use the iterative or recursive bisection (see, e.g., [19, 22]) of elements with the minimal refinement condition in the procedure **REFINE**. The **Marking Strategy** adopted in **Adaptive Algorithm** C was introduced by Dörfler [12] and Morin et al. [20] and can be defined as follows.

Marking Strategy E

Given a parameter $0 < \theta < 1$:
1. Construct a minimal subset $\widehat{\mathcal{T}}_h$ from $\mathcal{T}_h$ by selecting some elements in $\mathcal{T}_h$ such that

$$\sum_{T\in\widehat{\mathcal{T}}_h} \eta_h^2(u_h, T) \geq \theta\eta_h^2(u_h, \Omega).$$

2. Mark all the elements in $\widehat{\mathcal{T}}_h$.

Now we state some convergence results of this type of AFEM for Steklov eigenvalue problems.

Lemma 3.2. *([18]) Assume the current eigenpair approximation* $(\lambda_{h_k}, u_{h_k}) \in \mathcal{R}\times V_{h_k}$ *has the following error estimates*

$$\begin{aligned} \|u - u_{h_k}\|_1 &\lesssim \varepsilon_{h_k}(\lambda), & (29)\\ \|u - u_{h_k}\|_{-1/2,\partial\Omega} &\lesssim \eta_a(h_0)\|u - u_{h_k}\|_1, & (30)\\ |\lambda - \lambda_{h_k}| &\lesssim \varepsilon_{h_k}^2(\lambda). & (31) \end{aligned}$$

Then after one adaptive step in **Adaptive Algorithm** *C, the resultant approximation* $(\lambda_{h_{k+1}}, u_{h_{k+1}}) \in \mathcal{R}\times V_{h_{k+1}}$ *has the following error estimates*

$$\begin{aligned} \|u - u_{h_{k+1}}\|_1 &\lesssim \varepsilon_{h_{k+1}}(\lambda), & (32)\\ \|u - u_{h_{k+1}}\|_{-1/2,\partial\Omega} &\lesssim \eta_a(h_0)\|u - u_{h_k}\|_1, & (33)\\ |\lambda - \lambda_{h_{k+1}}| &\lesssim \varepsilon_{h_{k+1}}^2(\lambda), & (34) \end{aligned}$$

where $\varepsilon_{h_{k+1}}(\lambda) := \eta_a(h_0)\varepsilon_{h_k}(\lambda) + \varepsilon_{h_k}^2(\lambda) + \delta_{h_{k+1}}(\lambda)$.

Theorem 3.1. *([18]) Assume* $\eta_a(H) \gtrsim \delta_{h_1}(\lambda) \geq \delta_{h_2}(\lambda) \geq \cdots \geq \delta_{h_n}(\lambda)$. *The obtained eigenpair approximation* (λ_{h_n}, u_{h_n}) *after n adaptive steps in* **Adaptive Algorithm** *C has the error estimate*

$$\begin{aligned} \|u_{h_n} - u\|_1 &\lesssim \varepsilon_{h_n}(\lambda), & (35)\\ |\lambda_{h_n} - \lambda| &\lesssim \varepsilon_{h_n}^2(\lambda), & (36) \end{aligned}$$

where $\varepsilon_{h_n}(\lambda) = \sum_{k=1}^{n} \eta_a(h_0)^{n-k}\delta_{h_k}(\lambda)$.

4. Numerical results

In this section, we give a numerical example to illustrate the efficiency of the **Adaptive Algorithm** C for the model Steklov eigenvalue problem. We set the computing domain as the L-shape one $\Omega = (-1,-1)\times(-1,1)/[-1,0]\times[-1,0]$ and

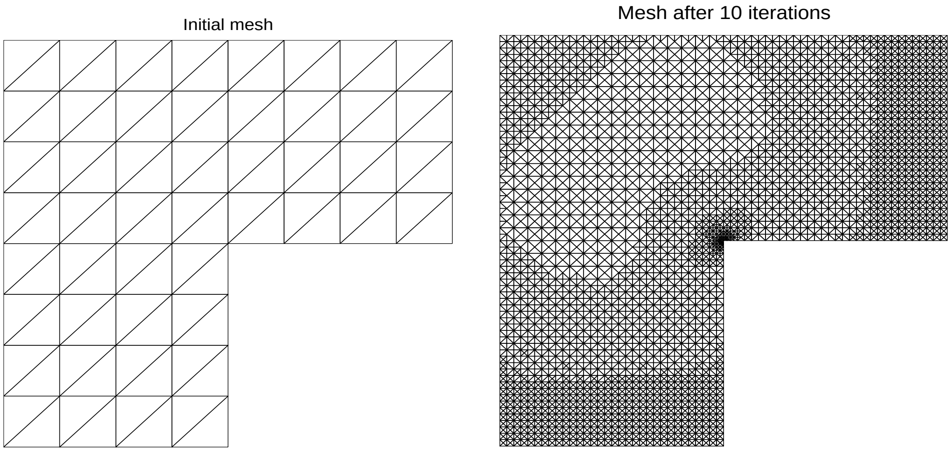

Figure 1: The initial triangulation and the one after 10 adaptive iterations.

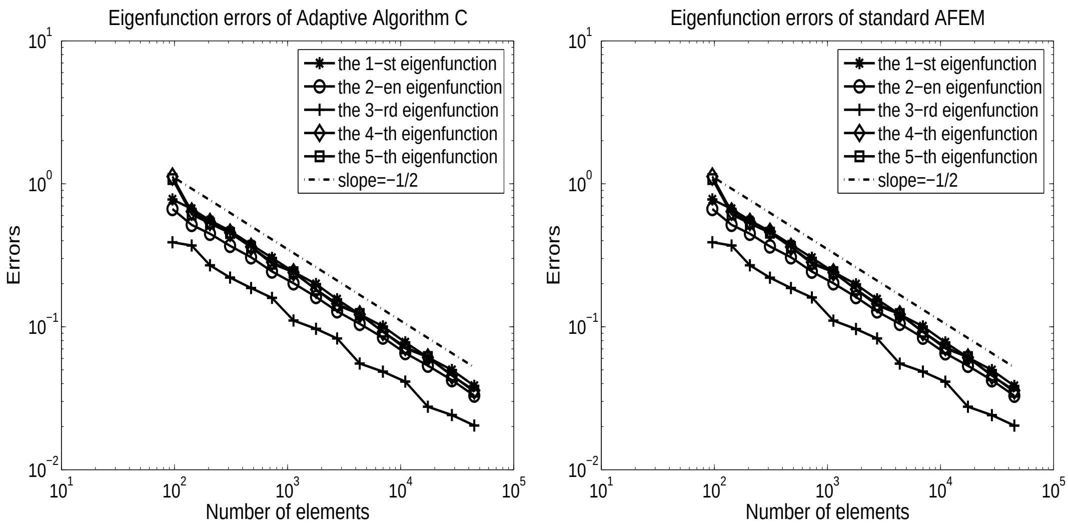

Figure 2: The a posteriori error estimates of eigenfunction approximations by **Adaptive Algorithm** C and standard AFEM.

compare the accuracy with the standard AFEM in [13]. Figure 1 shows the initial mesh and the mesh after 10 adaptive iterations of **Adaptive Algorithm** C. In order to check the efficiency of **Adaptive Algorithm** C, we compare the numerical results of **Adaptive Algorithm** C with those of standard AFEM. The corresponding numerical results are shown in Figure 2.

From the results presented in Figure 2, we find the accuracy of **Adaptive Algorithm** C is almost the same as the standard AFEM.

5. Concluding remarks

In this paper, we propose a type of AFEM for Steklov eigenvalue problems based on multilevel correction scheme. An numerical experiment is provided to demonstrate the efficiency of the AFEM. The convergence and optimality analysis should be the topic in our future work.

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