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UNIFORM L^1 ERROR BOUNDS FOR SEMI-DISCRETE FINITE ELEMENT SOLUTIONS OF EVOLUTIONARY INTEGRAL EQUATIONS

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Abstract

In this paper, we consider the second-order continuous time Galerkin approximation of the solution to the initial problem $u_t+\int_0^t \beta(t-s)Au(s)ds = 0,\ u(0) = v,\ t > 0$, where A is an elliptic partial-differential operator and $\beta(t)$ is positive, nonincreasing and log-convex on $(0,\infty)$ with $0 \le \beta(\infty) < \beta(0^+) \le \infty$. Error estimates are derived in the norm of $L^1_t(0,\infty;L^2_x)$, and some estimates for the first order time derivatives of the errors are also given.

1. Introduction

We study the discretization in space of the initial-boundary value problem (with $u_t = \partial u/\partial t$),

$$\begin{aligned} &u_t(t) + \int_0^t \beta(t-s)Au(s)ds = 0 \ \text{ in } \Omega,\ \text{ for } t > 0,\\ &u = 0 \ \text{ on } \partial\Omega,\ \text{ for } t > 0, \\ &u(0,\cdot) = u_0 \text{ in } \Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in R^d with a smooth boundary $\partial\Omega$, A is a linear selfadjoint positive definite second-order elliptic partial differential operator. For the real-valued kernel β we assume that

$$\begin{aligned} &\beta \in C(0,\infty) \cap L^1(0,1) \text{ is positive, nonincreasing, and}\\ &\text{log-convex on } (0,\infty), \text{with } 0 \le \beta(\infty) < \beta(0^+) \le \infty. \end{aligned} \tag{1.2}$$

Problems such as (1.1) occur, e.g., in modelling heat transfer in materials with memory. See, for example, [1, 2], and the references therein. The numerical solutions of the problems of type (1.1) have been studied in extensive literature. See, for instance, [3-14] for the finite element method, [15, 16, 17] the finite difference method, [18-21] the mixed finite element method, and [22] the finite volume element method. The kernel considered in [23, 24] is the positive type, and in, e.g., [25, 26] weakly singular, and [27, 28] completely monotonic, which is a particular case of (1.2) (cf. [29, Miller 1968, Lemma 2]). The kernels $\beta(t)$ with the condition (1.2) has been introduced in a very ingenious paper [30, Prüss 1987], which shows that $\beta(t)$ is completely positive (also, see [31, Clément and Nohel, 1979]), therefore $\beta(t)$ is of positive type (cf. [32, Clément and Mitidieri 1988, page 11]). Thus, the kernels satisfying (1.2) are intermediate between the classical completely monotonic and the positive type.

In the positive type, McLean and Thomée in [23] studied the finite element method, and obtained the error bounds for small t, and in [33] presented the exponential decay for a fully-discrete scheme in which the backward Euler method in combination with the convolution quadrature was used for the time discretization. In that paper, the kernel considered was under stronger assumptions and the exponential decay. Yan and Fairweather [24] analyzed the spline collocation method with the asymptotic error behavior, but with the decreasing exponentially weight. For the weakly singular kernel, the asymptotic error estimates were analyzed in, e.g., [25, 26]. Choi and MacCamy[25] gave the asymptotic error analysis in $L_t^2(0,\infty;L_x^2)$, the space of all measurable functions $f:[0,+\infty)\to L^2(\Omega)$ such that $\int_0^\infty \|f(t)\|^2dt<\infty$, where $\|\cdot\|$ denotes the norm in $L^2=L^2(\Omega)$. The results in [26] presented the optimal order error bounds for nonsmooth data $u_0\in L^2(\Omega)$.

In our earlier papers [27, 28], we considered the completely monotonic convolution kernel, and studied the backward Euler time discretization [28] and the finite element methods [27], respectively. The analysis in both of those papers was based on the methods of Carr and Hannsgen [34, 35] who considered the kernel satisfying

$$\begin{aligned}&\beta(t)\in C(R^+)\cap L^1(0,1),\ \text{not constant, and } \beta(t) \text{ is nonnegative,}\\&\text{nonincreasing, and convex on } R^+, 0<\beta(0^+)\le\infty, \text{and } \beta(\infty)\ge 0,\end{aligned}\tag{1.3}$$

rather than log-convex, and satisfies some additional conditions, for example, $-\beta'$ is convex. Carr and Hannsgen used the spectral theory for selfadjoint operators in Hilbert spaces and the results on the solutions of the parameter-dependent scalar Volterra equation,

$$\frac{du(t,\lambda)}{dt}+\lambda\int_0^t\beta(t-s)u(s,\lambda)ds=0,\ \ u(0,\lambda)=1.\tag{1.4}$$

For a general $\Omega\subset R^d$ we denote below by $\|\cdot\|_r$ the norms in the Sobolev spaces $H^r=H^r(\Omega)=W_2^r(\Omega)$, such that for any real-valued function v, and any positive integer r,

$$\|v\|_r = \|v\|_{H^r} = \left(\sum_{|\alpha| \le r} \|D^\alpha v\|^2 \right)^{1/2},$$

where $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_d)^{\alpha_d}$, $\alpha = (\alpha_1, \cdots, \alpha_d)$, denotes an arbitrary derivative with respect to x of order $|\alpha| = \sum_{j=1}^{d} \alpha_j$, so that the sum above contains all such derivatives of order at most r. As usual, we use the notation $H_0^1 = H_0^1(\Omega)$ to stand for the Sobolev space which consists of the functions v with $\nabla v = \operatorname{grad} v$ in $L^2(\Omega)$ and vanishing on $\partial \Omega$.

Under the assumption (1.3), and assuming that $-\beta'$ is convex, they have obtained the following two more estimates,

$$\int_0^\infty \|u(t)\| dt \le C \|u_0\|, \tag{1.5}$$

$$\int_0^\infty \|u'(t)\| dt \le C \|u_0\|_1, \tag{1.6}$$

where C is a positive constant independent of $u(t, x)$.

In our earlier work [27], we showed uniform $L_t^1(0, \infty; L_x^2)$ global error estimates for the linear finite element solutions, provided the initial data are appropriately smooth. The present paper is a continuation of the investigation in Xu [27], and the discretization (1.8) whose kernel satisfying (1.2) is considered here. We use the methods developed in Prüss [30, 36] to show uniform $L_t^1(0, \infty; L_x^2)$ global error estimates, and relaxes the regularity assumption on the initial data u_0.

It is noted that the approach of Prüss [30, 36] is quite different from Carr and Hannsgen [34, 35]. Prüss [30, 36] gave a new approach to questions such as those in Carr and Hannsgen [34, 35], avoiding a lot of messy estimates in overlapping cases. Indeed, by means of Laplace transform methods, operational calculus techniques and Banach algebra theory Prüss [30, 36] also derived (1.5), (1.6) and in particular the estimate

$$\int_0^\infty \|u''(t)\| dt \le C \|Au_0\|, \tag{1.7}$$

when $\beta(t)$ satisfies (1.2) and $\dot{\beta}(t)$ is absolutely continuous on $(0, \infty)$ in case $\mu = (\beta(0^+))^{\frac{1}{2}} < \infty$, and for (1.7) if $\mu + \kappa = \infty$, $-\int_0^1 \beta(t) \log t \; dt < \infty$ holds with $\kappa = -\dot{\beta}(0^+)/2\mu$; see [30, Theorem 11] and [36, Theorems 3.2 and 3.3].

For our aim, we assume that we are given a family S_h of the piecewise linear functions on a triangulation of Ω of standard type such that

$$\inf_{\chi \in S_h} \{ \|u - \chi\| + h \|u - \chi\|_1 \} \le C h^2 \|u\|_2 \quad \forall u \in H^2(\Omega) \cap H_0^1(\Omega),$$

where $A(\cdot, \cdot)$ denotes the bilinear form associated with A, and $(\cdot, \cdot)$ the inner product in $L^2(\Omega)$. Then, the spatially discrete problem is to find $u_h(t) \in S_h$ for $t \ge 0$ such that

$$
\begin{aligned}
&(u_{h,t}, \chi) + \int_0^t \beta(t-s) A(u_h(s), \chi) ds = 0, \text{ for } \chi \in S_h,\ t > 0, \\
&u_h(0) = u_{0h} \approx\ u_0.
\end{aligned}
\tag{1.8}
$$

For this problem, it was shown in [27] that with $u_{0h} = P_{0h} u_0$, where P_{0h} is the $L^2(\Omega)$-projection onto S_h, there holds

$$
\int_0^\infty \|u_h(t) - u(t)\| dt \le C h^2 \|u_0\|_4, \tag{1.9}
$$

where $\beta(t)$ satisfies (1.3) and $-\dot{\beta}(t)$ is convex (c.f. [27, Theorem 1.1 and Remark 2.3]).

Our purpose is to study the discretization (1.8) with the kernel (1.2) and derive some estimates similar to (1.9).

Theorem 1.1. *Suppose $\beta(t)$ satisfies (1.2) and $\dot{\beta}(t)$ is absolutely continuous in case $\mu < \infty$. Then, for the solutions of (1.1) and (1.8), with $u_{0h} = P_{0h} u_0$, we have*

$$
\begin{aligned}
&(i) \text{ if } \mu + \kappa = \infty, \ \int_0^\infty \|u_h(t) - u(t)\| dt \le C h^2 \|u_0\|_2, \\
&(ii) \text{ if } \mu + \kappa < \infty, \ \int_0^\infty \|u_h(t) - u(t)\| dt \le C h^2 \|u_0\|_3.
\end{aligned}
\tag{1.10}
$$

Theorem 1.1 partly recovers and extends results of Xu [27], since logarithmic convex functions are in particular convex, and relaxes the regularity assumption on the initial data u_0 to a sharper case given by McLean and Thomée [33, Theorem 5.1].

The exact solution of (1.1) can be represented as

$$
u(x,t) = \sum_{j=1}^\infty u_{\lambda_j}(t) (\varphi_j, u_0) \varphi_j, \text{ for } t \ge 0,
$$

where $u_{\lambda_j}(t)$ is the solution of the corresponding scalar problem (1.4) and $\{\lambda_j\}_1^\infty$ and $\{\varphi_j\}_1^\infty$ are the eigenvalues (in nondecreasing order) and ($L^2(\Omega)$ orthonormal) eigenfunctions of the associated elliptic problem

$$
Aw = \lambda w \text{ in } \Omega,\ w = 0 \text{ on } \partial\Omega.
$$

The eigenvalues are positive and tend to infinity when $j \to \infty$.

We now introduce the solution operator T of the elliptic problem:

$$
Aw = f \text{ in } \Omega,\ w = 0 \text{ on } \partial\Omega,
$$

by $w = Tf$. This operator can be represented by its eigenfunction expansion as

$$
Tf = \sum_{j=1}^\infty \lambda_j^{-1} (f, \varphi_j) \varphi_j,
$$

with λ_j and φ_j as above, and it follows at once that T is a bounded operator from $L^2(\Omega)$ into $H^2(\Omega) \cap H_0^1(\Omega)$. In terms of T we may write the problem (1.1) as

$$Tu_t(t) + \int_0^t \beta(t-s)u(s)ds = 0, \; u(0) = u_0. \tag{1.11}$$

Define the discrete elliptic operator $A_h : S_h \to S_h$ of A by $(A_h\psi, \chi) = A(\psi, \chi) \; \forall \, \psi$, $\chi \in S_h$. Related to the definition of the discrete elliptic operator A_h is that of the solution operator $T_h : L^2(\Omega) \to S_h$ of the discrete elliptic problem, namely

$$A(T_h f, \chi) = (f, \chi) \;\; \forall \chi \in S_h, f \in L^2(\Omega).$$

T_h approximates the exact solution operator $T = A^{-1} : L^2(\Omega) \to H_0^1(\Omega) \cap H^2(\Omega)$ in the sense that (see [37, Chapter 2])

$$\|T_h f - Tf\| \leq Ch^2 \|f\|, \text{ for } f \in L^2(\Omega). \tag{1.12}$$

The operator T is selfadjoint and positive definite on $L^2(\Omega)$, and T_h is selfadjoint, positive semidefinite on $L^2(\Omega)$ and positive definite on S_h. The semidiscrete problem (1.8) can now be written in the form

$$T_h u_{h,t}(t) + \int_0^t \beta(t-s)u_h(s)\, ds = 0, \; u_h(0) = u_{0h} \in S_h, \tag{1.13}$$

where u_{0h} is a suitable approximation to u_0. It is easy to see that the finite-dimensional problem (1.13) has a unique solution.

We also recall the elliptic regularity property $T : L^2(\Omega) \to H_0^1(\Omega) \cap H^2(\Omega)$ and the associated inequality

$$\|Tf\|_2 \leq C\|f\|, \text{ for } f \in L^2(\Omega). \tag{1.14}$$

We derive the estimates for the first order time derivative of the error as follows.

Theorem 1.2. *Assume that $\beta(t)$ satisfies (1.2) and let $\dot{\beta}(t)$ be absolutely continuous on (0, ∞) in case $\mu < \infty$. Then, for the solutions of (1.1) and (1.8), with $u_{0h} = P_{0h}u_0$, we have*

$$\int_0^\infty \|T_h u_h'(t) - Tu'(t)\| dt \leq Ch^2 \|Au_0\|. \tag{1.15}$$

In the following, $\hat{a}(s) = \hat{\beta}(s)/s$ denotes the Laplacian transform of $a(t) = \int_0^t \beta(\tau)\, d\tau$. We remark that the error estimate in Theorem 1.1 (ii) requires $u_0 \in \dot{H}^3(\Omega)$, such that the initial data must satisfy $u_0 = Au_0 = 0$ on $\partial\Omega$.

The remainder of this paper is then devoted to the proofs of Theorems 1.1 and 1.2. Our proofs are based on the transform methods developed in [30, 36], together with some basic error estimates for the finite element approximations of the elliptic problems, for example (1.12), and the Paley-Wiener Lemma which is true in any Banach algebra $L^1(R^+, B(X))$, the space of all measurable functions $f: R^+ \to B(X)$ such that $\int_0^\infty \|f(t)\|_{B(X)}\, dt < \infty$ with $\|\cdot\|_{B(X)}$ denoting the norm in $B(X)$ which is the space of all bounded linear operators from a Banach space X to X, and is stated as follows.

Lemma 1.1. *Suppose that $K \in L^1(R^+, B(X))$ is such that $I - \widehat{K}(s)$ is invertible for each $Re\, s \geq 0$. Then, there is a unique solution $L \in L^1(R^+, B(X))$ such that $L = K + K * L = K + L * K$ holds in $L^1(R^+, B(X))$.*

For the proof of this result we refer the readers to Prüss [2, Theorem 0.7].

2. The proof of Theorem 1.1.

This section is organized as follows. First, in Section 2.1. we prove Theorem 1.1. (ii), that is, the regular case $\mu + \kappa < \infty$. In Section 2.2., we turn to the singular case, i.e., $\mu + \kappa = \infty$.

2.1. The regular case

In this section we assume that $\mu + \kappa < \infty$ and prove Theorem 1.1 (ii).

It follows from the proof of the regular case of Theorem 11 in [30] that

$$u(t) = u_0(t) + u_1(t) = (U_0(t) + U_1(t))\, u_0 = U(t)\, u_0,$$

where

$$u_1(t) = U_1(t)u_0 = C(\mu\, t) \exp\left(-\kappa\, t/\mu\right) u_0, \tag{2.1}$$

with the cosine family $C(\mu t) = \cos\left(A^{\frac{1}{2}}\mu t\right)$. Let $w_1(t) = C(\mu t)u_0$. Then, $w_1(t)$ satisfies

$$w_{1,t}(t) + \mu^2 \int_0^t A w_1(s) ds = 0, \quad w_1(0) = u_0. \tag{2.2}$$

Similarly, we can also write

$$u_h(t) = u_{0h}(t) + u_{1h}(t) = (U_{0h}(t) + U_{1h}(t))\, P_{0h}u_0,$$

where

$$u_{1h}(t) = U_{1h}(t)P_{0h}u_0 = C_h(\mu t) \exp\left(-\kappa\, t/\mu\right) P_{0h}u_0, \tag{2.1$_h$}$$

with $C_h(\mu t) = \cos\left(A_h^{\frac{1}{2}}\mu t\right)$. Now, let $w_{1h}(t) = C_h(\mu t)P_{0h}u_0$. Then, $w_{1h}(t)$ satisfies

$$w_{1h,t}(t) + \mu^2 \int_0^t A_h w_{1h}(s) ds = 0, \quad w_{1h}(0) = P_{0h}u_0. \tag{2.2$_h$}$$

Hence, by Theorem 2.1. in [23],

$$\|w_{1h}(t) - w_1(t)\| \leq Ch^2 \left\{ \|u_0\|_2 + \int_0^t \|w_{1,t}(s)\|_2\, ds \right\}, \quad \text{for } t > 0, \tag{2.3}$$

since the function 1 is a positive-definite kernel. Moreover, we note that

$$w_{1,t}(t) = -\sin\left(A^{\frac{1}{2}}\mu t\right)\mu A^{\frac{1}{2}}u_0,$$

and

$$\|w_{1,t}(s)\|_2 \le C\|u_0\|_3, \qquad \int_0^t \|w_{1,t}(s)\|_2 ds \le Ct\|u_0\|_3,$$

and

$$\|w_{1h}(t) - w_1(t)\| \le Ch^2\left\{\|u_0\|_2 + t\|u_0\|_3\right\}, \quad \text{for } t > 0. \tag{2.4}$$

Thus, we establish that

$$\begin{aligned} &\|u_{1h}(t) - u_1(t)\| = \|(w_{1h}(t) - w_1(t))\exp\left(-\kappa\, t/\mu\right)\| \\ &\le Ch^2\left\{\|u_0\|_2 + t\|u_0\|_3\right\}\exp\left(-\kappa\, t/\mu\right), \quad \text{for } t > 0, \end{aligned} \tag{2.5}$$

and it suffices to prove that

$$\int_0^\infty \|u_{0h}(t) - u_0(t)\| dt \le Ch^2\|u_0\|_3. \tag{2.6}$$

To do this, we use the fact that $\hat{C}(s) = s(s^2I + A)^{-1}$ and the operational calculus to get

$$\hat{U}_1(s) = \frac{1}{\mu}\left(\hat{g}^{-1}(s) + \hat{g}(s)A\right)^{-1}, \qquad \operatorname{Re} s \ge 0, \tag{2.7}$$

where $\hat{g}(s) = \mu^2(\mu s + \kappa)^{-1}$ is the transform of $g(t) = \mu\exp(-\kappa t/\mu)$. Similarly,

$$\widehat{U_{1h}}(s) = \frac{1}{\mu}\left(\hat{g}^{-1}(s) + \hat{g}(s)A_h\right)^{-1}, \quad \operatorname{Re} s \ge 0. \tag{2.8}$$

We shall obtain a convolution equation for $U_0(t) - U_{0h}(t)P_{0h}$ and use the Paley-Wiener Lemma to deduce (2.6). In fact, following the argument in [30], we have that

$$\hat{U}_0(s) = \hat{R}_1(s) + \hat{R}_2(s)\hat{U}_0(s), \tag{2.9}$$

where

$$\begin{aligned} \hat{R}_1(s) &= (1 + \kappa\hat{\beta}(s))^{-1}\hat{R}(s)\hat{U}_1(s), \\ \hat{R}_2(s) &= (1 + \kappa\hat{\beta}(s))^{-1}\left(\hat{R}(s) + \kappa\hat{\beta}(s)\right), \\ \hat{R}(s) &= \mu^{-4}\left\{\mu^2\hat{\ddot{\beta}} + 2\kappa\mu\hat{\dot{\beta}}(s) + \kappa^2\hat{\beta}(s)\right\}\hat{U}_1(s) - \mu^{-3}\left\{\mu\hat{\dot{\beta}}(s) + \kappa\hat{\beta}(s)\right\}. \end{aligned}$$

Also, we have

$$\widehat{U_{0h}}(s) = \widehat{R_{1h}}(s) + \widehat{R_{2h}}(s)\widehat{U_{0h}}(s). \tag{2.10}$$

This formula for $\widehat{U_{oh}}(s)$ is of the same form as (2.9) with A_h instead of A. Subtracting (2.10) from (2.9) we get

$$\begin{aligned}\hat{U}_0(s) - \hat{U}_{0h}(s)P_{0h} &= (\hat{R}_1(s) - \hat{R}_{1h}(s)P_{0h}) + (\hat{R}_2(s) - \hat{R}_{2h}(s)P_{0h})\hat{U}_0(s) \\ &+\hat{R}_{2h}(s)(P_{0h} - I)\hat{U}_0(s) + \hat{R}_{2h}(s)\left(\hat{U}_0(s) - \hat{U}_{0h}(s)P_{0h}\right). \end{aligned} \tag{2.11}$$

As in [30] we see that $R_{2h}(t) \in L^1(R^+, L(S_h))$, $\int_0^\infty \|R_{2h}(t)\|dt \le C$ and

$$I - \widehat{R_{2h}}(s) = \left(1 + \kappa\hat{\beta}(s)\right)^{-1}\left(s + \hat{\beta}(s)A_h\right)\widehat{U_{1h}}(s), \quad \operatorname{Re} s \ge 0,$$

so $I - \widehat{R_{2h}}(s)$ is invertible for each $s \in \Pi = \{s : \operatorname{Re} s \ge 0\}$ and $h > 0$. Now, by Lemma 1.1., there exists $Q_{1h}(t) \in L^1(R^+, L(S_h))$ such that

$$Q_{1h}(t) = R_{2h}(t) + Q_{1h} * R_{2h}(t) = R_{2h}(t) + R_{2h} * Q_{1h}(t), \quad t \ge 0.$$

Following the proof of the continuous case [2, Theorem 0.7] we can obtain that $\int_0^\infty \|Q_{1h}(t)\|dt \le C$. Thus, solving (2.11) for $\hat{U}_0(s) - \widehat{U_{0h}}(s)P_{0h}$, we have that

$$\begin{aligned}\hat{U}_0(s) - \hat{U}_{0h}(s)P_{0h} &= \left(\hat{R}_1(s) - \hat{R}_{1h}(s)P_{0h}\right) + \hat{R}_{2h}(s)(P_{0h} - I)\hat{U}_0(s) \\ &+ \left(\hat{R}_2(s) - \hat{R}_{2h}(s)P_{0h}\right)\hat{U}_0(s) \\ &+\hat{Q}_{1h}(s)\Big[\left(\hat{R}_1(s) - \hat{R}_{1h}(s)P_{0h}\right) \\ &+ \hat{R}_{2h}(s)(P_{0h} - I)\hat{U}_0(s) + \left(\hat{R}_2(s) - \hat{R}_{2h}(s)P_{0h}\right)\hat{U}_0(s)\Big]. \end{aligned} \tag{2.12}$$

Define

$$\begin{aligned}\hat{l}(s) = &\quad (1 + \kappa\hat{\beta}(s))^{-1}\mu^{-4}\left\{\mu^2\ddot{\hat{\beta}} + 2\kappa\mu\dot{\hat{\beta}}(s) + \kappa^2\hat{\beta}(s)\right\}, \\ \hat{m}(s) = &\quad (1 + \kappa\hat{\beta}(s))^{-1}\mu^{-3}\left\{\mu\,\dot{\hat{\beta}}(s) + \kappa\,\hat{\beta}(s)\right\}.\end{aligned}$$

From the proof of [30, Theorem 11] we know that $l(t)$ and $m(t)$ belong to $L^1(R^+)$.

Since

$$\begin{aligned}\hat{R}_1(s) - \hat{R}_{1h}(s)P_{0h} &= (1 + \kappa\hat{\beta}(s))^{-1}\left[\hat{R}(s)\hat{U}_1(s) - \hat{R}_h(s)\hat{U}_{1h}(s)P_{0h}\right] \\ &= \hat{l}(s)\Big[\left(\hat{U}_1(s) - \hat{U}_{1h}(s)P_{0h}\right)\hat{U}_1(s) + \hat{U}_{1h}(s)P_{0h} \\ &\left(\hat{U}_1(s) - \hat{U}_{1h}(s)P_{0h}\right)\Big] - \hat{m}(s)\left[\hat{U}_1(s) - \hat{U}_{1h}(s)P_{0h}\right],\end{aligned} \tag{2.13}$$

and by (2.1)

$$\int_0^\infty \|A^{\frac{3}{2}}U_1(t)u_0\|dt \le C\|A^{\frac{3}{2}}u_0\|, \tag{2.14}$$

it follows from (2.13), (2.14), (2.5) and [30, Theorem 11] that

$$\int_0^\infty \|\left(R_1(t) - R_{1h}(t)P_{0h}\right)u_0\|dt \le Ch^2\|A^{\frac{3}{2}}u_0\|. \tag{2.15}$$

Next we write

$$\hat{R}_2(s) - \hat{R}_{2h}(s)P_{0h} = \hat{l}(s)\left[\hat{U}_1(s) - \hat{U}_{1h}(s)P_{0h}\right], \tag{2.16}$$

which, together with (2.5), implies

$$\int_0^\infty \|\left(R_2(t) - R_{2h}(t)P_{0h}\right)u_0\|dt \le Ch^2\|A^{\frac{3}{2}}u_0\|. \tag{2.17}$$

Following from the proof of Theorem 11 in [30], we can also obtain

$$\int_0^\infty \|A^{\frac{3}{2}}U_0(t)u_0\|dt \le C\|A^{\frac{3}{2}}u_0\|, \tag{2.18}$$

and

$$\int_0^\infty \|(I - P_{0h})U_0(t)u_0\|dt \le Ch^2\int_0^\infty \|Au_0(t)\|dt \le Ch^2\|Au_0\|. \tag{2.19}$$

Thus, combining (2.12), (2.13) and (2.16) with the estimates (2.15), (2.17), (2.18) and (2.19), we can gain our desired estimate (2.6).

2.2. The singular case

In this subsection, we consider the singular case that $\mu + \kappa = \infty$. First of all, we introduce some notations and recall results in Section 3 of [30]. Define $h_0(s,x)$ and $h(s,x)$ for the kernel $\beta(t)$ (see, for example, page 327 of [30]) as follows:

$$h_0(s,x) = \exp\left(-x/\hat{a}(s)^{\frac{1}{2}}\right), \quad h(s,x) = \frac{1}{s\hat{a}(s)^{\frac{1}{2}}}h_0(s,x). \tag{2.20}$$

By Theorems 3 and 4 of [30], we can write

$$\widehat{w_{0t}}(s,x) = h_0(s,x), \quad \widehat{w}_t(s,x) = h(s,x). \tag{2.21}$$

See Theorem 3 of [30] for the definitions of the functions $w_0(t,x)$ and $w(t,x)$. Notice, in particular, that for each $x > 0$, $w_0(t,x)$ and $w(t,x)$ are nondecreasing, continuous functions of $t \ge 0$, and are absolutely continuous for $t \ne x/\mu$ in the case that $\mu < \infty$.

Now let ω be a positive number. Define $U_\omega(t)$ and $R_\omega(t)$ as those in (7.7) and (7.9) of [30], respectively, by

$$U_\omega(t) = \int_0^\infty e^{-\omega\tau}C(\tau)w_t(t,\tau)d\tau, \quad R_\omega(t) = \int_0^\infty e^{-\omega\tau}C(\tau)w_{0t}(t,\tau)d\tau.$$

Similarly, we define $R_{\omega,h}(t)$ by

$$R_{\omega,h}(t) = \int_0^\infty e^{-\omega\tau}C_h(\tau)w_{0t}(t,\tau)d\tau,$$

where $C_h(\tau) = \cos\left(A_h^{\frac{1}{2}}\tau\right)$ is cosine family, defined by the discrete elliptic operator A_h. We write $u(t) = U(t)u_0$ and $u_h(t) = U_h(t)P_{0h}u_0$.

Since $\hat{C}(s) = s(s^2 + A)^{-1}$, Re $s > 0$ (see [38, Proposition 2.6]). Using (2.20) and (2.21) (see [30, the formula (8.3)]), we have via some calculations as those in [30] that

$$\hat{U}(s) = h(s)\hat{U}_\omega(s) + \omega(1 + h(s))\hat{R}_\omega(s)\hat{U}(s), \tag{2.22}$$

where

$$\begin{aligned}
&\hat{U}(s) = (I + \hat{a}(s)A)^{-1}/s = \left(sI + \hat{\beta}(s)A\right)^{-1},\\
&\hat{U}_\omega(s) = s^{-1}h(s)\left(I + h^2(s)\hat{a}(s)A\right)^{-1},\\
&\hat{R}_\omega(s) = \hat{a}(s)^{\frac{1}{2}}h(s)\left(I + h^2(s)\hat{a}(s)A\right)^{-1}.
\end{aligned}$$

Here, as that on page 341 of [30], $h(s)$ denotes

$$h(s) = \left(1 + \omega\hat{a}(s)^{\frac{1}{2}}\right)^{-1}.$$

Now we multiply (2.22) by $\omega(\hat{a}(s))^{\frac{1}{2}}h(s)$ to yield

$$\begin{aligned}
\hat{U}(s) \ &= \omega(\hat{a}(s))^{\frac{1}{2}}h(s)h_1(s)\hat{R}_\omega(s)\\
&\quad + \left[h(s)I + \omega\hat{a}(s)^{\frac{1}{2}}h(s)\omega(1 + h(s))\hat{R}_\omega(s)\right]\hat{U}(s),
\end{aligned} \tag{2.23}$$

where like that on page 341 of [30] $h_1(s)$ is defined as

$$h_1(s) = h(s)/s\hat{a}(s)^{\frac{1}{2}}.$$

Also, we have that

$$\begin{aligned}
\hat{U}_h(s) \ &= \omega(\hat{a}(s))^{\frac{1}{2}}h(s)h_1(s)\widehat{R_{\omega,h}}(s)\\
&\quad + \left[h(s)I + \omega\hat{a}(s)^{\frac{1}{2}}h(s)\omega(1 + h(s))\widehat{R_{\omega,h}}(s)\right]\hat{U}_h(s).
\end{aligned} \tag{2.24}$$

Set

$$\begin{aligned}
\widehat{Z_{1,h}}(s) \ &= h_1(s)\omega\hat{a}(s)^{\frac{1}{2}}h(s)\left[\widehat{R_{\omega,h}}(s)P_{0h} - \widehat{R_\omega}(s)\right],\\
\widehat{Z_{2,h}}(s) \ &= \omega(1 + h(s))\omega\hat{a}(s)^{\frac{1}{2}}h(s)\left[\widehat{R_{\omega,h}}(s)P_{0h} - \widehat{R_\omega}(s)\right]\hat{U}(s),\\
\widehat{Z_{3,h}}(s) \ &= h(s)I + \omega(1 + h(s))\omega\hat{a}(s)^{\frac{1}{2}}h(s)\widehat{R_{\omega,h}}(s),\\
\widehat{Z_{4,h}}(s) \ &= \omega(1 + h(s))\omega\hat{a}(s)^{\frac{1}{2}}h(s)\widehat{R_{\omega,h}}(s)(I - P_{0h})\hat{U}(s).
\end{aligned}$$

Subtracting (2.23) from (2.24) we can obtain

$$\widehat{U_h}(s)\,P_{0h} - \hat{U}(s) = \widehat{Z_{1,h}}(s) + \widehat{Z_{2,h}}(s) + \widehat{Z_{4,h}}(s) + \widehat{Z_{3,h}}(s)\left(\widehat{U_h}(s)P_{0h} - \hat{U}(s)\right). \tag{2.25}$$

Now, as that shown on page 341 of [30], we have

$$Z_{3,h}(t) \in L^1(R^+ L(S_h)), \quad \int_0^\infty \|Z_{3,h}(t)\| dt \le C < \infty,$$

and

$$\begin{aligned} I - \widehat{Z_{3,h}}(s) &= \frac{h(s)\hat{a}(s)^{\frac{1}{2}}\omega h^2(s)}{I + h^2(s)\hat{a}(s)A_h}(I + \hat{a}(s)A_h) \\ &= \omega h^2(s)(I + \hat{a}(s)A_h)\widehat{R_{\omega,h}}(s), \quad \operatorname{Re} s \ge 0. \end{aligned} \tag{2.26}$$

Thus, $I - \widehat{Z_{3,h}}(s)$ is invertible for $\operatorname{Re} s \ge 0$ and $h > 0$. Lemma 1.1 indicates that there is a $Y_{3,h}(t) \in L^1(R^+, L(S_h))$ such that

$$Y_{3,h}(t) = Z_{3,h}(t) + Y_{3,h} * Z_{3,h}(t) = Z_{3,h}(t) + Z_{3,h} * Y_{3,h}(t), \quad t \ge 0, \tag{2.27}$$

and we can follow the proof of [2, Theorem 0.7] to obtain

$$\int_0^\infty \|Y_{3,h}(t)\| dt \le C < \infty.$$

Therefore, solving (2.25) for $\widehat{U_h}(s)P_{0h} - \hat{U}(s)$, we obtain

$$\begin{aligned} \widehat{U_h}(s)P_{0h} - \hat{U}(s) &= \widehat{Z_{1,h}}(s) + \widehat{Z_{2,h}}(s) + \widehat{Z_{4,h}}(s) \\ &\quad + \widehat{Y_{3,h}}(s)\left(\widehat{Z_{1,h}}(s) + \widehat{Z_{2,h}}(s) + \widehat{Z_{4,h}}(s)\right). \end{aligned} \tag{2.28}$$

Also, we have

$$\begin{aligned} \widehat{Z_{1,h}}(s) &= \omega h_1(s)\hat{a}(s)^{\frac{1}{2}}h(s)\left[\frac{1}{\hat{a}(s)^{\frac{1}{2}}h(s)}\left(\frac{P_{0h}}{\hat{q}(s)I + A_h} - \frac{1}{\hat{q}(s)I + A}\right)\right] \\ &= \omega h_1(s)\left[\frac{T_h}{I + \hat{q}(s)T_h} - \frac{T}{I + \hat{q}(s)T}\right] \\ &= \omega h_1(s)\left[\left(\frac{T_h}{I + \hat{q}(s)T_h}(P_{0h} - I)A^{-1}\right)A \right. \\ &\quad \left. + \frac{T_h}{I + \hat{q}(s)T_h}(T - T_h)A + \left(\frac{T_h^2}{I + \hat{q}(s)T_h} - \frac{T^2}{I + \hat{q}(s)T}\right)A\right] \\ &= \omega h_1(s)\left\{\hat{a}(s)^{\frac{1}{2}}h(s)\widehat{R_{\omega,h}}(s)P_{0h}\left[(P_{0h} - I)A^{-1} + 2(T - T_h)\right]A \right. \\ &\quad \left. + \hat{a}(s)^{\frac{1}{2}}h(s)(T_h - T)\hat{R}_\omega(s)A + \widehat{R_{\omega,h}}(s)P_{0h}(T - T_h)\hat{R}_\omega(s)A\right\}, \end{aligned} \tag{2.29}$$

where $\hat{q}(s) = \frac{1}{\hat{a}(s)h^2(s)}$. It follows from (1.12) and the proof shown on pages 341–342 in [30] that

$$\int_0^\infty \|Z_{1,h}(t)u_0\| dt \le Ch^2\|Au_0\|. \tag{2.30}$$

Similarly, we can obtain

$$\int_0^\infty \|Z_{i,h}(t)u_0\| dt \le Ch^2 \int_0^\infty \|AU(t)u_0\| dt \le Ch^2\|Au_0\|, \quad i = 2, 4, \tag{2.31}$$

where the second inequality of (2.31) follows from Theorem 11 of [30]. Combining (2.29)–(2.31) with (2.28) yields

$$\int_0^\infty \|u_h(t) - u(t)\| dt \le Ch^2 \|Au_0\|, \tag{2.32}$$

and thus Theorem 1 (i) is proved.

3. The proof of Theorem 1.2.

First of all, we consider the regular case that $\mu + \kappa < \infty$, and then we discuss the singular case that $\mu + \kappa = \infty$.

3.1. The regular case

Define

$$V_1(t) = \mu \dot{C}(\mu t) \exp(\kappa t/\mu),$$

and write

$$V(t) = V_0(t) + V_1(t),$$

where $u'(t) = V(t)u_0$. Note that $\hat{\dot{C}}(s) = -A(s^2 I + A)^{-1}$. So, we have

$$\hat{V}_1(s) = \hat{\dot{C}}(\hat{g}(s)^{-1}) = -A(\hat{g}^{-2}(s)I + A)^{-1}, \tag{3.1}$$

where, as before, $\hat{g}(s) = \mu^2(\mu s + \kappa)^{-1}$. We multiply (3.1) with T to get

$$T\hat{V}_1(s) = \frac{-\hat{g}^2(s)T}{\hat{g}^2(s)I + T}. \tag{3.2}$$

Similarly, with $u_{0h} = P_{0h}u_0$ and $u_h'(t) = V_h(t)P_{0h}u_0$, we have $V_h(t) = V_{0h}(t) + V_{1h}(t)$ and $V_{1h}(t) = \mu \dot{C}_h(\mu t) \exp(-\kappa t/\mu)$, where $\dot{C}_h(\mu t) = -\mu \sin(\mu t A_h^{\frac{1}{2}}) A_h^{\frac{1}{2}}$, and

$$T_h\hat{V}_{1h}(s) = \frac{-\hat{g}^2(s)T_h}{\hat{g}^2(s)I + T_h}. \tag{3.3}$$

Subtracting (3.2) from (3.3) and doing some simple computations, we can obtain that

$$\begin{aligned} &T_h\hat{V}_{1h}(s)P_{0h} - T\hat{V}_1(s) = \left(\hat{V}_{1h}(s)T_hP_{0h}T - \hat{V}_1(s)T^2\right)A \\ &= \left[\hat{V}_{1h}(s)T_h(P_{0h} - I)T + \hat{V}_{1h}(s)T_h(T - T_h) + \hat{V}_{1h}(s)T_h^2 - \hat{V}_1(s)T^2\right]A \\ &= \left[\hat{V}_{1h}(s)T_h(P_{0h} - I)T + \hat{V}_{1h}(s)T_h(T - T_h)\right]A \\ &\quad + \left[\hat{V}_{1h}(s)T_h(T_h - T) + (T_h - T)\hat{V}_1(s)T + \mu\hat{U}_{1h}(s)(T_h - T)\mu\hat{U}_1(s)\right]A. \end{aligned} \tag{3.4}$$

By means of the spectral theorems and the definitions in [30, 36], we can also obtain that

$$\begin{aligned}
&\int_0^\infty \|V_{1h}(t)T_h\|dt \le C < \infty, \qquad \int_0^\infty \|U_{1h}(t)\|dt \le C < \infty,\\
&\int_0^\infty \|V_1(t)T\|dt \le C < \infty, \qquad \int_0^\infty \|U_1(t)\|dt \le C < \infty.
\end{aligned} \tag{3.5}$$

In addition, from (1.13), (3.4), and (3.5) we know that

$$\int_0^\infty \| \left(V_{1h}(t)T_hP_{0h} - V_1(t)T\right) u_0\|dt \le Ch^2\|Au_0\|. \tag{3.6}$$

Thus, to complete the proof of Theorem 1.2 in the regular case, it suffices to show that

$$\int_0^\infty \| \left(V_{0h}T_hP_{0h} - V_0(t)T\right) u_0\|dt \le Ch^2\|Au_0\|. \tag{3.7}$$

Combining (2.7) with (3.2) leads to

$$A^{-1}\hat{V}_1(s) = -\mu\hat{g}(s)\hat{U}_1(s). \tag{3.8}$$

Since $\hat{V}(s) = \hat{\beta}(s)(-A)\left(sI + \hat{\beta}(s)A\right)^{-1}$, we have

$$\hat{V}_0(s) = \left(\hat{V}_1^{-1}(s) - \hat{V}^{-1}(s)\right)\hat{V}_1(s)\hat{V}(s) = -\left\{\hat{g}^{-2}(s) - s/\hat{\beta}(s)\right\}A^{-1}\hat{V}_1(s)\hat{V}(s),$$

which, together with some manipulations, yields

$$\hat{V}_0(s)T = \hat{R}_4(s)T + \hat{R}_3(s)\hat{V}_0(s)T, \tag{3.9}$$

where

$$\begin{aligned}
\hat{R}_3(s) &= \kappa^2\mu^{-3}\hat{g}(s)\hat{U}_1(s) + \mu\hat{r}_0(s)\hat{U}_1(s),\\
\hat{R}_4(s) &= \hat{R}_3(s)\hat{V}_1(s),
\end{aligned}$$

and $r_0(t)$ is the scalar function whose transform is

$$\hat{r}_0(s) = \hat{g}(s)\left\{\hat{g}^{-2}(s) - s/\hat{\beta}(s) - \kappa^2/\mu^4\right\}.$$

Moreover, from Lemma 10.1 in [2] we know that

$$r_0(t) \in L^1(R^+) \tag{3.10}$$

(see [39, Hannsgen and Wheeler (1990), page 506, the proof of (3.11)]).

Similar to the arguments of Theorem 1.1 we can also get that

$$\begin{aligned}
\hat{V}_{0h}(s)T_hP_{0h} - \hat{V}_0(s)T &= \hat{R}_{4h}(s)T_hP_{0h} - \hat{R}_4(s)T\\
&+ \left(\hat{R}_{3h}(s)P_{0h} - \hat{R}_3(s)\right)\hat{V}_0(s)T + \hat{R}_{3h}(s)\left(\hat{V}_{0h}(s)T_hP_{0h} - \hat{V}_0(s)T\right)\\
&+ \hat{R}_{3h}(s)\left(I - P_{0h}\right)\hat{V}_0(s)T.
\end{aligned} \tag{3.11}$$

We use (3.10) and $(2.1)_h$ to gain

$$R_{3h}(t) \in L^1(R^+, L(S_h)) \quad \text{and} \quad \int_0^\infty \|R_{3h}(t)\| dt \le C < \infty.$$

To see that $I - \hat{R}_{3h}(s)$ is invertible for $s \in \Pi = \{s : \operatorname{Re} s \ge 0\}$ and $h > 0$, we note that $\hat{V}_h(s) - \hat{V}_{1h}(s) = \hat{V}_{0h}(s) = \hat{R}_{3h}(s)\hat{V}_h(s)$. So, we have that $I - \hat{R}_{3h}(s) = \hat{V}_{1h}(s)\hat{V}_h^{-1}(s)$, which clearly indicates that $I - \hat{R}_{3h}(s)$ is invertible for $s \in \Pi = \{s : \operatorname{Re} s \ge 0\}$ and $h > 0$.

According to Lemma 1.1, there exists $Q_{2h}(t) \in L^1(R^+, L(S_h))$ such that

$$Q_{2h}(t) = R_{3h}(t) + Q_{2h} * R_{3h}(t) \quad \text{for } t \ge 0, \quad \text{and} \int_0^\infty \|Q_{2h}(t)\| dt \le C < \infty.$$

Therefore, solving (3.11) for $\hat{V}_{0h}(s)T_h P_{0h} - \hat{V}_0(s)T$, we obtain

$$\begin{aligned} \hat{V}_{0h}(s)T_h P_{0h} - \hat{V}_0(s)T &= \hat{R}_{3h}(s)\left(I - P_{0h}\right)\hat{V}_0(s)T + \hat{Z}_{4h}(s) + \hat{Z}_{5h}(s) \\ &+ \hat{Q}_{2h}(s)\left[\hat{R}_{3h}(s)\left(I - P_{0h}\right)\hat{V}_0(s)T + \hat{Z}_{4h}(s) + \hat{Z}_{5h}(s)\right], \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} \hat{Z}_{4h}(s) &= \hat{R}_{4h}(s)T_h P_{0h} - \hat{R}_4(s)T, \\ \hat{Z}_{5h}(s) &= \left(\hat{R}_{3h}(s)P_{0h} - R_3(s)\right)\hat{V}_0(s)T. \end{aligned}$$

Next we show that

$$\int_0^\infty \|Z_{4h}(t)u_0\| dt \le Ch^2 \|Au_0\|, \tag{3.13}$$

and

$$\int_0^\infty \|Z_{5h}(t)u_0\| dt \le Ch^2 \|Au_0\|. \tag{3.14}$$

Then, (3.7) follows from (3.12), (3.13), and (3.14). So, we reduce (3.7) to (3.13) and (3.14).

Write

$$\begin{aligned} \hat{Z}_{5h}(s)u_0 &= \kappa^2\mu^{-3}\hat{g}(s)\left[\hat{U}_{1h}(s)P_{0h} - \hat{U}_1(s)\right]\hat{V}_0(s)Tu_0 \\ &+ \mu\hat{r}_0(s)\left[\hat{U}_{1h}(s)P_{0h} - \hat{U}_1(s)\right]\hat{V}_0(s)Tu_0. \end{aligned}$$

We use (2.5), (3.10), and Theorem 3.2 in [36] to get

$$\begin{aligned} \int_0^\infty \|Z_{5h}(t)u_0\| dt &\le Ch^2 \int_0^\infty \left\| A^{\frac{3}{2}} V_0(t)Tu_0 \right\| dt \\ &= Ch^2 \int_0^\infty \left\| \left(V_0(t)T^{\frac{1}{2}}\right) Au_0 \right\| dt \le Ch^2 \|Au_0\|. \end{aligned} \tag{3.15}$$

Thus, (3.14) is established. Furthermore, from (3.9) we can derive

$$\begin{aligned}\hat{Z}_{4h}(s)u_0 &= \left[\hat{R}_{3h}(s)\hat{V}_{1h}(s)T_hP_{0h} - \hat{R}_3(s)\hat{V}_1(s)T\right]u_0 \\ &= \left(\hat{R}_{3h}(s)P_{0h} - R_3(s)\right)\hat{V}_1(s)Tu_0 \\ &\quad +\hat{R}_{3h}(s)P_{0h}\left(\hat{V}_{1h}(s)T_hP_{0h} - \hat{V}_1(s)T\right)u_0.\end{aligned}$$

So, (3.6) and (3.15) with $V_1(t)$ instead of $V_0(t)$ imply (3.13). This, in turn, gives us (3.7), which completes the proof of Theorem 1.2 in the regular case.

3.2. The singular case

Let $\mu+\kappa=\infty$. As in the proof of Theorem 3.1 of [39] we define $V_\omega(t)$ as follows:

$$V_\omega(t) = \int_0^\infty e^{-\omega\tau}\dot{C}(\tau)w_{0t}(t,\tau)d\tau.$$

Similarly, like the derivation of (3.9), we can write

$$\hat{V} - \hat{V}_\omega = \left(\hat{V}_\omega^{-1} - \hat{V}^{-1}\right)\hat{V}_\omega\hat{V}.$$

It follows from (2.20), (2.21), and $\hat{\dot{C}}(s) = -A(s^2I+A)^{-1}$ for $\operatorname{Re} s > 0$ that

$$\hat{V} - \hat{V}_\omega = -\left(\omega^2 + 2\omega\left(\frac{s}{\hat{\beta}(s)}\right)^{\frac{1}{2}}\right)A^{-1}\hat{V}_\omega\hat{V}.$$

Since

$$-A^{-1}\hat{V}_\omega = \left(\omega + \hat{a}(s)^{-\frac{1}{2}}\right)^{-1}\hat{R}_\omega,$$

it is easy to verify that

$$\hat{V}(s)T = \hat{V}_\omega(s)T + \hat{S}_2(s)\hat{V}(s)T, \tag{3.16}$$

where

$$\hat{S}_2(s) = \omega(1+h(s))\hat{R}_\omega(s).$$

The proof of the remainder part is almost identical to that of (3.7). We only need to derive the following two more estimates,

$$\int_0^\infty \|\left(R_{\omega h}P_{0h} - R_\omega\right) * V(t)Tu_0\|dt \le Ch^2\|Au_0\|, \tag{3.17}$$

and

$$\int_0^\infty \|\left(V_{\omega h}(t)T_hP_{0h} - V_\omega(t)T\right)u_0\|dt \le Ch^2\|Au_0\|. \tag{3.18}$$

To verify the claim (3.17), we note that

$$\begin{aligned}
&\int_0^\infty \| \left(R_{\omega h}(t)P_{0h} - R_\omega(t)\right) v\| dt \\
&\quad = \int_0^\infty \left\| \int_0^\infty e^{-\omega\tau} \left(C_h(\tau)P_{0h} - C(\tau)\right) w_{0t}(t,\tau) v d\tau \right\| dt \\
&\quad \le C \int_0^\infty \int_0^\infty e^{-\omega\tau} w_{0t}(t,\tau)\tau h^2 \left\| A^{\frac{3}{2}} v \right\| d\tau dt \le C h^2 \left\| A^{\frac{3}{2}} v \right\|,
\end{aligned} \tag{3.19}$$

where we have used (2.4) and Theorem 3 of [30]. Thus, combining (3.19) and (1.6) leads to (3.17).

Next, our discussions turn to (3.18). From the derivation of (3.16) we know that

$$\hat{V}_\omega(s) = -A\hat{a}(s)^{\frac{1}{2}} h(s) \hat{R}_\omega(s) = -A\hat{a}(s)h(s)^2 \frac{T}{\hat{a}(s)h(s)^2 I + T},$$

from which we claim that

$$\begin{aligned}
&\hat{V}_{\omega h}(s) T_h P_{0h} - \hat{V}_\omega(s) T \\
&\quad = \left[\hat{V}_{\omega h}(s) T_h (P_{0h} - I) A^{-1} + \hat{V}_{\omega h}(s) T_h (T - T_h)\right] A \\
&\quad + \left[\hat{V}_{\omega h}(s) T_h (T - T_h) + (T - T_h)\hat{V}_\omega(s) T + \hat{R}_{\omega h}(s)(T_h - T)\hat{R}_\omega(s)\right] A.
\end{aligned} \tag{3.20}$$

We recall from the proofs of Theorem 11 in [30] and Theorem 3.2 in [36] that

$$\begin{aligned}
&\int_0^\infty \|V_\omega(t) T\| dt \le C < \infty, \quad \int_0^\infty \|V_{\omega h}(t) T_h\| dt \le C < \infty, \\
&\int_0^\infty \|R_\omega(t)\| dt \le C < \infty, \quad \int_0^\infty \|R_{\omega h}(t)\| dt \le C < \infty.
\end{aligned} \tag{3.21}$$

These estimates, together with (1.12) and (3.20), yield (3.18). And thus, we complete the proof of Theorem 1.2.

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