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A SUFFICIENT CONDITION FOR NON-OSCILLATORY BEHAVIOR OF SOME INPUT/OUTPUT MODELS

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Abstract

A simple condition sufficient for non-oscillatory behavior of input/output systems is formulated and discussed.

1. Introductory remarks

Input-output systems form a special subclass of the class of compartmental systems. The interest may be attracted for several reasons. As first, it is a simple but nicely structured system.As second, it applies in many particular situations covering both traditional as well as newly appearing areas of research. As third, one can register still growing amount of new results and publications and huge growth of areas of application.

2. Definitions, notation and auxiliary results

We are going to examine input-output systems, i.e. dynamical systems whose generators are maps on finite dimensional spaces and whose structure is based on a partial order of the basic space.

Let $\mathcal{E} = \mathcal{R}^N$ be the N-dimensional vector space of N-tuples of reals. This space can be partially ordered via the natural order in the set of reals. We define $x \in \mathcal{R}^N$ to be *nonnegative* i.e. $x^T = (x_1, ..., x_N) \geq 0$ if $x_j \geq 0, j = 1, ..., N$. In other words, $x \geq 0$ whenever $x \in \mathcal{E}_+ = \mathcal{R}^N_+$. We define partial order by setting $x \leq y, x, y \in \mathcal{E}$ whenever $y - x \in \mathcal{E}_+$ and, equivalently, $y \geq x$. We thus have $\mathcal{E} = \mathcal{R}^N = \mathcal{R}^N_+ - \mathcal{R}^N_+ = \mathcal{E}_+ - \mathcal{E}_+$, i.e. $x = x^+ - x^-, x^+, x^- \in \mathcal{R}^N_+$. An operator A mapping $\mathcal{E}$ into $\mathcal{E}$ satisfying $A\mathcal{E}_+ \subset \mathcal{E}_+$ is called nonnegative. This fact is symbolically denoted by by writing $A \geq 0$. The set of all linear maps of $\mathcal{E}$ into $\mathcal{E}$ can be partially ordered by setting $A \leq B$ means that $(B - A)\mathcal{E}_+ \subset \mathcal{E}_+$ and equivalently $B \geq A$.

3. An example: Michaelis-Menten Kinetics

3.1. Michaelis-Menten-Kinetics

This example comes from Michaelis-Menten-Kinetics which follows the network

$$X + E_0 \underset{k_{-0}}{\overset{k_0}{\rightleftharpoons}} E_1 \xrightarrow{\kappa_0} E_0 + P. \tag{3.1}$$

Here and in the following capital letters denote chemical species ($E_0,\ E_1,\ X,\ P$) and the respective small letters ($e_0(t),\ e_1(t),\ x(t),\ p(t)$) the corresponding concentrations at time $t \geq 0$. The network (3.1) talks about a substrate X being transformed into a product P by means of an enzyme E_0 which has one binding side for X to form a complex E_1, the so called loaded form of the enzyme E_0.

The above situation is described via the the following system

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = A((u))u(t), u(0) = u_0,$$

where [1]

$$u^T = \left([\eta^{(1)}]^T, [\eta^{(2)}]^T\right)$$

and

$$\eta^{(1)} = \begin{pmatrix} e_0, \\ e_1 \end{pmatrix}, \eta^{(2)} = \begin{pmatrix} x \\ e_1 \\ p \end{pmatrix},$$

Further, let

$$A = \begin{pmatrix} A^{(1)} & 0 \\ 0 & A^{(2)} \end{pmatrix}$$

where

$$A^{(1)} = \begin{pmatrix} -k_0 x & \kappa_0 + k_{-0} \\ k_0 x & -\kappa_0 - k_{-0} \end{pmatrix}, A^{(2)} = \begin{pmatrix} -k_0 e_0 & k_{-0} & 0 \\ k_0 e_0 & -\kappa_0 - k_{-0} & 0 \\ 0 & \kappa_0 & 0 \end{pmatrix}. \tag{3.2}$$

where k_0, k_{-0}, κ_0 are given positive constants.

Now, assume $e_0(0) > 0, x(0) > 0, e_1(0) = 0$. We are going to show that this system satisfies the following conditions

$$\frac{\mathrm{d}}{\mathrm{d}t}e_0(t) \leq 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}e_1(t) \geq 0 \tag{3.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) \leq 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}p(t) \geq 0 \tag{3.4}$$

for all $t \geq 0$.

[1]Superscript C^T means the transposed of $M \times N$ real valued matrix C.

It is easy to see that

$$e_0(t) + e_1(t) = c_0$$

and

$$x(t) + p(t) + e_1(t) = c_1,$$

where c_0 and c_1 are positive reals independent of $t \geq 0$. It follows that e_1 is increasing. To complete the proof of (3.3)-(3.4) it is enough to check that

$$p(t) = \kappa_0 \int_0^t e_1(\tau)\mathrm{d}\tau.$$

System such as the Michaelis-Menten model presented above is a typical representative of input-output models. Though such models are quite simple their behavior may exhibit many very curios effects typical for rather complicated dynamical systems. By producing such effects the compartmental systems can be very helpful in studying many systems whose fundamental variables are based on concept of (may be deterministic) probability.

4. Some facts concerning input-output models

In this note we are going to examine input-output systems arising in Chemistry and Cell Biology. An instructive example is described in Section 3.

Let $A = A(x)$ be an $N \times N$ matrix whose elements are real numbers as representatives of generally nonlinear functions of the variable vector $x = x(t)$. To study structured dynamical systems one has to understand the structure well. Below we bring some basic concepts of the theory of nonnegative operators needed in the further explanation.

A nonnegative linear operator A is called *irreducible* if it possesses the following property: For any two vectors $0 \neq x, 0 \neq y, x, y \in \mathcal{E}_+$ there exists a positive integer $p = p(x, y)$ such that the standard inner product $(A^p x, y) > 0$.

We are going to investigate the following dynamical system

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = A(u(t))u(t),\ u(0) = u_0. \tag{4.1}$$

We can assume that there exists a unique solution to (4.1) $u = u(t), u(t)^T = (u_1, ..., u_N)$ because, as a rule, the entries of matrix $A = A(u)$ are sufficiently smooth. We further assume that there exists a vector $\hat{x}$ with all its components positive such that

$$A'\hat{x} = (A(u))'\hat{x} = 0 \tag{4.2}$$

where A' denotes the dual operator of A with respect to the dual space $\mathcal{E}'$ assuming $\mathcal{E}$ is the inner product space equipped with the standard inner product on $\mathcal{E} = \mathcal{R}^N$. Let $X = X(t)$ be a solution to problem (4.1). We derive easily that

$$(X(t), \hat{x}') = (X(0), \hat{x}') = c. \tag{4.3}$$

In this way we arrived at the *full concentration preservation law*.

The operator-generator appearing in an input-output model is characterized by the following property. Operator $A(x)$ can be split as

$$A(x) = F(x) - D(x), F(x) \geq 0, D(x) \geq 0, x^T = (x_1, ..., x_N), x_j > 0, j = 1, ..., N.$$

It follows that

$$A(x)D(x)^{-1} = F(x)D(x)^{-1} - I, I = \text{diag}\{1, ..., 1\}.$$

according to (4.2) we get that

$$0 = \left[A(x)D(x)^{-1}\right]' \hat{x}' = [F(x)]'D(x)^{-1}\hat{x}' - \hat{x}'$$

implying that

$$B(x) \equiv D(x)^{-1}F(x) \tag{4.4}$$

satisfies

$$[B(x)]'\hat{x}' = \hat{x}'. \tag{4.5}$$

4.1. Remark Quite frequently the role of vector $\hat{x}'$ is played by the constant vector $e^T = (1, ..., 1)$. In such case, matrix $B(x)$ is column stochastic. Consequently, a solution to (4.1) is concentration i.e. a probability.

4.2. Remark One of the most important results concerned with the input-output models is the boundedness of solutions to (4.1). It is a consequence of the full concentration preservation law see [4] and [1].

5. A sufficient condition for non-oscillatory behavior

In this section we prove main result. To this purpose we need some deeper knowledge of the operators on partially ordered spaces. In particular, an enormous influence upon the behavior of the solutions has the substructuring of the matrix representing the generator of solutions.

5.1. Proposition *Let $A = A(x)$ be the generator of the system of solutions to (4.1). For any vector $x^T = (x_{(1)}^T, ..., x_{(p)})$, each $x_{(j)}$ possessing positive elements there exists a permutation matrix $U = U(x)$ such that*

$$A(x) = U(x) \begin{pmatrix} A_{(0):(0)} & 0 & & 0 \\ A_{(1):(0)} & A_{(1):(1)} & & 0 \\ & & \cdot & \\ & & \cdot & \\ & & \cdot & \\ A_{(p):(0)} & 0 & & A_{(p):(p)} \end{pmatrix} U(x)^{-1}, \; A_{j:j} = F_{(j):(k)} - D_{(j)}$$

where blocks $F_{(j):(k)}, (j) \neq (k)$ are elementwise nonnegative and the diagonal blocks $F_{(j):(j)} - D_{(j)}(x)$ are square irreducible matrices, $D_{(j):(j)}(x)$ are diagonal with positive elements.

5.2. Remark In practical applications in Chemistry, Biology and other areas the off-diagonal block and block $A_{(0):(0)}$ are either missing or they can be detected and separated, though the separation procedures may be experimentally and numerically tedious. In our theoretical considerations the substochastic block $A_{(0):(0)}$, and consequently all the off-diagonal blocks, will be absent. We thus assume that each generator $A_{(j):(j)}(x)$ for each x positive generates a stochastic semigroup of operators. This fact concerns the structure of the generator and formally it has nothing to do with the solution itself.

Our model is thus block stochastic. In addition, we will assume that $B_{(j)}(x)$ – the diagonal block of matrix $B(x)$ defined in(4.4) – is independent of x_r with r belonging to the multi-index (j). E.g. the elements of matrix $A^{(1)}$ in (3.2) are independent of e_1 and e_0.

Now, we can formulate and prove our result.

5.3. Theorem *Assume the generator $A = A(x)$ of Problem (4.1) satisfies the hypotheses of this and previous sections. In addition, let the derivatives*

$$\frac{\mathrm{d}}{\mathrm{d}t}x_j(t), j = 1, ..., N,$$

do not change the signs within the interval $0 < t < +\infty$. Then system (4.2) is nonoscillatory.

Proof. To prove the validity of the statement it is enough to recall that the solution vector is uniformly norm bounded and apply a classical result saying that a nonnegative monotone bounded continuous function on interval $[0, +\infty)$ satisfies

$$\lim_{t \to +\infty} x(t) = x(+\infty) < +\infty.$$

The proof is complete.

6. Concluding remarks

The condition discussed in this contribution is closely related to the conditions formulated in [4]. Seemingly, none of these conditions is more general than the other if applied to the input/output systems. On the other hand, the author suspects the condition introduced in this paper as a candidate to be also necessary for nonoscillatory behavior of input/output systems.

It is interesting to note that the main result formulated in Theorem 5.3. can formally be generalized to input/output systems in which the order is induced into the models via generalized notion of nonnegativity in the spirit of [3]. In such a case the generalized theory goes far beyond the frame of standard input/output systems. In particular, the interpretation of the solutions to (4.2) as probabilities may be lost.

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