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ALGEBRAIC CLASSIFICATION OF THE WEYL TENSOR: SELECTED APPLICATIONS

Vojtěch Pravda

Institute of Mathematics AS CR
Žitná 25, 115 67 Prague 1, Czech Republic
pravda@math.cas.cz

Abstract

Selected applications of the algebraic classification of tensors on Lorentzian manifolds of arbitrary dimension are discussed. We clarify some aspects of the relationship between invariants of tensors and their algebraic class, discuss generalization of Newman-Penrose and Geroch-Held-Penrose formalisms to arbitrary dimension and study an application of the algebraic classification to the case of quadratic gravity.

1. Introduction

This contribution focuses on selected applications of the algebraic classification of tensors on Lorentzian manifolds of arbitrary dimension introduced in [2, 15] and conveniently summarized in paper [20] in this volume.

The algebraic classification of tensors in higher dimensions was originally developed in the context of studying invariants of the Weyl tensor and naturally first applications were in this area [3]. However, this research is still in development and very recently new important insights were obtained in [12]. Therefore, in section 2 we overview the relationship between the algebraic class of a tensor and its polynomial invariants. In this context we also discuss VSI spacetimes - spacetimes with vanishing curvature invariants.

In section 3, we focus on the generalization of the Newman-Penrose (NP) formalism and the Geroch-Held-Penrose (GHP) formalism to the case of arbitrary dimension [19, 16, 8]. The main goal of this section is to show how assuming the spacetime to be algebraically special often leads to a dramatic simplification of PDEs of this formalism. To illustrate this point we discuss two problems: i) we solve the Sachs equation for non-degenerate algebraically special spacetimes and ii) we use GHP formalism to prove a proposition about type N Einstein spacetimes.

In section 4, we show how assuming the Weyl tensor to belong to certain algebraically special classes leads to a considerable simplification of the field equations of a particular generalization of the Einstein theory, so called quadratic gravity. This approach allows us to identify a new class of exact solutions of this theory.

2. Invariants of a tensor

As pointed out in [2, 15], the algebraic classification based on the alignment theory developed there can be applied to an arbitrary tensor on a Lorentzian manifold. Using notation of [12], the tensor $\boldsymbol{T}$ can be decomposed according to boost weight of its components

$$\boldsymbol{T} = \sum_b (T)_b \tag{1}$$

with $(T)_b$ being components of boost weight b. By definition, $\boldsymbol{T}$ is said to be of type II if there exists a frame for which all positive boost weight components vanish and type III if only negative boost weight components are non-zero.

Some information encoded in the tensor $\boldsymbol{T}$ can be invariantly expressed in terms of its polynomial invariants. For a rank two tensor, examples of such invariants are its trace $T^a_{\ a}$ or $T_{ab}T^{ab}$. Note that in the case of the Riemannian signature $T_{ab}T^{ab}$ is essentially a sum of squares of all components of $\boldsymbol{T}$ and is thus non-vanishing. In the Lorentzian case $T_{ab}T^{ab}$ can vanish for (special) non-trivial $\boldsymbol{T}$ but in principle some more complicated invariants, such as $T_{ab}T^{ac}T^b_{\ c}$, could survive. What are the necessary and sufficient conditions for vanishing of *all* polynomial invariants of $\boldsymbol{T}$?

Very recently the following proposition (discussed previously as the *algebraic VSI conjecture* in [3]) was proven in [12]

Proposition 1. *All polynomial invariants of a tensor $\boldsymbol{T}$ of arbitrary rank on a Lorentzian manifold of arbitrary dimension vanish if and only if $\boldsymbol{T}$ is of type III.*

Thus clearly a tensor $\boldsymbol{T}$ of type III contains more information than its polynomial invariants. One can then ask under what conditions polynomial invariants of a tensor $\boldsymbol{T}$ contain less information than $\boldsymbol{T}$ or in other words when $\boldsymbol{T}$ is not characterized by its invariants. In terms of the algebraic classification the answer is again surprisingly simple [12].

Proposition 2. *Assume that a tensor $\boldsymbol{T}$ is not characterized by its polynomial invariants. Then it is of type II or more special.*

The elegant proof [12] of this proposition (in fact of a more general statement applying to invariants of a set of tensors) combines the use of invariant theory, group theory and real analysis.

Definition 1. *Curvature invariant of order p is a polynomial invariant constructed from metric, curvature tensors (the Riemann, Ricci, and Weyl tensors) and their covariant derivatives up to order p.*

Definition 2. *We say that a manifold M with a metric of arbitrary signature is VSI (vanishing scalar invariants) if all curvature invariants of all orders vanish at all points of M.*

The VSI condition is obviously very restrictive and in the case of positive definite metric (the Riemannian case) the only such manifold is a flat space. However, in the Lorentzian case, the set of VSI manifolds is non-trivial.

Theorem 1. *VSI Theorem: A Lorentzian manifold of arbitrary dimension is VSI if and only if the following two conditions are satisfied:*
(A) The spacetime possesses a non-expanding, twist-free, shear-free, geodesic null congruence and consequently belongs to the Kundt class.
(B) Relative to the above null congruence, all curvature tensors are of algebraic type III or more special.

In four dimensions this theorem has been proven in [18]. In [3] it has been proven that conditions (A) and (B) imply VSI in arbitrary dimension. The part of the proof showing that VSI property implies (A) and (B) has been given there only under the assumption that the algebraic VSI conjecture holds[1]. Thanks to the Proposition 1. the proof of the VSI theorem is now complete.

Motivated by the VSI theorem, various authors have studied Kundt spacetimes in arbitrary dimension and explicit metrics of Kundt type III and type N spacetimes are now known (see e.g. [1]).

Apart from differential geometry [10], VSI spacetimes are of interest in various physical theories, such as general relativity [9] and supergravity [6] or when studying quantum corrections of these theories [5]. Possible applications of VSI spacetimes in string theory are also discussed in [4].

Recently VSI spacetimes with more general signatures were studied in [13].

3. Generalization of NP and GHP formalisms to higher dimensions

In four dimensions Newman-Penrose formalism and Geroch-Held-Penrose formalism are essential tools for finding exact solutions of the Einstein field equations and analyzing their properties. Using the higher dimensional classification of the Weyl tensor, these methods were generalized to arbitrary dimension in [19, 16] (NP) and [8] (GHP).

In the NP (GHP) formalism, Einstein equations can be rewritten as a particular set of first order partial differential equations. These equations are considerably simplified when searching for algebraically special solutions. In four dimensions, this approach has led to discovery of many exact solutions of the Einstein equations including the famous Kerr solution describing gravitational field of a rotating black hole.

In higher dimensions, a similar simplification for algebraically special solutions occurs, however, the search for new exact solutions is still in the initial phase. We will therefore illustrate advantages of the NP (GHP) formalism on a few *selected* equations from the complete set of PDEs.

[1] An omission in the original proof that VSI imply (A) and (B) has been clarified in [12].

We will use a null frame consisting of two null vectors $\boldsymbol{\ell} = \boldsymbol{m^{(1)}} = \boldsymbol{m_{(0)}}$ and $\boldsymbol{n} = \boldsymbol{m^{(0)}} = \boldsymbol{m_{(1)}}$ and $n-2$ spacelike vectors $\boldsymbol{m^{(i)}} = \boldsymbol{m_{(i)}}$ obeying

$$\ell^a \ell_a = n^a n_a = 0, \quad \ell^a n_a = 1, \quad m^{(i)a} m^{(j)}_a = \delta_{ij}, \quad a = 0 \dots n-1 \tag{2}$$

with $i, j, k = 2 \dots n-1$. Thus the metric can be expressed as

$$g_{ab} = 2\ell_{(a} n_{b)} + \delta_{ij} m^{(i)}_a m^{(j)}_b. \tag{3}$$

Let us denote covariant derivatives of the basis vectors as

$$L_{\mu\nu} = \nabla_\nu \ell_\mu, \qquad N_{\mu\nu} = \nabla_\nu n_\mu, \qquad \overset{i}{M}_{\mu\nu} = \nabla_\nu m_{(i)\mu}. \tag{4}$$

The projections onto the basis are the scalars L_{ab}, N_{ab}, $\overset{i}{M}_{ab}$. As a consequence of (2), they are subject to the following conditions

$$N_{0a} + L_{1a} = 0, \quad \overset{i}{M}_{0a} + L_{ia} = 0, \quad \overset{i}{M}_{1a} + N_{ia} = 0, \quad \overset{i}{M}_{ja} + \overset{j}{M}_{ia} = 0, \tag{5}$$

$$L_{0a} = N_{1a} = \overset{i}{M}_{ia} = 0. \tag{6}$$

Vector field $\boldsymbol{\ell}$ is tangent to a null geodesic if and only if $\kappa_i \equiv L_{i0} = 0$ and in such a case one can always choose an affine parameterization with $L_{10} = 0$. Then expansion, shear and twist of the congruence are determined by trace, trace-free symmetric and antisymmetric parts of $\rho_{ij} \equiv L_{ij}$, respectively.

Let us also introduce covariant derivatives along the frame vectors by

$$D \equiv \ell^a \nabla_a, \qquad \triangle \equiv n^a \nabla_a, \qquad \delta_i \equiv m^a_{(i)} \nabla_a. \tag{7}$$

It is often more convenient to introduce compact GHP derivative operators þ and ð which still obey the Leibnitz rule. The full definition of these operators can be found in [8]. Here we just give few illustrative examples

$$\begin{aligned}
þ\rho_{ij} &= D\rho_{ij} - L_{10}\rho_{ij} + \overset{k}{M}_{i0}\rho_{kj} + \overset{k}{M}_{j0}\rho_{ik}, && (8)\\
þ\Phi_{ij} &= D\Phi_{ij} + \overset{s}{M}_{i0}\Phi_{sj} + \overset{s}{M}_{j0}\Phi_{is}, && (9)\\
þ\Psi_{ijk} &= D\Psi_{ijk} + L_{10}\Psi_{ijk} + \overset{s}{M}_{i0}\Psi_{sjk} + \overset{s}{M}_{j0}\Psi_{isk} + \overset{s}{M}_{k0}\Psi_{ijs}, && (10)\\
þ\Omega'_{ij} &= D\Omega'_{ij} + 2L_{10}\Omega'_{ij} + 2\Psi_{(i|s}\overset{s}{M}_{|j)0}. && (11)
\end{aligned}$$

3.1. Ricci equations

Contractions of the Ricci identity $v_{a;bc} - v_{a;cb} = R_{sabc} v^s$ with various combinations of the frame vectors and with v^a being either ℓ^a, n^a or $m^a_{(i)}$ lead to a set of first order differential equations which are in full given in [16].

The point of this section is to illustrate how a clever choice of a frame for Einstein spacetimes admitting a Weyl aligned null direction (WAND) leads to a considerable simplification of these equations. Here let us discuss only the case of the Sachs equation (12) which can be set to a strikingly simple form (14).

The Sachs equation in full generality reads

$$\begin{aligned} DL_{ij} - \delta_j L_{i0} &= L_{10}L_{ij} - L_{i0}(2L_{1j} + N_{j0}) - L_{i1}L_{j0} \\ &\quad +2L_{k[0|}\overset{k}{M}_{i|j]} - L_{ik}(L_{kj} + \overset{k}{M}_{j0}) - C_{0i0j} - \tfrac{1}{n-2}R_{00}\delta_{ij}. \end{aligned} \tag{12}$$

By choosing a frame parallelly propagated along the geodetic congruence $\boldsymbol{\ell}$ it reduces to

$$DL_{ij} = -L_{ik}L_{kj} - C_{0i0j} - \tfrac{1}{n-2}R_{00}\delta_{ij} \tag{13}$$

and for Einstein spaces of type I (or more special algebraic types) it becomes

$$DL_{ij} = -L_{ik}L_{kj}, \quad \text{in matrix form:} \quad D\mathbf{L} = -\mathbf{L}^2. \tag{14}$$

For invertible matrix $\mathbf{L}$, the Sachs equation implies [17]

$$D\mathbf{L}^{-1} = \mathbf{I} \quad \Rightarrow \quad \mathbf{L}^{-1} = r\mathbf{I} - \mathbf{b}, \quad D\mathbf{b} = 0. \tag{15}$$

where $\mathbf{I}$ is the identity matrix, r is an affine parameter along the geodetic congruence $\boldsymbol{\ell}$ and $\mathbf{b}$ is a matrix constant along each geodetic (and thus independent on r).

Note that

$$L^{-1}_{[ij]} = -b_{[ij]}, \tag{16}$$

and thus $L_{[ij]} = 0 \Leftrightarrow b_{[ij]} = 0$. Therefore the antisymmetric part of $\mathbf{b}$ is responsible for twist.

3.2. Bianchi equations

For Einstein spacetimes, the Ricci tensor is proportional to the metric and consequently $\nabla_\rho R_{\mu\nu} = 0$. Therefore Bianchi identity $\nabla_{[\tau|}R_{\mu\nu|\rho\sigma]} = 0$ implies that

$$\nabla_{[\tau|}C_{\mu\nu|\rho\sigma]} = 0. \tag{17}$$

The frame components of these equations lead to a set of complicated first order PDEs which can be found in [8]. These equations can be greatly simplified by assuming algebraically special spacetimes.

Let us further discuss the simplest non-trivial case, type N Einstein spacetimes, to provide an illustration of the use of NP/GHP formalism. The following proposition for the Ricci-flat case was proven in [19]. Considerably shorter proof applying also to Einstein spaces was given in [8]. Hereafter we thus follow [8].

Proposition 3. *The multiple WAND $\boldsymbol{\ell}$ of type N Einstein spacetime is necessarily geodetic and the optical matrix $\boldsymbol{\rho}$ can be cast to the form*

$$\boldsymbol{\rho} = \left(\begin{array}{c|c} \frac{1}{2}\begin{pmatrix} \rho & a \\ -a & \rho \end{pmatrix} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array} \right) \tag{18}$$

Proof. In a type N Einstein spacetime the only non-vanishing components of the Weyl tensor are determined by a symmetric traceless matrix Ω'_{ij}. Therefore Bianchi equation (B5) from [8]

$$\begin{aligned} þ'\Psi_{ijk} - 2ð_{[j|}\Phi_{i|k]} &= 2(\Psi'_{[j|}\delta_{il} - \Psi'_{[j|il)})\rho_{l|k]} + (2\Phi_{i[j}\delta_{k]l} - 2\delta_{il}\Phi^{\mathrm{A}}_{jk} - \Phi_{iljk})\tau_l \\ &\quad +2(\Psi_i\delta_{[j|l} - \Psi_{i[j|l)})\rho'_{l|k]} + 2\Omega_{i[j}\kappa'_{k]}, \end{aligned} \tag{19}$$

implies

$$\Omega'_{i[j}\kappa_{k]} = 0, \tag{20}$$

where the square brackets denote antisymmetrization. By tracing (20) over i and k we obtain

$$\Omega'_{ij}\kappa_i = 0, \tag{21}$$

while by multiplying (20) by Ω'_{ik} and using (20) we arrive to

$$(\Omega'_{ik}\Omega'_{ik})\kappa_j = 0. \tag{22}$$

Since for type N Ω'_{ik} possesses at least one non-vanishing component we conclude that $\kappa_j = 0$ and thus *the multiple WAND for Einstein type N spacetimes is always geodetic.*

Now the remaining Bianchi equations [8] imply

$$\begin{aligned} þ\Omega'_{ij} &= -\Omega'_{ik}\rho_{kj}, & (23) \\ \Omega'_{i[j}\rho_{kl]} &= 0, & (24) \\ \Omega'_{i[k|}\rho_{j|l]} &= \Omega'_{j[k|}\rho_{i|l]}. & (25) \end{aligned}$$

Let us denote symmetric and antisymmetric parts of $\boldsymbol{\rho}$ as $\mathbf{S}$ and $\mathbf{A}$, respectively. Tracing (24) and (25) over i and k leads to

$$\begin{aligned} \mathbf{\Omega}'\mathbf{A} + \mathbf{A}\mathbf{\Omega}' &= 0, & (26) \\ \mathbf{\Omega}'\boldsymbol{\rho} + \boldsymbol{\rho}\mathbf{\Omega}' &= (\mathrm{tr}\boldsymbol{\rho})\mathbf{\Omega}', & (27) \end{aligned}$$

respectively. Using (26), eq. (27) reduces to

$$\mathbf{\Omega}'\mathbf{S} + \mathbf{S}\mathbf{\Omega}' = (\mathrm{tr}\mathbf{S})\mathbf{\Omega}'. \tag{28}$$

The antisymmetric part of (23) reads

$$0 = -[\mathbf{\Omega}', \mathbf{S}] - (\mathbf{\Omega}'\mathbf{A} + \mathbf{A}\mathbf{\Omega}'), \tag{29}$$

which, together with (26), gives

$$[\mathbf{\Omega}', \mathbf{S}] = 0. \tag{30}$$

This allows us to use rotations of the $m_{(i)}$ to diagonalize simultaneously both $\mathbf{\Omega}'$ and $\mathbf{S}$.

Let us denote the number of non-vanishing eigenvalues of $\mathbf{\Omega}'$ by N. We can shuffle the vectors $m_{(i)}$ so that

$$\mathbf{\Omega}' = \mathrm{diag}(\psi_{(2)}, ..., \psi_{(N+1)}, 0, ..., 0), \quad \mathbf{S} = \mathrm{diag}(s_{(2)}, ..., s_{(n-1)}), \tag{31}$$

with all the $\psi_{(\alpha)}$ being non-zero[2]. Note that for type N spacetime $N \geq 1$. Substituting (31) into (28) leads to

$$\psi_{(i)} s_{(i)} = \frac{1}{2} \psi_{(i)} (\mathrm{tr}\mathbf{S}) \quad \text{(no summation over } i) \tag{32}$$

for all i and thus

$$s_{(\alpha)} = \frac{\mathrm{tr}\mathbf{S}}{2} \quad \text{for} \quad \alpha = 2, ..., N+1. \tag{33}$$

The αI component of (26) implies that $A_{I\alpha} = 0 = A_{\alpha I}$, and thus $\boldsymbol{\rho}$ is block diagonal with blocks of size N and $n-2-N$. The $ijkl = I\alpha J\beta$ component of the Bianchi equation (25) implies $\Omega'_{\alpha\beta}\rho_{IJ} = 0$ and therefore $\rho_{IJ} = 0$.

So far we have shown that

$$\boldsymbol{\rho} = \left(\begin{array}{c|c} \frac{\rho}{2}\mathbf{1}_N + \mathbf{A}_N & 0 \\ \hline 0 & 0 \end{array} \right), \tag{34}$$

where $\mathrm{tr}\mathbf{S} = \rho$, $\mathbf{1}_N$ is the $N \times N$ identity matrix, and $\mathbf{A}_N$ is an antisymmetric $N \times N$ matix. The trace of the above equation leads to

$$\rho = N\rho/2 \tag{35}$$

and thus either (i) $N = 2$ or (ii) $\rho = 0$.

For $N = 2$ we have shown that $\boldsymbol{\rho}$ must be of the form (18) for some a.

In the second case (ii) vanishing of ρ implies $\mathbf{S} = \mathbf{0}$. The trace of the Sachs equation reads $\text{Þ}(\mathrm{tr}\mathbf{S}) = -\mathrm{tr}(\mathbf{S}^2) - \mathrm{tr}(\mathbf{A}^2)$. Consequently $\mathrm{tr}(\mathbf{A}^2) = -A_{ij}A_{ij}$ also vanishes, implying $\mathbf{A} = \mathbf{0}$. Such spacetime is thus Kundt. □

4. Quadratic gravity

In this section we would like to follow [14] to point out that field equations of various theories generalizing the Einstein gravity can be considerably simplified by choosing sufficiently special algebraic type of the metric.

In perturbative quantum gravity, corrections have to be added to the Einstein action. Since we require coordinate invariance, these corrections consist of various curvature invariants. One important class of such modified gravities is quadratic gravity whose action contains general quadratic terms in curvature [7]

$$S = \int \mathrm{d}^n x \sqrt{-g} \left(\frac{1}{\kappa} (R - 2\Lambda_0) + \alpha R^2 + \beta R_{ab}^2 + \gamma \left(R_{abcd}^2 - 4R_{ab}^2 + R^2 \right) \right). \tag{36}$$

[2]From now on indices $\alpha, \beta, ...$ range over $2, ..., N+1$ and $I, J, ...$ range over $N+2, ..., n-1$.

The action (36) leads to vacuum quadratic gravity field equations [11]

$$\frac{1}{\kappa}\left(R_{ab}-\frac{1}{2}Rg_{ab}+\Lambda_0 g_{ab}\right)+2\alpha R\left(R_{ab}-\frac{1}{4}Rg_{ab}\right)+(2\alpha+\beta)\left(g_{ab}\nabla^c\nabla_c-\nabla_a\nabla_b\right)R$$
$$+2\gamma\left(RR_{ab}-2R_{acbd}R^{cd}+R_{acde}R_b{}^{cde}-2R_{ac}R_b{}^c-\frac{1}{4}g_{ab}\left(R^2_{cdef}-4R^2_{cd}+R^2\right)\right)$$
$$+\beta\nabla^c\nabla_c\left(R_{ab}-\frac{1}{2}Rg_{ab}\right)+2\beta\left(R_{acbd}-\frac{1}{4}g_{ab}R_{cd}\right)R^{cd}=0. \tag{37}$$

Obviously these fourth order non-linear PDEs are far more complicated than Einstein equations

$$R_{ab}=\frac{2\Lambda}{n-2}g_{ab} \tag{38}$$

and it seems hopeless to attempt to solve this system without starting with some simplifying assumptions.

If we restrict our interest to Einstein spacetimes obeying (38), the equations of quadratic gravity reduce to [14]

$$\mathcal{B}g_{ab}-\gamma\left(C_a{}^{cde}C_{bcde}-\frac{1}{4}g_{ab}C^{cdef}C_{cdef}\right)=0, \tag{39}$$

where

$$\mathcal{B}=\frac{\Lambda-\Lambda_0}{2\kappa}+\Lambda^2\left(\frac{(n-4)}{(n-2)^2}(n\alpha+\beta)+\frac{(n-3)(n-4)}{(n-2)(n-1)}\gamma\right). \tag{40}$$

Note that in the above equations we have used

$$R_{abcd}=C_{abcd}+\frac{2}{n-2}(g_{a[c}R_{d]b}-g_{b[c}R_{d]a})-\frac{2}{(n-1)(n-2)}Rg_{a[c}g_{d]b} \tag{41}$$

to express the Riemann tensor in terms of the Weyl and Ricci tensors and the scalar curvature $R=\frac{2n}{n-2}\Lambda$.

It can be shown that for type N spacetimes

$$C_a{}^{cde}C_{bcde}=C^{cdef}C_{cdef}=0 \tag{42}$$

and thus in this case eqs. (39), (40) reduce to a simple algebraic constraint relating the effective cosmological constant Λ with parameters of the quadratic gravity α, β, γ, κ, Λ_0. We thus arrive at

Proposition 4. *In arbitrary dimension all Weyl type N Einstein spacetimes with cosmological constant* Λ *(chosen to obey* $\mathcal{B}=0$*) are exact solutions of quadratic gravity (37).*

Since many type N Einstein spacetimes are known, we obtained "for free" a rich class of exact solutions of quadratic gravity.

These results may be partially generalized to the case of type III spacetimes or to the case of type N spacetimes which are not Einstein but admit Ricci tensor of the form

$$R_{ab} = \frac{2\Lambda}{n-2} g_{ab} + \Phi \ell_a \ell_b. \tag{43}$$

See [14] for further details.

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