Conference Applications of Mathematics 2012
in honor of the 60th birthday of Michal Křížek.
Institute of Mathematics AS CR, Prague 2012

BOUNDARY ELEMENT METHOD FOR CONVEX BOUNDARY CONTROL PROBLEMS

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Abstract

In this paper, we discuss the numerical methods for a class of convex boundary control problems. The boundary element method is applied for the approximations of the problems. The a posteriori error estimators for the boundary element approximations are presented, which can be applied as the indicators of the adaptive mesh refinement of the related boundary element methods.

1. Introduction

In this paper, we consider the numerical methods for the boundary control problem governed by the elliptic partial differential equations. It is described as follows:

$$\min_{u\in\Lambda}\left\{\frac{1}{2}\left\|\frac{\partial y}{\partial n}-q_0\right\|^2_{-1,\partial\Omega}+\frac{\alpha}{2}\|u\|^2_{0,\partial\Omega}\right\} \tag{1.1}$$

subject to

$$\begin{aligned} -\Delta y &= 0 \quad \text{in} \quad \Omega \\ y &= Bu \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{1.2}$$

Note that the control is applied only on the boundary of domain, and the objective functional is also only defined by the boundary information of the control and state. It is reasonable to use the boundary element method instead of the finite element method to make the numerical approximation for above boundary control problem.

Although the boundary element method and adaptive boundary element method are useful methods for the numerical approximations of the partial differential equations and have been investigated deeply (see, e.g., [8], [9] and [15], for more details), there are only a few work on the boundary element methods for the optimal control problem governed by partial differential equations (see, e.g., [13]), where the a priori error estimates are presented. While there are many work on a priori and a posteriori error estimates of the finite element method for the optimal control problem governed by partial differential equations (see, e.g., [10], [12]), and some related work on the optimal control problem governed by integral equations (see, e.g., [4]).

In this paper, we provide the boundary element scheme for the the boundary control problem (1.1)-(1.2). The a posteriori error estimate are presented, which can be used as the indicator for the adaptive mesh refinement of the boundary element methods. Note that in this scheme, the control u only belongs to $L^2(\partial\Omega)$, then the condition of smooth boundary instead of piecewise Lipschitz boundary is required. This restricts the application of the scheme, and more research should be completed to extend the related results to more practical cases. The techniques on residual type a posteriori error estimates for boundary element methods on partial differential equations (see, e.g., [5]) are applied in this paper. But to our best knowledge, this kind of a posteriori error estimates for boundary element methods on boundary control problems is new.

The plan of the paper is as follows. In Section 2, the model problem of the boundary control problem governed by elliptic partial differential equations is described, and the boundary element scheme is presented for the model problem. Then the a posteriori error estimates for the boundary element method are discussed in Section 3.

2. Boundary control problem and boundary element scheme

Let Ω be a bounded open set in R^2 with smooth (C^∞) boundary $\partial\Omega$. We adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{m,q,\Omega}$ and semi-norm $|\cdot|_{m,q,\Omega}$. We denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$, with norm $\|\cdot\|_{m,\Omega}$ and semi-norm $|\cdot|_{m,\Omega}$. In addition, c and C denote generic positive constants which can be different in different places.

Let us consider the convex boundary control problem governed by elliptic partial differential equation:

$$\min_{u\in\Lambda}\left\{\frac{1}{2}\left\|\frac{\partial y}{\partial n}-q_0\right\|^2_{-1,\partial\Omega}+\frac{\alpha}{2}\|u\|^2_{0,\partial\Omega}\right\} \tag{2.1}$$

subject to

$$\begin{aligned} -\Delta y &= 0 \quad \text{in} \quad \Omega,\\ y &= Bu \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.2}$$

where B is a linear operator from $L^2(\partial\Omega)$ to $L^2(\partial\Omega)$, $q_0 \in H^{-1}(\partial\Omega)$ is a given function, Λ is a convex subset in the space $U := L^2(\partial\Omega)$, $\Omega \subset R^2$ is a bounded domain, n is the outward normal of $\partial\Omega$. In this paper, let

$$\Lambda = \{v \in L^2(\partial\Omega) : \; v \geq \beta\},$$

where β is a constant. Above boundary control problem (2.1)-(2.2) was discussed in page 77 of [11]. Because we only can apply Bu on the boundary by the control $u \in L^2(\partial\Omega)$, we have $y \in H^{\frac{1}{2}}(\Omega)$ and $\frac{\partial y}{\partial n} \in H^{-1}(\partial\Omega)$. The related regularity can be found in [11].

Let $\gamma(\cdot,\cdot)$ be the fundamental solution of equation (2.2), such that

$$\gamma(x,y) = \frac{1}{2\pi} \log|x-y|.$$

Moreover, set

$$\begin{aligned} V\phi(z) &= -2\int_{\partial\Omega} \phi(x)\gamma(z,x)ds_x, \\ K\phi(z) &= -2\int_{\partial\Omega} \phi(x)\frac{\partial}{\partial n_x}\gamma(z,x)ds_x, \\ K'\phi(z) &= -2\int_{\partial\Omega} \phi(x)\frac{\partial}{\partial n_z}\gamma(z,x)ds_x. \end{aligned}$$

Then it is well known (see e.g. [7]) that

$$V: \ H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega), \tag{2.3}$$

$$K: \ H^{\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega), \tag{2.4}$$

$$K': \ H^{-\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega) \tag{2.5}$$

are linear and continuous, K' is the dual of K, and V is symmetric.

Setting $q = \frac{\partial y}{\partial n}$, the equation (2.2) can be rewritten to the boundary integral equation:

$$Vq(z) = KBu(z) + Bu(z) \quad z \in \partial\Omega. \tag{2.6}$$

Similar to (2.3), we have $V: \ H^{-1}(\partial\Omega) \to L^2(\partial\Omega)$. Then, noting that Vq, KBu, $Bu \in L^2(\partial\Omega)$, the equation (2.6) can be rewritten to the standard Galerkin formulation:

$$(Vq,\phi) = (KBu,\phi) + (Bu,\phi) \quad \forall \phi \in L^2(\partial\Omega).$$

Moreover, we can use $\|V(q-q_0)\|^2_{0,\Omega}$ to replace $\|q-q_0\|^2_{-1,\partial\Omega} = \|\frac{\partial y}{\partial n} - q_0\|^2_{-1,\partial\Omega}$ in (2.1).

Therefore, the control problem (2.1)-(2.2) can be rewritten to

$$\min_{u\in\Lambda}\{\frac{1}{2}\|V(q-q_0)\|^2_{0,\partial\Omega} + \frac{\alpha}{2}\|u\|^2_{0,\partial\Omega}\} \tag{2.7}$$

subject to

$$(Vq,\phi) = (KBu,\phi) + (Bu,\phi) \quad \forall \phi \in L^2(\partial\Omega), \tag{2.8}$$

where $(\cdot,\cdot)$ presents the inner product in $L^2(\partial\Omega)$.

Using the standard method in [11], it can be proven that the problem (2.7)-(2.8) has a solution $(q,u) \in H^{-1}(\partial\Omega) \times L^2(\partial\Omega)$, and that the pair (q,u) is a solution of (2.7)-(2.8) if and only if there exists a co-state $p \in L^2(\partial\Omega)$ such that the triple (q, p, u) satisfies the optimality condition:

$$(Vq, \phi) = (KBu, \phi) + (Bu, \phi) \quad \forall \phi \in L^2(\partial\Omega) \tag{2.9}$$

$$(p, V\psi) = (V(q - q_0), V\psi) \quad \forall \psi \in H^{-1}(\partial\Omega) \tag{2.10}$$

$$(\alpha u + B^* K' p + B^* p, v - u) \geq 0 \quad \forall v \in \Lambda \subset L^2(\partial\Omega), \tag{2.11}$$

where B^* is the adjoint operator of B.

Next, let us consider the boundary element approximation of the control problem (2.7) and (2.8).

Let T^h be a partitioning of $\partial\Omega$ into disjoint segmental arc τ, so that $\partial\Omega = \bigcup_{\tau \in T^h} \bar{\tau}$. Set $W^h \subset H^1(\partial\Omega)$ to be a finite-dimensional subspace related on the partition T^h, such that $\chi|_\tau$ are polynomials of order m $(m \geq 1)$ for all $\chi \in W^h$ and $\tau \in T^h$ (see, i.e., [15] and [16]). It is easy to see that $W^h \subset H^1(\partial\Omega) \subset L^2(\partial\Omega) \subset H^{-1}(\partial\Omega)$.

Similarly, let T_U^h be a partitioning of $\partial\Omega$ into disjoint segmental arc τ_U, so that $\partial\Omega = \bigcup_{\tau_U \in T_U^h} \bar{\tau}_U$. Again, set $U^h \subset L^2(\partial\Omega)$ to be another finite-dimensional subspace related on the partition T_U^h, such that $\chi|_{\tau_U}$ are polynomials of order m $(m \geq 0)$ for all $\chi \in U^h$ and $\tau_U \in T_U^h$. Note that there is no continuity requirement for U^h. It is easy to see that $U^h \subset U = L^2(\partial\Omega)$.

Let h_τ (h_{τ_U}) denote the maximum length of the element τ (τ_U) in T^h (T_U^h). Let $h = \max_{\tau \in T^h} h_\tau$ $(h_U = \max_{\tau_U \in T_U^h} h_{\tau_U})$. Let $\Lambda^h = \Lambda \cap U^h$ be a close convex set. Note that the regularity of the optimal control u is limited. It is only in $H^1(\partial\Omega)$ in general, because of the structure of Λ and the inequality (2.11). Therefore, there will be no advantage in considering higher-order finite element spaces for U^h. We only consider the piecewise linear and piecewise constant finite element spaces for W^h and U^h, i.e., $W^h = \{w \in H^1(\partial\Omega) : w|_\tau \in P_1\}$ and $U^h = \{w \in L^2(\partial\Omega) : w|_{\tau_U} \in P_0\}$ in this paper, where P_1 denotes the linear function space, and P_0 denotes the 0-order polynomial space.

Using above boundary element space, the boundary element approximation of the control problem (2.7) and (2.8) is defined by

$$\min_{u_h \in \Lambda^h} \left\{ \frac{1}{2} \|V(q_h - q_0)\|_{0,\partial\Omega}^2 + \frac{\alpha}{2} \|u_h\|_{0,\partial\Omega}^2 \right\} \tag{2.12}$$

subject to

$$(Vq_h, \phi_h) = (KBu_h, \phi_h) + (Bu_h, \phi_h) \quad \forall \phi_h \in W^h. \tag{2.13}$$

Similar to the continuous problem (2.7)-(2.8), the control problem (2.12)-(2.13) has a solution (q_h, u_h), and that a pair (q_h, u_h) is a solution of (2.12)-(2.13) if and only if there exists a co-state $p_h \in V^h$ such that the triple (q_h, p_h, u_h) satisfies the following optimality conditions:

$$(Vq_h, \phi_h) = (KBu_h, \phi_h) + (Bu_h, \phi_h) \quad \forall \phi_h \in W^h \subset L^2(\partial\Omega) \tag{2.14}$$

$$(p_h, V\psi_h) = (V(q_h - q_0), V\psi_h) \quad \forall \psi_h \in W^h \subset H^{-1}(\partial\Omega) \tag{2.15}$$

$$(\alpha u_h + B^* K' p_h + B^* p_h, v_h - u_h) \geq 0 \quad \forall v_h \in \Lambda^h \subset \Lambda \subset U = L^2(\partial\Omega). \tag{2.16}$$

3. A posteriori error analysis

In this section, we will discuss the a posteriori error estimates of the boundary element methods provided in the last section. In order to do it, let us divide $\partial\Omega$ into three subdomains:

$$\begin{aligned}
\partial\Omega^- &:= \{x \in \partial\Omega : \ (B^*K'p_h + B^*p_h)(x) \le -\alpha\beta\},\\
\partial\Omega^+ &: \{x \in \partial\Omega : \ (B^*K'p_h + B^*p_h)(x) > -\alpha\beta, \ u_h > \beta\},\\
\partial\Omega^0 &: \{x \in \partial\Omega : \ (B^*K'p_h + B^*p_h)(x) > -\alpha\beta, \ u_h = \beta\}.
\end{aligned}$$

Then we have the following a posteriori error estimates:

Theorem 3.1. *Let (y, p, u) and (y_h, p_h, u_h) be the solutions of the systems (2.9)–(2.11) and (2.14)–(2.16), respectively. Then,*

$$\|q - q_h\|^2_{-1,\partial\Omega} + \|p - p_h\|^2_{0,\partial\Omega} + \|u - u_h\|^2_{0,\partial\Omega} \le C(\eta_1^2 + \eta_2^2 + \eta_3^2), \tag{3.1}$$

where

$$\begin{aligned}
\eta_1^2 &= \|B^*K'p_h + B^*p_h + \alpha u_h\|^2_{0,\partial\Omega^-\cup\partial\Omega^+},\\
\eta_2^2 &= \|Vq_h - KBu_h - Bu_h\|^2_{0,\partial\Omega},\\
\eta_3^2 &= \|p_h - V(q_h - q_0)\|^2_{0,\partial\Omega}.
\end{aligned}$$

Proof. Let $q(u_h)$ and $p(u_h)$ be the solutions of the auxiliary equations:

$$(Vq(u_h), \phi) = (KBu_h, \phi) + (Bu_h, \phi) \quad \forall \phi \in L^2(\partial\Omega) \tag{3.2}$$

$$(p(u_h), V\psi) = (V(q(u_h) - q_0), V\psi) \quad \forall \psi \in H^{-1}(\partial\Omega). \tag{3.3}$$

It follows from (2.11) that

$$\begin{aligned}
\alpha\|u - u_h\|^2_{0,\partial\Omega} =& (\alpha u, u - u_h) - (\alpha u_h, u - u_h)\\
\le& -(B^*K'p + B^*p, u - u_h) - \alpha(u_h, u - u_h)\\
=& (B^*K'(p - p(u_h)), u_h - u) + (B^*(p - p(u_h)), u_h - u)\\
&+ (B^*K'(p(u_h) - p_h), u_h - u) + (B^*(p(u_h) - p_h), u_h - u)\\
&+ (B^*K'p_h + B^*p_h + \alpha u_h, u_h - u).
\end{aligned} \tag{3.4}$$

It follows from (2.9)-(2.10) and (3.2)-(3.3) that

$$\begin{aligned}
&(B^*K'(p - p(u_h)), u_h - u) + (B^*(p - p(u_h)), u_h - u)\\
&= (B^*K'(p - p(u_h)) + B^*(p - p(u_h)), u_h - u)\\
&= (p - p(u_h), KB(u_h - u) + B(u_h - u))\\
&= (V(q(u_h) - q), p - p(u_h)) = (p - p(u_h), V(q(u_h) - q))\\
&= (V(q - q_0), V(q(u_h) - q)) - (V(q(u_h) - q_0), V(q(u_h) - q))\\
&= (V(q - q(u_h)), V(q(u_h) - q)) \le 0.
\end{aligned} \tag{3.5}$$

Hence, (3.4) and (3.5) imply that

$$\alpha\|u-u_h\|^2_{0,\partial\Omega} \le C\|p(u_h)-p_h\|_{0,\partial\Omega}\|u-u_h\|_{0,\partial\Omega}+(B^*K'p_h+B^*p_h+\alpha u_h, u_h-u). \quad (3.6)$$

Note that

$$(B^*K'p_h + B^*p_h + \alpha u_h, u_h - u) = \int_{\partial\Omega^-\cup\partial\Omega^+} (B^*K'p_h + B^*p_h + \alpha u_h)(u_h - u) + \int_{\partial\Omega^0} (B^*K'p_h + B^*p_h + \alpha u_h)(u_h - u), \quad (3.7)$$

and

$$\int_{\partial\Omega^-\cup\partial\Omega^+} (B^*K'p_h+B^*p_h+\alpha u_h)(u_h-u) \le \|B^*K'p_h+B^*p_h+\alpha u_h\|_{0,\partial\Omega^-\cup\partial\Omega^+}\|u-u_h\|_{\partial\Omega}. \quad (3.8)$$

Moreover, note that $B^*K'p_h + B^*p_h + \alpha\beta > 0$ and $u_h = \beta$ on $\partial\Omega^0$, and $u \ge \beta$ on whole $\partial\Omega$. We have that

$$\int_{\partial\Omega^0} (B^*K'p_h + B^*p_h + \alpha u_h)(u_h - u) = \int_{\partial\Omega^0} (B^*K'p_h + B^*p_h + \alpha\beta)(\beta - u) \le 0. \quad (3.9)$$

Summing up, it follows from (3.7)-(3.9) that

$$\begin{aligned}(B^*K'p_h + B^*p_h + \alpha u_h, u_h - u) &\le \|B^*K'p_h + B^*p_h + \alpha u_h\|_{0,\partial\Omega^-\cup\partial\Omega^+}\|u - u_h\|_{\partial\Omega} \\ &= \eta_1\|u - u_h\|_{\partial\Omega}. \end{aligned} \quad (3.10)$$

Thus, (3.6) and (3.10) lead to

$$\|u - u_h\|_{0,\partial\Omega} \le C\|p(u_h) - p_h\|_{0,\partial\Omega} + C\eta_1. \quad (3.11)$$

Next, let us consider the estimate of $\|p(u_h) - p_h\|_{0,\partial\Omega}$. Let $V\psi = p_h - p(u_h)$. It follows from (2.15) and (3.3) that

$$\begin{aligned}\|p_h - p(u_h)\|^2_{0,\partial\Omega} &= (p_h - p(u_h), V\psi) = (V(p_h - p(u_h)), \psi) \\ &= (Vp_h, \psi) - (V(q(u_h) - q_0), V\psi) \\ &= (p_h - V(q_h - q_0), V\psi) + (V(q_h - q(u_h)), V\psi) \\ &\le (\|p_h - V(q_h - q_0)\|_{0,\partial\Omega} + \|V(q_h - q(u_h))\|_{0,\partial\Omega})\|V\psi\|_{0,\partial\Omega} \\ &= (\eta_3 + \|V(q_h - q(u_h))\|_{0,\partial\Omega})\|p_h - p(u_h)\|_{0,\partial\Omega}.\end{aligned}$$

Then, we have that

$$\|p_h - p(u_h)\|_{0,\partial\Omega} \le (\eta_3 + \|V(q_h - q(u_h))\|_{0,\partial\Omega}). \quad (3.12)$$

Similarly, let $\phi = V(q_h - q(u_h))$. It follows from (2.14) and (3.2) that

$$\begin{aligned}\|V(q_h - q(u_h))\|^2_{0,\partial\Omega} &= (V(q_h - q(u_h)), \phi) \\ &= (Vq_h, \phi) - (KBu_h - Bu_h, \phi) \\ &\le \|Vq_h - KBu_h + Bu_h\|_{0,\partial\Omega}\|\phi\|_{0,\partial\Omega} \\ &= \eta_2\|V(q_h - q(u_h))\|_{0,\partial\Omega},\end{aligned}$$

and hence,

$$\|V(q_h - q(u_h))\|_{0,\partial\Omega} \leq \eta_2. \tag{3.13}$$

It can be deduced from (3.12) and (3.13) that

$$\|p_h - p(u_h)\|_{0,\partial\Omega}^2 + \|V(q_h - q(u_h))\|_{0,\partial\Omega}^2 \leq C(\eta_2^2 + \eta_3^2). \tag{3.14}$$

Then, (3.11) and (3.14) lead to

$$\|u - u_h\|_{0,\partial\Omega}^2 \leq C(\eta_1^2 + \eta_2^2 + \eta_3^2). \tag{3.15}$$

Let $V\psi = p - p(u_h)$. It follows from (2.10) and (3.3) that

$$\begin{aligned} \|p - p(u_h)\|_{0,\partial\Omega}^2 &= (p - p(u_h), V\psi) = (V(q - q(u_h)), V\psi) \\ &\leq \|V(q - q(u_h))\|_{0,\partial\Omega}\|V\psi\|_{0,\partial\Omega} \\ &= \|V(q - q(u_h))\|_{0,\partial\Omega}\|p - p(u_h)\|_{0,\partial\Omega}. \end{aligned} \tag{3.16}$$

Similarly, let $\phi = V(q - q(u_h))$. It can be deduced from (2.9) and (3.2) that

$$\begin{aligned} \|V(q - q(u_h))\|_{0,\partial\Omega}^2 &= (V(q - q(u_h)), \phi) = (KB(u - u_h) + B(u - u_h), \phi) \\ &\leq \|KB(u - u_h) + B(u - u_h)\|_{0,\partial\Omega}\|\phi\|_{0,\partial\Omega} \\ &\leq C\|u - u_h\|_{0,\partial\Omega}\|V(q - q(u_h))\|_{0,\partial\Omega}. \end{aligned} \tag{3.17}$$

Then (3.16) and (3.17) imply that

$$\|p - p(u_h)\|_{0,\partial\Omega} \leq \|V(q - q(u_h))\|_{0,\partial\Omega} \leq C\|u - u_h\|_{0,\partial\Omega}. \tag{3.18}$$

Moreover, note that

$$\|q_h - q\|_{-1,\partial\Omega} \leq \|q_h - q(u_h)\|_{-1,\partial\Omega} + \|q(u_h) - q\|_{-1,\partial\Omega}, \tag{3.19}$$

$$\|p_h - p\|_{0,\partial\Omega} \leq \|p_h - p(u_h)\|_{0,\partial\Omega} + \|p(u_h) - p\|_{0,\partial\Omega}. \tag{3.20}$$

Therefore, it follows from (3.14)-(3.15) and (3.18)-(3.20) that

$$\|p_h - p\|_{0,\partial\Omega}^2 + \|V(q_h - q)\|_{0,\partial\Omega}^2 \leq C(\eta_1^2 + \eta_2^2 + \eta_3^2). \tag{3.21}$$

Then, (3.1) is the direct result of (3.15) and (3.21).

Acknowledgements

The research was was supported by the National Natural Science Foundation of China under Grant 11171337 and the National Basic Research Program under the Grant 2010CB731505. Moreover, the authors would like to thank Professor C. Carstensen in Humboldt University of Berlin for his fruitful discussion. She also wish to acknowledge the referee and editor for their careful reviews and many valuable suggestions on this paper.

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