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ON KŘÍŽEK'S DECOMPOSITION OF A POLYHEDRON INTO CONVEX COMPONENTS AND ITS APPLICATIONS IN THE PROOF OF A GENERAL OSTROGRADSKIJ'S THEOREM

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I encountered Professor Křížek for the first time when he defended his CSc.-degree; I was a member of the committee. One of his results fascinated me. It has the following form:

Křížek's lemma (on a decomposition of a polygon and a polyhedron into convex components)

a) For every polygon $\overline{\Omega}$ there exists a finite number of convex polygons with mutually disjoint interiors the union of which is $\overline{\Omega}$.
b) For every polyhedron $\overline{\Omega}$ there exists a finite number of convex polyhedrons with mutually disjoint interiors the union of which is $\overline{\Omega}$.

Definition. a) By a polygon we understand every nonempty, bounded and closed domain in $\mathbb{R}^2$ the boundary of which can be expressed as a union of a finite number of segments.
b) By a polyhedron we understand every nonempty, bounded and closed domain in $\mathbb{R}^3$ the boundary of which can be expressed as a union of a finite number of polygons with mutually disjoint interiors.

Proof of Křížek's lemma. The proof is presented in the three-dimensional case; this part of Lemma will play a fundamental role in the proof of the Gauss–Ostrogradskij theorem. In the two-dimensional case the proof is analogous but simpler.

The proof is a part of the proof of a more general theorem (see [2]). However, because of the importance of the lemma we reproduce the corresponding part of Křížek's proof in a slightly extended form.

Let $\overline{\Omega}$ be an arbitrary polyhedron and let $\pi^1, \ldots, \pi^m$ be polygons the union of which is the boundary $\partial\Omega$. Let $\varrho^1, \ldots, \varrho^m$ be such planes that $\pi^i \subset \varrho^i$, $i = 1, \ldots, m$. It may happen that some of these planes coincide. Without loss of generality let us assume that $\varrho^1, \dots, \varrho^k$ $(k \le m)$ are mutually different planes and each ϱ^i $(k < i \le m)$ belongs to the set $\{\varrho^1, \dots, \varrho^k\}$. Let $\Omega_1, \dots, \Omega_r \subset R^3$ be all connected components of the set $\overline{\Omega} \setminus \bigcup_{i=1}^{k} \varrho^i$ (i.e., the connected components which arise after "cutting up" the polyhedron $\overline{\Omega}$ by the planes ϱ^i). The number of these components is finite (at most 2^k). We assert that $\overline{\Omega}_j$ $(j = 1, \dots, r)$ are the sought convex polyhedrons. First we show that Ω_j are open sets. As $\partial\Omega \subset \bigcup_{i=1}^{k} \varrho^i$ we have

$$\overline{\Omega} \setminus \bigcup_{i=1}^{k} \varrho^i = \Omega \setminus \bigcup_{i=1}^{k} \varrho^i.$$

This set is open because Ω is an open set and $\bigcup_{i=1}^{k} \varrho^i$ is a closed set, and components of an open set are open.

Further we prove the convexity of $\overline{\Omega}_j$. Let $j \in \{1, \dots, r\}$ be an arbitrary fixed integer. Each plane ϱ^i $(i = 1, \dots, k)$ divides the space $\mathbb{R}^3$ into two half-spaces. Let us denote by Q^i the closed half-space, which is bounded by the plane ϱ^i and which contains $\overline{\Omega}_j$, and let us denote $M := \bigcap_{i=1}^{k} Q^i$. Then we have $\overline{\Omega}_j \subset M$. The converse inclusion will be proved by contradiction. Let us assume that there exists a point $P \in M \setminus \overline{\Omega}_j$. As $\overline{\Omega}_j$ is a closed set we have $R = \operatorname{dist}(P, \overline{\Omega}_j) > 0$; this means that

$$M \setminus \overline{\Omega}_j \supset M \cap \mathcal{S}_{\mathcal{R}}(\mathcal{P}) \neq \emptyset,$$

where $\mathcal{S}_{\mathcal{R}}(\mathcal{P})$ is an open ball of the radius R and with the center at P. Let $X \in M \cap \mathcal{S}_{\mathcal{R}}(\mathcal{P})$ be a point that does not belong to any plane $\varrho^1, \dots, \varrho^k$ and let Y be an arbitrary interior point of $\overline{\Omega}_j$ (such a point certainly exists because Ω_j is a domain). Then inside the segment $\overline{XY}$ there exists such a point Z that $Z \in \partial\Omega_j$ (because $X \notin \overline{\Omega}_j$). As Z is a boundary point of $\overline{\Omega}_j$ there exists a plane ϱ^s $(1 \le s \le k)$ such that $Z \in \varrho^s$ and this plane separates the points X a Y because $X \notin \varrho^s$, $Y \notin \varrho^s$. This implies that $X \notin Q^s$, which contradicts the fact that $X \in M \subset Q^s$. Hence

$$\overline{\Omega}_j = \bigcap_{i=1}^{k} Q^i$$

and this intersection is evidently bounded and has at least one interior point. In other words, $\overline{\Omega}_j$ is a convex polyhedron.

Further, the definition of components Ω_j $(j = 1, \dots, r)$, i.e., the relation

$$\overline{\Omega} \setminus \bigcup_{i=1}^{k} \varrho^i = \bigcup_{j=1}^{r} \Omega_j,$$

implies immediately that $\overline{\Omega} = \bigcup_{j=1}^{r} \overline{\Omega}_j$. □

The rest of the paper is devoted to a very important application of Křížek's lemma – the proof of a general form of the Gauss-Ostrogradskij theorem.

1. The elementary form of the Gauss–Ostrogradskij theorem

Definition 1. a) A bounded domain $\Omega \subset \mathbb{R}^3$ is called *elementary with respect to the coordinate plane* (x, y) if every straight-line p parallel to the z-axis and such that $p \cap \overline{\Omega} \neq \emptyset$ intersects the boundary $\partial\Omega$ at two points or has with $\partial\Omega$ a common segment which can degenerate into a point.
b) Analogously we define *domains elementary with respect to the plane* (x, z), or *with respect to the plane* (y, z).
c) A bounded domain Ω is called *elementary* if it is elementary with respect to all three coordinate planes.

Remark 1. *Every bounded convex domain is elementary.*

Definition 2. a) We say that a set $\overline{S}$ is *a part of a surface* which is *regular with respect to the coordinate plane* (x, y), if the points $[x, y, z] \in \overline{S}$ satisfy

$$z = f(x, y), \quad [x, y] \in \overline{S}_{xy}$$

where $\overline{S}_{xy}$ is a simply connected two-dimensional bounded closed domain lying in the plane (x, y) which is bounded by a simple piecewise smooth closed curve ∂S_{xy}, and $f : \overline{S}_{xy} \to \mathbb{R}^1$ is a real function continuous on $\overline{S}_{xy}$ which has continuous first partial derivatives $f_x \equiv \frac{\partial f}{\partial x}$, $f_y \equiv \frac{\partial f}{\partial y}$ in S_{xy} (where the symbol S_{xy} denotes the interior of $\bar{S}_{xy}$, i.e., $S_{xy} = \bar{S}_{xy} \setminus \partial S_{xy}$; these derivatives can be unbounded in S_{xy}). The closed domain $\bar{S}_{xy}$ is called the orthogonal projection of the part $\bar{S}$ onto the plane (x, y).
b) Similarly we say that a set $\overline{S}$ is *a part of a surface* which is *regular with respect to the coordinate plane* (x, z) (or (y, z)), if the points $[x, y, z] \in \bar{S}$ satisfy

$$y = g(x, z), \quad [x, z] \in \bar{S}_{xz},$$

or

$$x = h(y, z), \quad [y, z] \in \bar{S}_{yz},$$

where the closed domains $\overline{S}_{xz}$, $\overline{S}_{yz}$ and the functions $g : \overline{S}_{xz} \to \mathbb{R}^1$, $h : \overline{S}_{yz} \to \mathbb{R}^1$ have analogous properties as the closed domain $\overline{S}_{xy}$ and the function $f : \overline{S}_{xy} \to \mathbb{R}^1$. The closed two-dimensional domains $\overline{S}_{xz}$ and $\overline{S}_{yz}$ are called orthogonal projections of the part $\bar{S}$ onto the planes (x, z) and (y, z).

Definition 3. We say that a part $\overline{S}$ has *property* (R) if it satisfies at least one of the following three conditions:
a) the part $\overline{S}$ is regular with respect to all three coordinate planes;
b) the orthogonal projection of the part $\overline{S}$ onto one of the three coordinate planes has the two-dimensional measure equal to zero; the part $\overline{S}$ is regular with respect to the remaining two coordinate planes;
c) two components of the vector $\mathbf{n}(x, y, z)$ equal zero for all points $[x, y, z] \in \overline{S}$.

Lemma 1. Let a domain Ω be elementary with respect to the plane (x, y) and let its boundary $\partial\Omega$ consist of a finite number of parts with property (R) which have mutually disjoint interiors. Then these parts can be divided into three groups with the following properties:
a) The union of parts belonging to the first group forms a part $\bar{D}^1$ whose points $[x, y, z]$ satisfy the equation

$$z = z_1(x, y), \quad [x, y] \in \bar{D}^1_{xy}, \tag{1}$$

where z_1 is a continuous function.
b) The union of parts belonging to the second group forms a part $\bar{D}^2$ whose points $[x, y, z]$ satisfy the equation

$$z = z_2(x, y), \quad [x, y] \in \bar{D}^2_{xy}, \tag{2}$$

where z_2 is a continuous function. At the same time we have

$$\bar{D}^1_{xy} = \bar{D}^2_{xy},$$
$$z_1(x, y) \le z_2(x, y) \quad \forall [x, y] \in \bar{D}^1_{xy}.$$

c) The normal vector $\mathbf{n} = (\cos\alpha, cos\beta, \cos\gamma)$ of the parts belonging to the third group satisfies

$$\cos\gamma \equiv 0.$$

The set of the parts belonging to the third group can be empty.

Proof. The assertion is evident. □

Theorem 1. Let the boundary $\partial\Omega$ of an elementary domain Ω be the union of a finite number of parts with property (R). Let functions P, Q, R be continuous on $\overline{\Omega}$ and let the derivatives $\partial P/\partial x$, $\partial Q/\partial y$, $\partial R/\partial z$ be continuous on $\overline{\Omega}$. Let the positive direction of the unit normal $\mathbf{n}$ be the direction of the outer normal. Then

$$\iiint_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) \mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint_{\partial\Omega} (P \,\mathrm{d}y\mathrm{d}z + Q \,\mathrm{d}x\mathrm{d}z + R \,\mathrm{d}x\mathrm{d}y). \tag{3}$$

Proof. By Lemma 1 and the Fubini theorem

$$\iiint_\Omega \frac{\partial R}{\partial z} \mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint_{D^1_{xy}} \left\{ \int_{z_1(x,y)}^{z_2(x,y)} \frac{\partial R}{\partial z} \mathrm{d}z \right\} \mathrm{d}x\mathrm{d}y =$$
$$= \iint_{D^2_{xy}} R(x, y, z_2(x, y)) \,\mathrm{d}x\mathrm{d}y - \iint_{D^1_{xy}} R(x, y, z_1(x, y)) \,\mathrm{d}x\mathrm{d}y. \tag{4}$$

Owing to the orientation of the normal, we have $\cos\gamma < 0$ on D^1 and $\cos\gamma > 0$ on D^2. Thus (4) can be rewritten in the form (where $\varepsilon_z = 1$ if $\gamma < \pi/2$ and $\varepsilon_z = -1$ if $\gamma > \pi/2$)

$$\iiint_\Omega \frac{\partial R}{\partial z}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \varepsilon_z \iint_{D^2_{xy}} R(x,y,z_2(x,y))\,\mathrm{d}x\mathrm{d}y + \varepsilon_z \iint_{D^1_{xy}} R(x,y,z_1(x,y))\,\mathrm{d}x\mathrm{d}y. \tag{5}$$

As the boundary $\partial\Omega$ can be expressed as the union of the surfaces (1), (2) and the parts for which $\cos\gamma = 0$, the right-hand side of (5) is equal to the surface integral $\iint_{\partial\Omega} R\,\mathrm{d}x\mathrm{d}y$. Hence

$$\iiint_\Omega \frac{\partial R}{\partial z}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint_{\partial\Omega} R\,\mathrm{d}x\mathrm{d}y. \tag{6}$$

Similarly we obtain

$$\iiint_\Omega \frac{\partial P}{\partial x}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint_{\partial\Omega} P\,\mathrm{d}y\mathrm{d}z, \tag{7}$$

$$\iiint_\Omega \frac{\partial Q}{\partial y}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint_{\partial\Omega} Q\,\mathrm{d}x\mathrm{d}z. \tag{8}$$

Summing (6)–(8), we obtain (3). □

Theorem 2. Let a domain $\overline{\Omega}$ be the union of a finite number of elementary domains $\overline{\Omega}^1, \dots, \overline{\Omega}^n$ which have mutually disjoint interiors. Let the boundary $\partial\Omega^i$ of each domain Ω^i $(i = 1, \dots, n)$ be the union of a finite number of parts with property (R). Let functions P, Q, R be continuous on $\overline{\Omega}$ and let the derivatives $\partial P/\partial x$, $\partial Q/\partial y$, $\partial R/\partial z$ be continuous on $\overline{\Omega}$. Let the unit normal $\mathbf{n}$ of the boundary $\partial\Omega$ be oriented in the direction of the outer normal. Then

$$\iiint_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint_{\partial\Omega} (P\,\mathrm{d}y\mathrm{d}z + Q\,\mathrm{d}x\mathrm{d}z + R\,\mathrm{d}x\mathrm{d}y). \tag{9}$$

Proof. The assumption concerning the normal $\mathbf{n}$ enables us to orient the normal of each boundary $\partial\Omega^i$ in the direction of the outer normal of Ω^i; hence

$$\begin{aligned}\iiint_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \sum_{i=1}^n \iiint_{\Omega^i} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \sum_{i=1}^n \iint_{\partial\Omega^i} (P\,\mathrm{d}y\mathrm{d}z + Q\,\mathrm{d}x\mathrm{d}z + R\,\mathrm{d}x\mathrm{d}y) \\ &= \iint_{\partial\Omega} (P\,\mathrm{d}y\mathrm{d}z + Q\,\mathrm{d}x\mathrm{d}z + R\,\mathrm{d}x\mathrm{d}y),\end{aligned}$$

because at every point $P \in \Omega$ which satisfies the relation $P \in \partial\Omega^j \cap \partial\Omega^k$ $(j \neq k)$ two opposite normals meet - one belonging to $\partial\Omega^j$ and the other to $\partial\Omega^k$. □

2. A more general form of the Gauss–Ostrogradskij theorem

Verifying the assumptions of Theorem 2 concerning the domain Ω is in most cases very difficult: Let us consider, for example, a domain (the so called "cheese ball with many bubbles")

$$\bar{\Omega} = \bar{K}_0 \setminus \bigcup_{i=1}^{n} K_i \,,$$

where $\bar{K}_0, \bar{K}_1, \ldots, \bar{K}_n$ are balls with properties

$$\bar{K}_i \subset K_0 \quad (i = 1, \ldots, n), \quad \bar{K}_i \cap \bar{K}_j = \emptyset \quad (i \neq j;\ i, j = 1, \ldots, n).$$

To make the Gauss–Ostrogradskij theorem applicable in general use we must substitute its assumption concerning the domain Ω by an assumption which would enable us to check only the properties of the boundary $\partial\Omega$.

Almost every Czech mathematician knows that satisfactory proofs of Ostrogradskij's theorem are introduced in [1] and [3]. As for me, after having been acquainted with Křížek's lemma I did not seek other proofs.

Definition 4. *We say that a part $\bar{S}$ has property (R^*) (or property (R^{**})) if it satisfies conditions a)–c) (or conditions a)–d)) where*
a) the part $\bar{S}$ has property (R);
b) if

$$z = f(x, y), \quad y = g(x, z), \quad x = h(y, z)$$

are functions appearing in the analytical expressions of the part $\bar{S}$ with respect to the coordinate planes then at least one of the three relations $f \in C^2(\bar{S}_{xy})$, $g \in C^2(\bar{S}_{xz})$, $h \in C^2(\bar{S}_{yz})$ holds;
c) if $\operatorname{meas}_2 S_{st} > 0$, then the boundary ∂S_{st} is piecewise of class C^2 and has no cusp-points;
d) at least one of the plane domains $\bar{S}_{xy}$, $\bar{S}_{xz}$, $\bar{S}_{yz}$ is starlike. (A domain $\bar{D}$ is starlike if there exists at least one point $Q \in D$ such that every half-line starting from this point intersects ∂D at just one point.)

Theorem 3 (Gauss–Ostrogradskij)**.** Let $\overline{\Omega}$ be a three-dimensional bounded closed domain whose boundary $\partial\Omega$ is the union of a finite number of parts with property (R^*), which have mutually disjoint interiors. Let functions

$$P,\ Q,\ R,\ \partial P/\partial x,\ \partial Q/\partial y,\ \partial R/\partial z$$

be continuous and bounded in a bounded three-dimensional domain $\widetilde{\Omega}$ satisfying $\widetilde{\Omega} \supset \overline{\Omega}$. Let the unit normal $\mathbf{n}$ of the boundary $\partial\Omega$ be oriented in the direction of the outer normal of $\partial\Omega$, which exists at almost all points of $\partial\Omega$. Then

$$\iiint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) \mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint_{\partial\Omega} (P\,\mathrm{d}y\mathrm{d}z + Q\,\mathrm{d}x\mathrm{d}z + R\,\mathrm{d}x\mathrm{d}y). \qquad (10)$$

Sketch of the proof. In a detailed proof (see [4], or [5], Chapter 20) the theorem is first proved in the case that the parts forming $\partial\Omega$ have property (R^{**}). At the end it is shown how to change the proof when these parts have only property (R^{*}).

A) Let us choose $\delta > 0$ arbitrary but fixed ($\delta < 1$). In this part of the proof it is shown (in details see [4] or [5], Chapter 20) how to approximate a part with property (R^{**}) by a "panel-shaped" surface which consists of triangular panels whose longest side has a length which is less or equal to δ. This approximation will be constructed in such a way that if

$$\partial\Omega = \bigcup_{i=1}^{n} \bar{S}_i, \quad S_i \cap S_j = \emptyset \ (i \neq j) \tag{11}$$

is a decomposition of $\partial\Omega$ into parts with property (R^{**}) and $\bar{S}_i^{\delta}$ is a panel-shaped surface approximating $\bar{S}_i$, then

$$\partial\Omega^{\delta} := \bigcup_{i=1}^{n} \bar{S}_i^{\delta} \tag{12}$$

is a boundary of a polyhedron satisfying

$$S_i^{\delta} \cap S_j^{\delta} = \emptyset \quad (i \neq j;\ i,j = 1,\dots,n) \tag{13}$$

and with vertices lying on $\partial\Omega$. The closed bounded three-dimensional domain with the boundary $\partial\Omega^{\delta}$ will be denoted by $\bar{\Omega}^{\delta}$.

B) As $\bar{\Omega}^{\delta}$ is a polyhedron, we can express it by Křížek's lemma in the form

$$\bar{\Omega}^{\delta} = \bigcup_{j=1}^{m} \bar{U}_j, \tag{14}$$

where $\bar{U}_1, \dots, \bar{U}_m$ are closed convex polyhedrons. Let us orientate the normal to ∂U_j as the outer normal of $\bar{U}_j$ ($j = 1, \dots, m$). Relation (14) and the proof of Theorem 2 yield

$$\begin{aligned}\iiint_{\Omega^{\delta}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z &= \sum_{j=1}^{m} \iiint_{U_j} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \sum_{j=1}^{m} \iint_{\partial U_j} (P\,\mathrm{d}y\mathrm{d}z + Q\,\mathrm{d}x\mathrm{d}z + R\,\mathrm{d}x\mathrm{d}y) \\ &= \iint_{\partial\Omega^{\delta}} (P\,\mathrm{d}y\mathrm{d}z + Q\,\mathrm{d}x\mathrm{d}z + R\,\mathrm{d}x\mathrm{d}y),\end{aligned} \tag{15}$$

because the surface integrals over $\partial U_j \cap \partial U_k$ altogether cancel.

C) It remains to prove that

$$\lim_{\delta\to 0} \iiint_{\Omega^{\delta}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{16}$$

and

$$\lim_{\delta \to 0} \iint_{\partial\Omega^\delta} (P\,\mathrm{d}y\mathrm{d}z + Q\,\mathrm{d}x\mathrm{d}z + R\,\mathrm{d}x\mathrm{d}y) = \iint_{\partial\Omega} (P\,\mathrm{d}y\mathrm{d}z + Q\,\mathrm{d}x\mathrm{d}z + R\,\mathrm{d}x\mathrm{d}y). \quad (17)$$

The proof of (17) is long and complicated and we refer to [4], or [5], Chapter 20.

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