Conference Applications of Mathematics 2012
in honor of the 60th birthday of Michal Křížek.
Institute of Mathematics AS CR, Prague 2012

FINITE ELEMENT ANALYSIS FOR A REGULARIZED VARIATIONAL INEQUALITY OF THE SECOND KIND

Tie Zhang[1], S. Zhang[2], H. Azari[3]

[1] Department of Mathematics, Northeastern University
Shenyang 110004, China
ztmath@163.com
[2] Research Center for Mathematics and Economics
Tianjin University of Finance and Economics 300222, Tianjin, China
szhang@tjufe.edu.cn (corresponding author)
[3] Department of Mathematics Science , Shahid Beheshti University
Tehran, Iran
h_azari@sbu.ac.ir

Abstract

In this paper, we investigate the a priori and the a posteriori error analysis for the finite element approximation to a regularization version of the variational inequality of the second kind. We prove the abstract optimal error estimates in the H^1- and L_2-norms, respectively, and also derive the optimal order error estimate in the L_∞-norm under the strongly regular triangulation condition. Moreover, some residual–based a posteriori error estimators are established, which can provide the global upper bounds on the errors. These a posteriori error results can be applied to develop the adaptive finite element methods. Finally, we supply some numerical experiments to validate the theoretical results.

1. Introduction

Many important physics and engineering problems, such as contact with friction, obstacle problems, problems in plasticity and viscoplasticity, etc.(see, for example, [4, 8, 10–13]) can be formulated as variational inequalities. The aim of this article is to present some a priori and a posteriori error estimates based on the finite element approximation for the following variational inequality: Find $u \in V$ such that

$$a(u, v-u) + j_\gamma(v) - j_\gamma(u) \geq (f, v-u) \quad \forall v \in V, \tag{1.1}$$

where

$$a(u,v) = \int_\Omega (\nabla u \cdot \nabla v + \mu uv)dx, \qquad (f,v) = \int_\Omega f\, v\, dx, \tag{1.2}$$

$$j_\gamma(v) = \int_{\Gamma_N} \psi(v)ds, \quad V = \left\{ v \in H^1(\Omega) : \ v = 0, \text{ on } \Gamma \setminus \Gamma_N \right\}, \tag{1.3}$$

and $\Omega \subset R^2$ is a convex polygonal domain, $\mu > 0$ is a constant, $\Gamma = \partial\Omega$, $\Gamma_N \subset \Gamma$, $\mathrm{meas}(\Gamma_N) > 0$, and

$$\psi(v) = \begin{cases} gv - \frac{\gamma}{2}g^2, & v \geq \gamma g, \\ \frac{1}{2\gamma}v^2, & |v| \leq \gamma g, \\ -gv - \frac{\gamma}{2}g^2, & v \leq -\gamma g, \end{cases} \tag{1.4}$$

with the constant $g > 0$ and the small parameter $\gamma > 0$. Problem (1.1)–(1.4) is a regularization version of the variational inequality of the second kind:

$$a(u, v-u) + j(v) - j(u) \geq (f, v-u) \ \ \forall v \in V, \ \ j(v) = \int_{\Gamma_N} g|v|, \tag{1.5}$$

and when $\gamma \to 0$, its solution $u = u_\gamma$ converges to the solution u of problem (1.5). See, for example, [8,10]. Since

$$|j_\gamma(v) - j(v)| \leq \frac{1}{2}\gamma g^2 \mathrm{meas}(\Gamma_N) \quad \forall v \in H^1(\Omega),$$

it is easy to see that

$$\|u_\gamma - u\|_1 \leq \sqrt{\gamma}\, g(\mathrm{meas}(\Gamma_N))^{\frac{1}{2}}.$$

Finite element methods for the variational inequalities of the second kinds (including their regularization versions) have been studied for many years simply because of their practical importance, but the bound on the discretization error in the literature is suboptimal [1, 8–10]. In existing work, the finite element discretizations are directly applied to the variational inequalities, which makes the finite element analysis very difficult because of the inequality constraint. In this paper, we establish the finite element discretization by a different way. We first transform the variational inequality problem (1.1) into an equivalent variational problem, and then construct the finite element approximation and give the unique existence and stability of the finite element solution. By this approach, we establish the abstract error estimates in the H^1- and L_2-norms, respectively, which imply the optimal convergence on both the approximation order of the finite element space and the regularity required for the exact solution. In addition, when the solution is smooth enough, we further derive the optimal order error estimate in the L_∞-norm under the strongly regular triangulation condition [17]. Moreover, we study the a posteriori error estimate of the finite element solution. We know that an a posteriori error estimate is set as a theoretical basis for the adaptive computations based on h, p, and hp finite element methods, and in this article, we give some residual-based a posteriori estimators which yield global upper bounds on the discretization errors in the H^1- and L_2-norms. It should be pointed out that for the finite element approximations to variational inequalities of the second kind (including their regularization forms), it is very difficult to obtain the optimal order error estimates in the L_2- and L_∞-norms. Hence, our method and result here provide some theoretical significance into the literature.

This paper is organized as follows. In Section 2, we transform the variational inequality problem (1.1) into an equivalent variational problem, and then construct the finite element discretization and discuss the unique existence and the stability of the finite element solution. In Section 3, some abstract error estimates are established and the optimal order error estimates are derived in the H^1-, L_2- and L_∞-norms, respectively. Section 4 is devoted to the a posteriori error analysis of the finite element solution. Finally, in Section 5, we present some numerical examples to illustrate our theoretical analysis.

In this paper, we adopt the standard notation W_p^m for the Sobolev space on the domain Ω with the corresponding norm $\|\cdot\|_{m.p}$, and when $p = 2$, $W_2^m = H^m$, $\|\cdot\|_{m,2} = \|\cdot\|_m$. Denote by $(\cdot,\cdot)$ and $\|\cdot,\cdot\|$ the inner product and the norm, respectively, in the L_2-space. We will also use the letter C to denote a generic positive constant independent of the mesh size h.

2. Equivalent problem and its finite element approximation

First we derive the equivalent variational form of problem (1.1). In (1.1) taking $v = u \pm t\,w,\ t > 0,\ w \in V$, we obtain

$$\pm a(u,w) + \int_{\Gamma_N} \frac{\psi(u \pm tw) - \psi(u)}{t} ds \geq \pm(f,w) \quad \forall w \in V.$$

Setting $t \to 0_+$ and noting that

$$\lim_{t\to 0_+} \frac{\psi(u \pm tw) - \psi(u)}{t} = \pm\psi'(u)w = \pm\varphi(u)w,$$

we see that the solution u of problem (1.1) satisfies

$$a(u,v) + \int_{\Gamma_N} \varphi(u)v\, ds = (f,v) \quad \forall v \in V, \tag{2.1}$$

where

$$\varphi(t) = \psi'(t) = \begin{cases} g, & t \geq \gamma g, \\ t/\gamma, & |t| \leq \gamma g, \\ -g, & t \leq -\gamma g. \end{cases}$$

Formula (2.1) gives the equivalent variational form of problem (1.1)–(1.4).

Lemma 2.1. *The function $\varphi(t) \in H^1(-\infty,\infty)$ and it satisfies the following Lipschitz's condition and the monotonicity condition:*

$$|\,\varphi(u) - \varphi(v)\,| \leq \frac{1}{\gamma}|\,u - v\,| \quad \forall u,\, v \in R, \tag{2.2}$$

$$(\varphi(u) - \varphi(v))(u - v) \geq 0 \quad \forall u,\, v \in R. \tag{2.3}$$

Proof. It follows from a straightforward calculation that

$$\varphi'(t) = \begin{cases} \dfrac{1}{\gamma}, & |t| \le \gamma g \\ 0, & |t| > \gamma g, \end{cases}$$

and

$$\varphi(u) - \varphi(v) = \int_v^u \varphi'(t)\,dt = \int_0^1 \varphi'(v + \tau(u-v))\,d\tau\,(u-v),$$

which, together with $\varphi'(t) \ge 0$, leads to (2.2)–(2.3). □

Corollary 2.1. *The solution of problem (2.1) is unique and satisfies the inequality*

$$\|u\|_1 \le \frac{1}{\mu_0}\|f\|, \quad \textit{where} \quad \mu_0 = \min\{\,1, \mu\,\}.$$

Proof. Assume that u_1 and u_2 are the solutions of problem (2.1). Then, we have

$$a(u_1 - u_2, v) + \int_{\Gamma_N} (\varphi(u_1) - \varphi(u_2))v\,ds = 0,\ v \in V.$$

Taking $v = u_1 - u_2$, the uniqueness is obtained by using Lemma 2.1 and the coercivity of $a(u,v)$. Now setting $v = u$ in (2.1), from Lemma 2.1 we know that $\varphi(u)u \ge 0$ (noting that $\varphi(0) = 0$), which yields

$$a(u,u) \le (f,u).$$

Thus, the stability estimate is derived. □

From book [8], we have known that there exists a solution $u \in V$ to problem (1.1). Then according to Corollary 2.1 and the equivalence of problems (1.1) and (2.1), we can conclude that problem (2.1) has a unique solution which is also the unique solution of problem (1.1).

Let $J_h = \cup\{e\}$ be a regular finite element triangulation of domain Ω parameterized by the mesh size $h = \max h_e$ so that $\overline{\Omega} = \cup_{e\in J_h}\{\,\overline{e}\,\}$, where h_e is the diameter of the element e. We assume that the triangulation is made such that the vertices of Γ_N are also the mesh points of the triangulation J_h. Introduce the finite element space $V_h \subset V$ as follows:

$$V_h = \{\,v_h \in C(\overline{\Omega}) : v_h|_e \in P_k(e),\ v_h|_{\Gamma\setminus\Gamma_N} = 0 \quad \forall e \in J_h\,\},$$

where $P_k(e)$ is the set of polynomials of degree at most k on e. The finite element approximation of problem (2.1) is defined to seek $u_h \in V_h$ such that

$$a(u_h, v_h) + \int_{\Gamma_N} \varphi(u_h)v_h\,ds = (f, v_h) \quad \forall v_h \in V_h. \tag{2.4}$$

Theorem 2.1. *Problem (2.4) admits a unique solution, which satisfies the stability estimate*

$$\|u_h\|_1 \le \frac{1}{\mu_0}\|f\|.$$

Proof. It follows from taking $v_h = u_h$ in (2.4), using Lemma 2.1 and the coercivity of $a(u,v)$ that we can immediately obtain the stability. Below we discuss the unique existence.

Let $\{\,\psi_i\,\}_{i=1}^N$ be the basis function system of space V_h. Then we can set $u_h = \sum_{i=1}^N u_i\psi_i(x)\ \forall u_h \in V_h$. Now we rewrite equation (2.4) as

$$A\vec{u} + \vec{\varphi}(\vec{u}) = \vec{f} \quad \text{or} \quad \vec{u} = T\vec{u} = A^{-1}(\vec{f} - \vec{\varphi}(\vec{u})), \tag{2.5}$$

where $A = (a(\psi_i, \psi_j))_{N\times N}$ is a positive definite matrix and

$$\vec{u} = (\,u_1, u_2, \cdots, u_N\,)^T, \qquad \vec{\varphi} = (\,\varphi_1, \varphi_2, \cdots, \varphi_N\,)^T,$$
$$\varphi_j = \int_{\Gamma_N} \varphi(\textstyle\sum u_i\psi_i(x))\psi_j\,ds, \quad j = 1, 2, \cdots, N.$$

From Lemma 2.1 we know that $\vec{\varphi}(\vec{u})$ is Lipschitz continuous, and hence the mapping $T:\ R^N \to R^N$ is a compact mapping. Furthermore, from the stability estimate, we see that any solution $\vec{u}$ of equation:

$$\vec{u} = \sigma T\vec{u},\ \sigma \in [0,1],$$

lies in a bounded set of R^N. Then, by the Brouwer fixed point theory (see Theorem 10.3 in [6]), the mapping T has a fixed point in R^N, that is, the solution to the discrete system of equations (2.5) exists. The proof of uniqueness is similar to that of Corollary 2.1. □

3. A priori error analysis in various norms

Let c_0 be the positive constant in the trace theorem such that

$$\|u\|_{L_2(\Gamma_N)} \le c_0\|u\|_1 \quad \forall u \in H^1. \tag{3.1}$$

Theorem 3.1. *Let u and u_h be the solutions of problems (2.1) and (2.4), respectively, $\mu_1 = \max\{\,1, \mu\,\}$. Then, we have the following abstract error estimates in the H^1-norm and the L_2-norm:*

$$\|u - u_h\|_1 \le \frac{1}{\mu_0}\left(\mu_1 + \frac{c_0^2}{\gamma}\right)\inf_{v_h\in V_h}\|u - v_h\|_1, \tag{3.2}$$

$$\|u - u_h\| \le \left(\mu_1 + \frac{c_0^2}{\gamma}\right)\sup_{q\in L_2(\Omega)}\left\{\frac{1}{\|q\|}\inf_{v_h\in V_h}\|w - v_h\|_1\right\}\|u - u_h\|_1, \tag{3.3}$$

where, for given $q \in L_2(\Omega)$, $w \in V$ is the unique solution of the elliptic problem (3.5) below.

Proof. From (2.1) and (2.4) we derive the error equation

$$a(u-u_h, v_h) + \int_{\Gamma_N} (\varphi(u) - \varphi(u_h))v_h \, ds = 0 \quad \forall v_h \in V_h. \tag{3.4}$$

Hence, using Lemma 2.1 and the trace theorem, we have for $v_h \in V_h$ that

$$\begin{aligned}
\mu_0\|u-u_h\|_1^2 \leq & a(u-u_h, u-u_h) \\
= & a(u-u_h, u-v_h) - \int_{\Gamma_N} (\varphi(u) - \varphi(u_h))(v_h - u_h) \, ds \\
= & a(u-u_h, u-v_h) - \int_{\Gamma_N} (\varphi(u) - \varphi(u_h))(v_h - u) \, ds \\
& - \int_{\Gamma_N} (\varphi(u) - \varphi(u_h))(u - u_h) \, ds \\
& \leq \mu_1\|u-u_h\|_1\|u-v_h\|_1 + \frac{1}{\gamma}\int_{\Gamma_N} |u-u_h| \, |v_h - u| \, ds \\
& \leq \mu_1\|u-u_h\|_1\|u-v_h\|_1 + \frac{c_0^2}{\gamma}\|u-u_h\|_1\|u-v_h\|_1.
\end{aligned}$$

Then, estimate (3.2) is obtained.

In order to derive the error estimate in the L_2-norm, let us consider the auxiliary problem: For any given $q \in L_2(\Omega)$, find $w \in V$ such that

$$A(w,v) = (q,v) \ \ \forall v \in V, \tag{3.5}$$

where

$$A(w,v) = a(w,v) + \int_{\Gamma_N} \beta w \, v \, ds, \quad \beta(x) = \frac{\varphi(u) - \varphi(u_h)}{u - u_h}. \tag{3.6}$$

From Lemma 2.1 we know that $\beta(x) \in L_\infty(\Omega)$ and

$$0 \leq \beta(x) \leq \frac{1}{\gamma}, \ \ x \in \overline{\Omega}.$$

This implies from the coercivity of $a(u,v)$ and (3.1) that

$$\mu_0\|w\|_1^2 \leq A(w,w), \qquad |A(w,v)| \leq \left(\mu_1 + \frac{1}{\gamma}c_0^2\right)\|w\|_1\|v\|_1 \quad \forall w, v \in V.$$

Thus, we see that $A(w,v)$ is a coercive, symmetric and bounded bilinear form on $V \times V$, so that the solution w of problem (3.5) uniquely exists. Now, it follows from taking $v = \theta = u - u_h$ in (3.5), and utilizing equation (3.4), the definition of β, Lemma 2.1 and the trace theorem that for $v_h \in V_h$ we have

$$
\begin{aligned}
(u-u_h,q) &= a(\theta,w)+\int_{\Gamma_N}\beta w\,\theta\,ds \\
&= a(\theta,w-v_h)-\int_{\Gamma_N}(\varphi(u)-\varphi(u_h))\,v_h\,ds+\int_{\Gamma_N}\beta w\,\theta\,ds \\
&= a(\theta,w-v_h)+\int_{\Gamma_N}(\varphi(u)-\varphi(u_h))\,(w-v_h)\,ds \\
&\le \mu_1\|u-u_h\|_1\|w-v_h\|_1+\frac{1}{\gamma}c_0^2\|u-u_h\|_1\|w-v_h\|_1.
\end{aligned}
$$

Because both $q\in L_2(\Omega)$ and $v_h\in V_h$ are arbitrary, we arrive at the conclusion claimed in (3.3). □

Let $u_I\in V_h$ be the usual interpolation approximation of a continuous function u with the approximation properties [2]:

$$
\|u-u_I\|_{0,p,e}+h_e\|u-u_I\|_{1,p,e}\le Ch_e^{1+s}\|u\|_{1+s,p,e},\ 0<s\le k,\ 2\le p\le\infty,\ e\in J_h. \tag{3.7}
$$

Then, from Theorem 3.1, we immediately obtain the following conclusion.

Corollary 3.1. *Let u and u_h be the solutions of problems (2.1) and (2.4), respectively. Then, u_h converges to u in the H^1-norm, and if $u\in H^{1+s}(\Omega)$ we have*

$$
\|u-u_h\|_1\le Ch^s\|u\|_{1+s},\ 0<s\le k\,. \tag{3.8}
$$

Furthermore, assuming that the solution w of problem (3.5) belongs to $H^{1+\alpha}(\Omega)$, $0<\alpha\le 1$ and $\|w\|_{1+\alpha}\le C\|q\|$, we have

$$
\|u-u_h\|\le Ch^{\alpha+s}\|u\|_{1+s},\ 0<s\le k,\ 0<\alpha\le 1. \tag{3.9}
$$

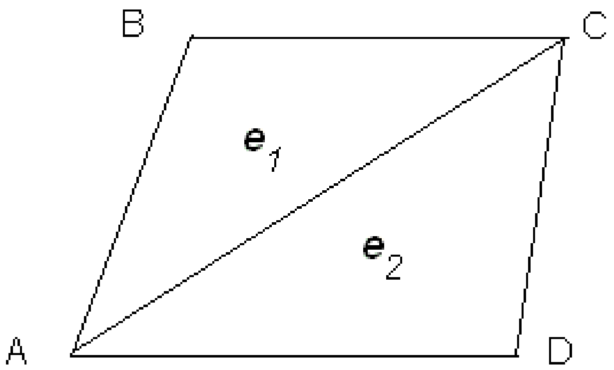
Obviously, the error estimates (3.8) and (3.9) (with $\alpha=1$) are optimal on both the approximation order of the finite element space and the regularity required for the exact solution.

Remark 3.1. According to the regularity theory of elliptic problems [6, 7], when the domain Ω and function $\beta(x)$ satisfy some smooth conditions, we indeed have that the solution $w\in H^2(\Omega)$ and $\|w\|_2\le C(\Omega)\|q\|$.

Below we will discuss the error estimate in the L_∞-norm by using the linear finite element space. We need an additional assumption on the triangulation J_h (see, for example, [17]).

Definition 31 *A quadrilateral $\diamondsuit ABCD$ is called an approximate parallelogram if (see Figure 1)*

$$
|\,\overrightarrow{AB}-\overrightarrow{DC}\,|\le Ch^2\,,\quad |\,\overrightarrow{BC}-\overrightarrow{AD}\,|\le Ch^2.
$$



Definition 32 *A triangulation J_h is called strongly regular, if any two adjacent triangular elements of J_h form an approximate parallelogram (see Figure 1).*

Under the strongly regular triangulation condition, we have the well known interpolation elementary estimate for the linear finite element space V_h (see Theorem 4.8 in [17]).

$$|\, a(u-u_I, v_h)\,| \le Ch^2(\,\|u\|_{2,\infty} + \|u\|_3)\,\|v_h\|_1,\;\; v_h \in V_h. \tag{3.10}$$

By means of this estimate, we can prove the following result.

Theorem 3.2. *Let u and u_h be the solutions of problems (2.1) and (2.4), respectively. Assume that the triangulation J_h is strongly regular, V_h is the linear finite element space, and $u \in W^2_\infty \cap H^3$. Then we have*

$$\|u-u_h\|_{0,\infty} \le C\left(1+\frac{1}{\gamma}\right) h^2 |\ln h\,|^{\frac{1}{2}}(\,\|u\|_{2,\infty} + \|u\|_3\,).$$

Proof. It follows from equation (3.4), Lemma 2.1, the elementary estimate (3.10) and the trace theorem that

$$\begin{aligned}
\mu_0\|u_I-u_h\|_1^2 &\le a(u_I-u_h, u_I-u_h)\\
&= a(u_I-u, u_I-u_h) - \int_{\Gamma_N} (\varphi(u)-\varphi(u_I)+\varphi(u_I)-\varphi(u_h))(u_I-u_h)\,ds\\
&\le a(u_I-u, u_I-u_h) - \int_{\Gamma_N} (\varphi(u)-\varphi(u_I))(u_I-u_h)\,ds\\
&\le Ch^2(\,\|u\|_{2,\infty}+\|u\|_3)\|u_I-u_h\|_1 + \frac{1}{\gamma}\|u-u_I\|_{L_2(\Gamma_N)}\|u_I-u_h\|_{L_2(\Gamma_N)}\\
&\le Ch^2(\,\|u\|_{2,\infty}+\|u\|_3)\|u_I-u_h\|_1 + C\frac{1}{\gamma}h^2\|u\|_3\|u_I-u_h\|_1,
\end{aligned}$$

where we have used the fact that

$$\|u-u_I\|_{L_2(\Gamma_N)} \le Ch^2\|u\|_{2,\partial\Omega} \le Ch^2\|u\|_{3,\Omega},$$

which results in a super-approximation estimate,

$$\|u_I-u_h\|_1 \le C\left(1+\frac{1}{\gamma}\right)h^2(\,\|u\|_{2,\infty}+\|u\|_3). \tag{3.11}$$

Now, we have by using the weak embedding inequality in the finite element space [17] that,

$$\|v_h\|_{0,\infty} \le C|\ln h|^{\frac{1}{2}}\|v_h\|_1,\ v_h \in V_h. \tag{3.12}$$

Thus, we obtain from (3.11)–(3.12) that

$$\begin{aligned}\|u-u_h\|_{0,\infty} &\le \|u-u_I\|_{0,\infty} + C|\ln h|^{\frac{1}{2}}\|u_I-u_h\|_1\\ &\le \|u-u_I\|_{0,\infty} + Ch^2|\ln h|^{\frac{1}{2}}(\,\|u\|_{2,\infty}+\|u\|_3),\end{aligned}$$

which, together with $\|u-u_I\|_{0,\infty} \le h^2\|u\|_{2,\infty}$, completes the proof. □

4. A posteriori error analysis

In this section, we will derive some residual-based a posteriori error estimators which provide global upper bounds and local lower bounds on the error $u-u_h$. To this end, we need to introduce some notions.

Let J_h be a regular finite element triangulation of domain Ω. We denote by $\mathcal{E}_h = \cup\{\, l \subset \partial e:\ e \in J_h\}$ the union of all the edges of J_h, and $\mathcal{E}_h^0 = \cup\{\, l \subset \partial e\backslash\partial\Omega:\ e \in J_h\}$ the union of all the interior edges of J_h. Let l be an edge shared by two adjacent elements e_1 and e_2 of J_h, and $n_i = n|_{\partial e_i}$ the unit normal vector external to ∂e_i. For a piecewise smooth function v on triangulation J_h, we define the jump $\left[\dfrac{\partial v}{\partial n}\right]$ of $\dfrac{\partial v}{\partial n}$ on $l \in \mathcal{E}_h^0$ as follows:

$$\left[\frac{\partial v}{\partial n}\right] = \nabla v_1\cdot n_1 + \nabla v_2\cdot n_2, \quad \text{on} \quad l \in \mathcal{E}_h^0,$$

where $\nabla v_i = \nabla v|_{\partial e_i}$ is the trace of ∇v from the interior of e_i.

Below we assume that there exists an interpolation function $\pi_h v \in V_h$ satisfying that (e.g., the Clément interpolant (see, for example, [3]))

$$\|v-\pi_h v\|_{s,e} \le Ch_e^{1-s}\|v\|_{1,\omega_e},\ s=0,1, \quad v \in H^1(\omega_e), \quad e \in J_h, \tag{4.1}$$

where $\omega_e = \cup\{e' \in J_h : \overline{e}' \cap \overline{e} \neq \emptyset\}$. Define the error estimators as follows:

$$\eta^{(s)}(u_h) = \Big(\sum_{e \in J_h} \|h_e^s (f + \Delta u_h - \mu u_h)\|_{0,e}^2 \Big)^{\frac{1}{2}},$$

$$\eta_b^{(s)}(u_h) = \Big(\sum_{l \in \mathcal{E}_h^0} \Big\| h_l^{s-1/2} \Big[\frac{\partial u_h}{\partial n}\Big] \Big\|_{0,l}^2 \Big)^{\frac{1}{2}},$$

$$\eta_N^{(s)}(u_h) = \Big(\sum_{l \subset \Gamma_N} \Big\| h_l^{s-1/2} \Big(\frac{\partial u_h}{\partial n} + \varphi(u_h)\Big) \Big\|_{0,l}^2 \Big)^{\frac{1}{2}}, s \geq 1,$$

where h_l is the length of the edge $l \in \mathcal{E}_h$. Obviously, all these quantities are computable in terms of the finite element solution u_h.

Let $u \in H^1(\Omega)$ be the solution of problem (2.1). Following a variational argument, it is easy to see that u is characterized by the following boundary value problem in the distribution sense:

$$-\Delta u + \mu u = f, \ \text{ in } \ \Omega, \tag{4.2}$$

$$u = 0, \ \text{ on } \ \Gamma \setminus \Gamma_N, \ \frac{\partial u}{\partial n} + \varphi(u) = 0, \ \text{ on } \ \Gamma_N. \tag{4.3}$$

Theorem 4.1. *Let u and u_h be the solutions of problems (2.1) and (2.4), respectively, $u \in H^1(\Omega)$. Then, there exists a constant C, independent of u, u_h, γ and the mesh size h, such that*

$$\|u - u_h\|_1 \leq C(\eta^{(1)}(u_h) + \eta_b^{(1)}(u_h) + \eta_N^{(1)}(u_h)). \tag{4.4}$$

Proof. Denote the error function $\theta = u - u_h$. Using the error equation (3.4) and integration by parts, we have that for $v_h \in V_h$,

$$\begin{aligned}
\mu_0 \|\theta\|_1^2 \leq & a(\theta,\theta) = a(\theta, \theta - v_h) + a(\theta, v_h) \\
= & a(\theta, \theta - v_h) - \int_{\Gamma_N} \big(\varphi(u) - \varphi(u_h)\big) v_h \\
= & \sum_{e \in J_h} (-\Delta\theta + \mu\theta, \theta - v_h)_e + \sum_{e \in J_h} \int_{\partial e} \frac{\partial \theta}{\partial n}(\theta - v_h) - \int_{\Gamma_N} \big(\varphi(u) - \varphi(u_h)\big) v_h \\
= & \sum_{e \in J_h} (f + \Delta u_h - \mu u_h, \theta - v_h)_e + \sum_{l \in \mathcal{E}_h^0} \int_l \Big[\frac{\partial \theta}{\partial n}\Big] (\theta - v_h) \\
& + \int_{\Gamma_N} \frac{\partial \theta}{\partial n}(\theta - v_h) - \int_{\Gamma_N} \big(\varphi(u) - \varphi(u_h)\big) v_h
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{e\in J_h}(f+\Delta u_h-\mu u_h,\theta-v_h)_e-\sum_{l\in\mathcal{E}_h^0}\int_l\left[\frac{\partial u_h}{\partial n}\right](\theta-v_h)\\
&\quad+\int_{\Gamma_N}\left(\frac{\partial\theta}{\partial n}+\varphi(u)-\varphi(u_h)\right)(\theta-v_h)-\int_{\Gamma_N}\big(\varphi(u)-\varphi(u_h)\big)\theta\\
&\le \sum_{e\in J_h}(f+\Delta u_h-\mu u_h,\theta-v_h)_e-\sum_{l\in\mathcal{E}_h^0}\int_l\left[\frac{\partial u_h}{\partial n}\right](\theta-v_h)\\
&\quad-\int_{\Gamma_N}\left(\frac{\partial u_h}{\partial n}+\varphi(u_h)\right)(\theta-v_h),
\end{aligned}
$$

where in the last inequality, we have utilized the boundary value condition (4.3) and property (2.3) of function φ. By taking $v_h=\pi_h\theta$, we obtain

$$
\begin{aligned}
\mu_0\|\theta\|_1^2 \le& \sum_{e\in J_h}\|f+\Delta u_h-\mu u_h\|_{0,e}\,\|\theta-\pi_h\theta\|_{0,e}\\
&+\sum_{l\in\mathcal{E}_h^0}\left\|\left[\frac{\partial u_h}{\partial n}\right]\right\|_{0,l}\|\theta-\pi_h\theta\|_{0,l}\\
&+\sum_{l\subset\Gamma_N}\left\|\frac{\partial u_h}{\partial n}+\varphi(u_h)\right\|_{0,l}\|\theta-\pi_h\theta\|_{0,l},
\end{aligned}
$$

from which and the well known trace inequality (see Lemma 2.3 in [16]), we obtain

$$
\int_{\partial e}|u|^2ds\le C\left(h_e^{-1}\|u\|_{0,e}^2+h_e\|\nabla u\|_{0,e}^2\right)\quad\forall u\in H^1(e). \tag{4.5}
$$

This, together with the approximation property (4.1), completes the proof. □

In order to derive the a posteriori error estimate in the L_2-norm, we need to introduce the auxiliary problem once again in the distribution sense:

$$
-\Delta w+\mu\, w=u-u_h\quad\text{in }\Omega, \tag{4.6}
$$

$$
w=0\quad\text{on }\Gamma\setminus\Gamma_N,\quad \frac{\partial w}{\partial n}+\beta(x)w=0\quad\text{on }\Gamma_N, \tag{4.7}
$$

where $\beta(x)=(\varphi(u)-\varphi(u_h))/(u-u_h)$. We assume that problem (4.6)–(4.7) admits a solution $w\in H^{1+\alpha}(\Omega)$ satisfying

$$
\|w\|_{1+\alpha}\le C\|u-u_h\|,\ 0<\alpha\le 1.
$$

Theorem 4.2. *Let u and u_h be the solutions of problems (2.1) and (2.4), respectively, $u\in H^1(\Omega)$. Then, there exists a constant C, independent of u, u_h and the mesh size h, such that*

$$
\|u-u_h\|\le C\left(\eta^{(1+\alpha)}(u_h)+\eta_b^{(1+\alpha)}(u_h)+\eta_N^{(1+\alpha)}(u_h)\right),\ 0<\alpha\le 1. \tag{4.8}
$$

Proof. Let $w_I \in V_h$ be the interpolation of function w, $\theta = u - u_h$. From equations (4.6)–(4.7) and the error equation (3.4), we have

$$
\begin{aligned}
\|\theta\|^2 =& a(w,\theta) - \int_{\Gamma_N} \frac{\partial w}{\partial n}\theta = a(w,\theta) + \int_{\Gamma_N} \beta w\,\theta \\
=& a(\theta, w - w_I) - \int_{\Gamma_N} \big(\varphi(u) - \varphi(u_h)\big) w_I + \int_{\Gamma_N} \beta w\,\theta \\
=& a(\theta, w - w_I) + \int_{\Gamma_N} \big(\varphi(u) - \varphi(u_h)\big)(w - w_I) \\
=& \sum_{e \in J_h} (f + \Delta u_h - \mu u_h, w - w_I)_e + \sum_{e \in J_h} \int_{\partial e} \frac{\partial \theta}{\partial n}(w - w_I) \\
&+ \int_{\Gamma_N} \left(\varphi(u) - \varphi(u_h)\right)(w - w_I) \\
=& \sum_{e \in J_h} (f + \Delta u_h - \mu u_h, w - w_I)_e + \sum_{l \in \mathcal{E}_h^0} \int_l \left[\frac{\partial \theta}{\partial n}\right](w - w_I) \\
&+ \int_{\Gamma_N} \left(\frac{\partial \theta}{\partial n} + \varphi(u) - \varphi(u_h)\right)(w - w_I) \\
=& \sum_{e \in J_h} (f + \Delta u_h - \mu u_h, w - w_I)_e - \sum_{l \in \mathcal{E}_h^0} \int_l \left[\frac{\partial u_h}{\partial n}\right](w - w_I) \\
&- \int_{\Gamma_N} \left(\frac{\partial u_h}{\partial n} + \varphi(u_h)\right)(w - w_I).
\end{aligned}
$$

Then, it follows from using the trace inequality (4.5), the approximation property (3.7), and noting $\|w\|_{1+\alpha} \le C\|\theta\|$ that the proof is completed. □

In practical finite element computations, it is desirable to implement them in an adaptive fashion. A typical procedure is to start with a coarse mesh first, and then use some a posteriori error estimators, say, provided in this section, as a guidance to properly refine the mesh locally or globally to achieve the desired accuracy. Many articles have discussed such adaptive algorithms, see, e.g., [5, 14] and the references therein, so we omit the further discussion here.

5. Numerical experiments

In this section we present some numerical examples to validate our theoretical analysis. The related data in problem (1.1)–(1.4) are $\Omega = (0,1) \times (0,1)$, $\mu = 1$, $g = 1$, $\Gamma_N = \{x = 1\}$. We take the exact solution as

$$
u_\gamma(x,y) = \left(\sin x - \frac{\gamma \cos 1 + \sin 1}{1 + \gamma} x\right) \sin \pi y,\ \ \gamma > 0,\ \ (x,y) \in \Omega,
$$

and the corresponding source term as

$$f = \left((2+\pi^2)\sin x - (1+\pi^2)\frac{\gamma\cos 1 + \sin 1}{1+\gamma} x \right) \sin \pi y.$$

We first partition Ω into a uniform square of mesh size h, and then divide each square into two right triangles in the same configuration. The linear finite element is used in the experiments. Denote by e_h the error between the exact solution and the numerical solution on mesh size h under some suitable norm, and the numerical convergence order is computed by

$$\alpha = \frac{\ln(e_h/e_1)}{\ln h},$$

In Table 1 and Table 2, we display the error and orders of convergence for numerical solutions in the L_2- and L_∞-norms, respectively. We see that the computation accuracy is very high and the theoretical orders of convergence are achieved, noticing that $\gamma << 1$.

mesh size	$\gamma = 0.01$		$\gamma = 0.001$	
h	error	order	error	order
1	3.5617e-1	-	3.6272e-1	-
1/20	0.95e-4	2.7470	0.532e-3	2.1780
1/40	0.59e-4	2.3600	0.487e-3	1.7927
1/60	0.52e-4	2.1571	0.476e-3	1.6208
1/80	0.50e-4	2.0244	0.471e-3	1.5168
1/100	0.42e-4	1.9642	0.368e-3	1.4968

Table 1: Order of convergence in the L_2-norm.

In Table 3 we display the approximate a posteriori error bound in the L_2-norm with $\gamma = 0.01$ and the effectivity index (see (4.8)),

$$\sigma = \frac{\eta_{\text{tol}}(u_h)}{\|u_\gamma - u_h\|}, \quad \eta_{\text{tol}}(u_h) = \eta^{(2)}(u_h) + \eta_b^{(2)}(u_h) + \eta_N^{(2)}(u_h).$$

As expected, we see that the error estimator $\eta_{\text{tol}}(u_h)$ is effective, that is, $\sigma \sim 1$.

Acknowledgements

This work was supported by the National Basic Research Program (2012CB95 5804), the Shahid Beheshti University and the Scientific Computation Research Group, and the National Natural Science Foundation of China (11071033 and 11171251).

mesh size h	$\gamma = 0.01$ error	order	$\gamma = 0.001$ error	order
1	3.8343e-1	-	3.8851e-1	-
1/20	0.180e-3	2.5582	0.167e-2	1.8178
1/40	0.172e-3	2.0899	0.168e-2	1.4758
1/60	0.172e-3	1.8829	0.168e-2	1.3295
1/80	0.172e-3	1.7593	0.168e-2	1.2422
1/100	0.172e-3	1.6741	0.168e-2	1.1820

Table 2: Order of convergence in the L_∞-norm.

mesh size h	1	1/20	1/40	1/60	1/80	1/100
$\eta_{\text{tol}}(u_h)$	1.9407	0.503e-3	0.280e-3	0.172e-3	0.157e-3	0.120e-3
σ	5.4487	5.2950	4.7481	3.3101	3.1397	2.8477

Table 3: A posteriori error estimators and the effectivity index.

References

[1] Atkinson, K. and Han, W.: *Theoretical numerical analysis: a functional analysis framework*. Springer, New York, 2001.

[2] Ciarlet, P. G.: *The finite element method for elliptic problems*. North-Holland, Amsterdam, 1978.

[3] Clément, P.: Approximation by finite element functions using local regularization. RARIO Anal. Numer. **9** (1975), 77–84.

[4] Djoko, J. K.: Discontinuous Galerkin finite element methods for variational inequalities of first and second kinds. Numer. Meth. for PDE **24** (2008), 296–311.

[5] Eriksson, K., Estep, D., Hansbo, P., and Johnson, C.: Introduction to adaptive methods for differential equations. Acta Numer. **4** (1995), 105–158.

[6] Gilbarg, D. and Trudinger, N. S.: *Elliptic partial differential equations of second order*. Springer-Verlag, Berlin, 1977.

[7] Grisvard, P.: Behavior of solutions of an elliptic boundary value problem in polygonal or polyhedral domains. In: B. Hubbard, (Ed.), *Numerical Solutions of Partial Differential Equations III*, pp. 207–274. Academic Press, New York, 1976.

[8] Glowinski, R., Lions, J. L., and Tremolieres, R.: *Numerical analysis of variational inequalities*. North-Holland, Amsterdam, 1981.

[9] Glowinski, R.: *Numerical methods for nonlinear variational problems*. Springer-Verlag, New York, 1984.

[10] Han, W. and Cheng, X. L.: *An introduction to variational inequalities: elementary theory, numerical analysis and applications*. Higher Education Press, Beijing, 2007.

[11] Han, W. and Sofonea, M.: *Quasistatic contact problems in viscoelasticity and viscoplasticity*. American Math. Soc. Inter. Press, 2002.

[12] Kikuchi, N. and Oden, J. T.: *Contact problem in elasticity: a study of variational inequalities and finite element methods*. SIAM, Philadelphia, 1988.

[13] Panagiotopoulos, P. D.: *Inequality problems in mechanics and applications*. Birhauser, Boston, 1985.

[14] Verfürth, R.: *A review of a posteriori error estimation and adaptive mesh refinement*. Wiley, Teubner, 1996.

[15] Zhang, T. and Li, C. J.: Finite element approximation to the second type variational inequality. Math. Numer. Sin. **25** (2003), 257–264.

[16] Zhou, T. X.: L_p error analysis of finite element method by using mesh dependent norms. Math. Numer. Sin. **4** (1982), 398–408.

[17] Zhu, Q. D. and Lin, Q.: *Superconvergence theory of the finite element methods*. Hunan Science Press, Hunan, China, 1989.