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Fast and guaranteed a posteriori error estimator

Combination of the equilibrated residual method and the method of hypercircle

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A posteriori error estimates

Elliptic problem $\left\{ \begin{array}{l} \text{exact solution } u \\ \text{FE solution } u_h \end{array} \right\}$ error $e = u - u_h$

A posteriori error estimator $\mathcal{E}$: $\left\{ \begin{array}{ll} \|e\| \approx \mathcal{E} & \\ \|e\| \leq \mathcal{E} & \text{guaranteed} \end{array} \right.$

Computable and fast

Methods

- *The equilibrated residual method*
  – locally computable, not guaranteed upper bound.

  Ladeveze and Leguillon (1983), Kelly (1984),
  Bank and Weiser (1985).

- *The method of hypercircle*
  – guaranteed upper bound, not locally computable.

  Synge (1957).

- *The combined method*
  – guaranteed upper bound, locally computable.

  Proposed by Ladeveze and Leguillon (1983)
  – piecewise constant data; not completely computable in 2D.

Linear elliptic model problem

$$
\begin{aligned}
-\nabla\cdot(\mathcal{A}\nabla\bar{u}) &= f && \text{in } \Omega,\\
\bar{u} &= g_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}},\\
(\mathcal{A}\nabla\bar{u})\cdot\nu &= g_{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}}.
\end{aligned}
$$

Notation:

$\Omega \subset \mathbb{R}^2$ ... polygonal domain,

$\nu = \nu(x_1, x_2)$... unite outer normal to $\partial\Omega$,

$\Gamma_{\mathrm{D}} \subset \partial\Omega$ ... Dirichlet part of $\partial\Omega$,

$\Gamma_{\mathrm{N}} \subset \partial\Omega$ ... Neumann part of $\partial\Omega$.

Model problem – weak formulation

Weak solution $\bar{u} \in H^1(\Omega)$, $\bar{u} = u + g_\mathrm{D}$ and $u \in V$ satisfies

$$(\mathcal{A}\nabla u, \nabla v) = (f, v) - (\mathcal{A}\nabla g_\mathrm{D}, \nabla v) + \langle g_\mathrm{N}, v\rangle \quad \forall v \in V,$$

where

$$V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_\mathrm{D}\},$$

$\mathcal{A} \in [L^\infty(\Omega)]^{2\times 2}$ ... symmetric, uniformly positive definite matrix,
$g_\mathrm{D} \in H^1(\Omega)$ ... prolongation of values on $\partial\Gamma_\mathrm{D}$ into interior of Ω,
$f \in L^2(\Omega)$ ... the right-hand side,
$g_\mathrm{N} \in L^2(\Gamma_N)$ ... the Neumann boundary condition,
$\Gamma_\mathrm{N} \subset \partial\Omega$ ... Neumann part of $\partial\Omega$.

$$(\mathcal{A}\nabla u, \nabla v) = \int_\Omega (\mathcal{A}\nabla u)\cdot\nabla v \,\mathrm{d}x, \quad \forall u, v \in V,$$

$$(f, v) = \int_\Omega f v \,\mathrm{d}x \quad \forall f, v \in L^2(\Omega),$$

$$\langle g_\mathrm{N}, v\rangle = \int_{\Gamma_\mathrm{N}} g_\mathrm{N} v \,\mathrm{d}s \quad \forall g_\mathrm{N}, v \in L^2(\Gamma_\mathrm{N}).$$

Finite element method

Finite element solution:

$\bar{u}_h = u_h + g_\mathsf{D}$, $\bar{u}_h \in H^1(\Omega)$ and $u_h \in V_h$ satisfies

$$(\mathcal{A}\nabla u_h, \nabla v_h) = (f, v_h) - (\mathcal{A}\nabla g_\mathsf{D}, \nabla v_h) + \langle g_\mathsf{N}, v_h \rangle \quad \forall v_h \in V_h.$$

T_h ... triangulation of Ω,

$V_h \subset V$ ... finite element space based on T_h

continuous and piecewise polynomial functions of degree p.

Residual equation:

$$(\mathcal{A}\nabla e, \nabla v) = \mathcal{R}(v) \quad \forall v \in V,$$

where

$$\mathcal{R}(v) = (f, v) - (\mathcal{A}\nabla \bar{u}_h, \nabla v) + \langle g_\mathsf{N}, v \rangle \text{... residuum,} \quad v \in V.$$

Notation: L^2-norm: $\|v\|^2_{0,\Omega} = (v, v)$, energy norm: $\|v\|^2 = (\mathcal{A}\nabla v, \nabla v)$.

The equilibrated residual method*

$$\text{Notation: } (\mathcal{A}\nabla u, \nabla v)_K = \int_K (\mathcal{A}\nabla u)\cdot\nabla v \,\mathrm{d}x, \quad \forall u, v \in H^1(K),$$

$$(f, v)_K = \int_K f v \,\mathrm{d}x \qquad \forall f, v \in L^2(K),$$

$$\langle g_K, v\rangle_{\partial K} = \int_{\partial K} g_K v \,\mathrm{d}s \qquad \forall g_K, v \in L^2(\partial K).$$

Error estimator:

$$\mathcal{E}_{\mathsf{EQ}}^2 = \sum_{K\in T_h} \|\Phi_K\|_K^2, \qquad \text{where } \|v\|_K^2 = (\mathcal{A}\nabla v, \nabla v)_K.$$

Local Neuman problems on triangles: $\Phi_K \in V(K)$,

$$(\mathcal{A}\nabla\Phi_K, \nabla v)_K = \underbrace{(f, v)_K - (\mathcal{A}\nabla\bar{u}_h, \nabla v)_K + \langle g_K, v\rangle_{\partial K}}_{\mathcal{R}_K^{\mathsf{EQ}}(v)} \quad \forall v \in V(K),$$

Upper bound $\|e\| \le \mathcal{E}_{\mathsf{EQ}}$. $\qquad V(K) = \left\{v \in H^1(K) : v = 0 \text{ on } \Gamma_{\mathsf{D}}\right\}$

*According to Ainsworth and Oden (2000).

Boundary fluxes g_K

- $g_K \in P^p(\gamma)$ ... polynomials of degree p on edges γ of elements K if $\gamma \not\subset \Gamma_N$.
- $g_K \approx (\mathcal{A}\nabla\bar{u}) \cdot \nu_K$, on ∂K.
- If K and K^* denote two adjacent elements then

$$\left.\begin{aligned} g_K + g_{K^*} &= 0 && \text{on } \partial K \cap \partial K^*, \\ g_K &= g_{\mathrm{N}} && \text{on } \partial K \cap \partial \Gamma_{\mathrm{N}}, \end{aligned}\right\} \Longrightarrow \quad \mathcal{R}(v) = \sum_{K \in T_h} \mathcal{R}_K^{\mathrm{EQ}}(v).$$

- p-th order equilibration condition:

$$\mathcal{R}_K^{\mathrm{EQ}}(\theta_K) = (f, \theta_K)_K - (\mathcal{A}\nabla\bar{u}_h, \nabla\theta_K)_K + \langle g_K, \theta_K \rangle_{\partial K} = 0$$

for all polynomials θ_K of degree p from $V(K)$.

- Boundary fluxes g_K can be computed quickly.

The equilibrated residual method – summary

- Compute boundary fluxes g_K – fast algorithm.

- Find approximate solutions to the local residual problems

$$(\mathcal{A}\nabla\Phi_K, \nabla v)_K = (f, v)_K - (\mathcal{A}\nabla\bar{u}_h, \nabla v)_K + \langle g_K, v\rangle_{\partial K} \quad \forall v \in V(K).$$

- Evaluate the estimator: $\|e\|^2 \leq \sum_{K\in T_h} \|\Phi_K\|_K^2 .$

This a posteriori error estimator is locally computable, but it is not guaranteed upper bound.

The method of hypercircle

Notation: $\|\mathbf{q}\|^2_{\mathcal{A}^{-1},\Omega} = (\mathcal{A}^{-1}\mathbf{q},\mathbf{q})$; $\quad H(\mathrm{div},\Omega) \subset \left[L^2(\Omega)\right]^2$

Substituting $v = e = \bar{u} - \bar{u}_h$ into the weak formulation we get:

$$-(\mathcal{A}\bar{u}, \nabla e) = -(f, e) - \langle g_{\mathsf{N}}, e\rangle\,.$$

Let us compute for any $\mathbf{q} \in H(\mathrm{div},\Omega)$:

$$\begin{aligned}
&\|\mathbf{q} - \mathcal{A}\nabla\bar{u}_h\|^2_{\mathcal{A}^{-1},\Omega}\\
&\quad = \left(\mathcal{A}^{-1}\mathbf{q} - \nabla\bar{u} - \nabla\bar{u}_h + \nabla\bar{u}\,,\, \mathbf{q} - \mathcal{A}\nabla\bar{u} - \mathcal{A}\nabla\bar{u}_h + \mathcal{A}\nabla\bar{u}\right)\\
&\quad = \left\|\mathbf{q} - \mathcal{A}\nabla\bar{u}\right\|^2_{\mathcal{A}^{-1},\Omega} + 2\left(\mathbf{q} - \mathcal{A}\nabla\bar{u}, \nabla\bar{u} - \nabla\bar{u}_h\right) + \left\|\bar{u} - \bar{u}_h\right\|^2\\
&\quad = \left\|\mathbf{q} - \mathcal{A}\nabla\bar{u}\right\|^2_{\mathcal{A}^{-1},\Omega} + 2\,(\mathbf{q}, \nabla e) - 2\,(f, e) - 2\,\langle g_{\mathsf{N}}, e\rangle + \left\|\bar{u} - \bar{u}_h\right\|^2.
\end{aligned}$$

$$Q(f, g_{\mathsf{N}}) = \{\mathbf{q} \in H(\mathrm{div},\Omega) : (\mathbf{q}, \nabla v) = (f, v) + \langle g_{\mathsf{N}}, v\rangle \quad \forall v \in V\}$$

$$\Downarrow$$

$$\|\mathbf{q} - \mathcal{A}\nabla\bar{u}_h\|^2_{\mathcal{A}^{-1},\Omega} = \|\mathbf{q} - \mathcal{A}\nabla\bar{u}\|^2_{\mathcal{A}^{-1},\Omega} + \|\bar{u} - \bar{u}_h\|^2, \quad \forall \mathbf{q} \in Q(f, g_{\mathsf{N}}).$$

The method of hypercircle – guaranteed upper bound

Thus,

$$\|e\| \leq \|\mathbf{q} - \mathcal{A}\nabla\bar{u}_h\|_{\mathcal{A}^{-1},\Omega} \quad \forall \mathbf{q} \in Q(f, g_\mathsf{N}).$$

$$Q(f, g_\mathsf{N}) = \bar{\mathbf{p}} + \mathbf{curl}\, W \quad \text{with} \quad \bar{\mathbf{p}} \in Q(f, g_\mathsf{N}) \text{ fixed,}$$

$$W = \left\{ w \in H^1(\Omega) : w = 0 \text{ on } \Gamma_\mathsf{N} \right\},$$

$$\mathbf{curl} = (\partial/\partial x_2, -\partial/\partial x_1)^\top.$$

Hence,

$$\|e\| \leq \|\bar{\mathbf{p}} + \mathbf{curl}\, y - \mathcal{A}\nabla\bar{u}_h\|_{\mathcal{A}^{-1},\Omega} \quad \forall y \in W$$

$$\bar{\mathbf{p}} = \mathbf{F} + \mathbf{curl}\, w, \text{ where } \mathbf{F}(x_1, x_2) = \left(-\int_0^{x_1} f(s, x_2)\,\mathrm{d}s, 0 \right)^\top,$$

$$\mathbf{curl}\, w \cdot \nu = \nabla w \cdot \tau = g_\mathsf{N} - F \cdot \nu \quad \text{on } \Gamma_\mathsf{N},$$

$$\tau = (-\nu_2, \nu_1)^\top$$

The method of hypercircle – summary

$$\|e\| \le \|\mathbf{q} - \mathcal{A}\nabla\bar{u}_h\|_{\mathcal{A}^{-1},\Omega} \qquad \forall \mathbf{q} \in Q(f, g_{\mathsf{N}})$$

$$\Updownarrow$$

$$\|e\| \le \|\bar{\mathbf{p}} + \mathbf{curl}\, y - \mathcal{A}\nabla\bar{u}_h\|_{\mathcal{A}^{-1},\Omega} \quad \forall y \in W$$

Replace W by a finite dimensional subspace $W_h \subset W$.
The optimal choice $y_h \in W_h$ minimizes the estimator over W_h:

$$(\mathcal{A}^{-1}\,\mathbf{curl}\, y_h, \mathbf{curl}\, v_h) = (\nabla\bar{u}_h - \mathcal{A}^{-1}\bar{\mathbf{p}}, \mathbf{curl}\, v_h) \quad \forall v_h \in W_h.$$

Trouble: evaluation of this estimator involves solution of a global problem, i.e., this estimator is not local – not fast.

THE COMBINED METHOD

- Compute boundary fluxes g_K by the equilibrated residual method.

- Apply the method of hypercircle to the local residual problem

$$(\mathcal{A}\nabla\Phi_K, \nabla v)_K = (f, v)_K - (\mathcal{A}\nabla\bar{u}_h, \nabla v)_K + \langle g_K, v\rangle_{\partial K} \quad \forall v \in V(K).$$

The combined method – error expression

Local residual problem with $v = \Phi_K$:

$$-(\mathcal{A}\nabla\Phi_K, \nabla\Phi_K)_K = -(f, \Phi_K)_K + (\mathcal{A}\nabla\bar{u}_h, \nabla\Phi_K)_K - \langle g_K, \Phi_K\rangle_{\partial K}$$

Let us compute for any $\mathbf{q} \in H(\text{div}, K)$:

$$\begin{aligned}
&\|\mathbf{q}\|^2_{\mathcal{A}^{-1},K} \\
&= \|\mathbf{q} - \mathcal{A}\nabla\Phi_K\|^2_{\mathcal{A}^{-1},K} + 2\left(\mathbf{q} - \mathcal{A}\nabla\Phi_K, \nabla\Phi_K\right)_K + \|\Phi_K\|^2_K \\
&= 2\,(\mathbf{q}, \nabla\Phi_K)_K - 2\,(f, \Phi_K)_K + 2\,(\mathcal{A}\nabla\bar{u}_h, \nabla\Phi_K)_K - 2\,\langle g_K, \Phi_K\rangle_{\partial K} \\
&\quad + \|\mathbf{q} - \mathcal{A}\nabla\Phi_K\|^2_{\mathcal{A}^{-1},K} + \|\Phi_K\|^2_K\,.
\end{aligned}$$

$$Q_K(f, g_K, \bar{u}_h) = \Big\{\mathbf{q} \in H(\text{div}, K) :$$
$$(\mathbf{q}, \nabla v)_K = (f, v)_K - (\mathcal{A}\nabla\bar{u}_h, \nabla v)_K + \langle g_K, v\rangle_{\partial K} \quad \forall v \in V(K)\Big\}$$

$$\Downarrow$$

$$\|\mathbf{q}\|^2_{\mathcal{A}^{-1},K} = \|\mathbf{q} - \mathcal{A}\nabla\Phi_K\|^2_{\mathcal{A}^{-1},K} + \|\Phi_K\|^2_K \quad \forall \mathbf{q} \in Q_K(f, g_K, \bar{u}_h)$$

The combined method – guaranteed upper bound

From the error expression we have

$$\|\Phi_K\|_K \le \|\mathbf{q}\|_{\mathcal{A}^{-1},K} \quad \forall \mathbf{q} \in Q_K(f, g_K, \bar{u}_h),$$

We conclude that

$$\|e\|^2 \le \sum_{K \in T_h} \|\Phi_K\|_K^2 \le \sum_{K \in T_h} \|\mathbf{q}\|_{\mathcal{A}^{-1},K}^2 = \sum_{K \in T_h} \|\bar{\mathbf{p}}_K + \mathbf{curl}\, y_K\|_{\mathcal{A}^{-1},K}^2$$

$$\forall y_K \in W(K) \text{ with } \mathbf{q} = \bar{\mathbf{p}}_K + \mathbf{curl}\, y_K,$$

because

$$Q_K(f, g_K, \bar{u}_h) = \bar{\mathbf{p}}_K + \mathbf{curl}\, W(K),$$

$$W(K) = \left\{ v \in H^1(K) : v = 0 \text{ on } \partial K \setminus \Gamma_\mathsf{D} \right\},$$

$$\bar{\mathbf{p}}_K \in Q_K(f, g_K, \bar{u}_h) \text{ is fixed.}$$

The combined method – computable estimator

Finite dimensional subspace: $W_h(K) \subset W(K)$.

Minimizer $y_{Kh} \in W_h(K)$, over $W_h(K)$ satisfies

$$\left(\mathcal{A}^{-1}\,\mathbf{curl}\,y_{Kh}, \mathbf{curl}\,v\right)_K = -\left(\mathcal{A}^{-1}\bar{\mathbf{p}}_K, \mathbf{curl}\,v\right)_K \quad \forall v \in W_h(K).$$

We chose $\bar{\mathbf{p}}_K \in Q_K(f, g_K, \bar{u}_h)$ as $\bar{\mathbf{p}}_K = \mathbf{F} + \mathbf{curl}\,w_K - \mathcal{A}\nabla\bar{u}_h,$ where

$$\mathbf{F}(x_1, x_2) = \left(-\int_0^{x_1} f(s, x_2)\,\mathrm{d}s, 0\right)^\top$$

$$w_K \in H^1(K) \quad \text{and} \quad \mathbf{curl}\,w_K \cdot \nu_K = \frac{\partial w_K}{\partial \tau_K} = g_K - \mathbf{F}\cdot\nu_K \quad \text{on } \partial K \setminus \Gamma_\mathrm{D}.$$

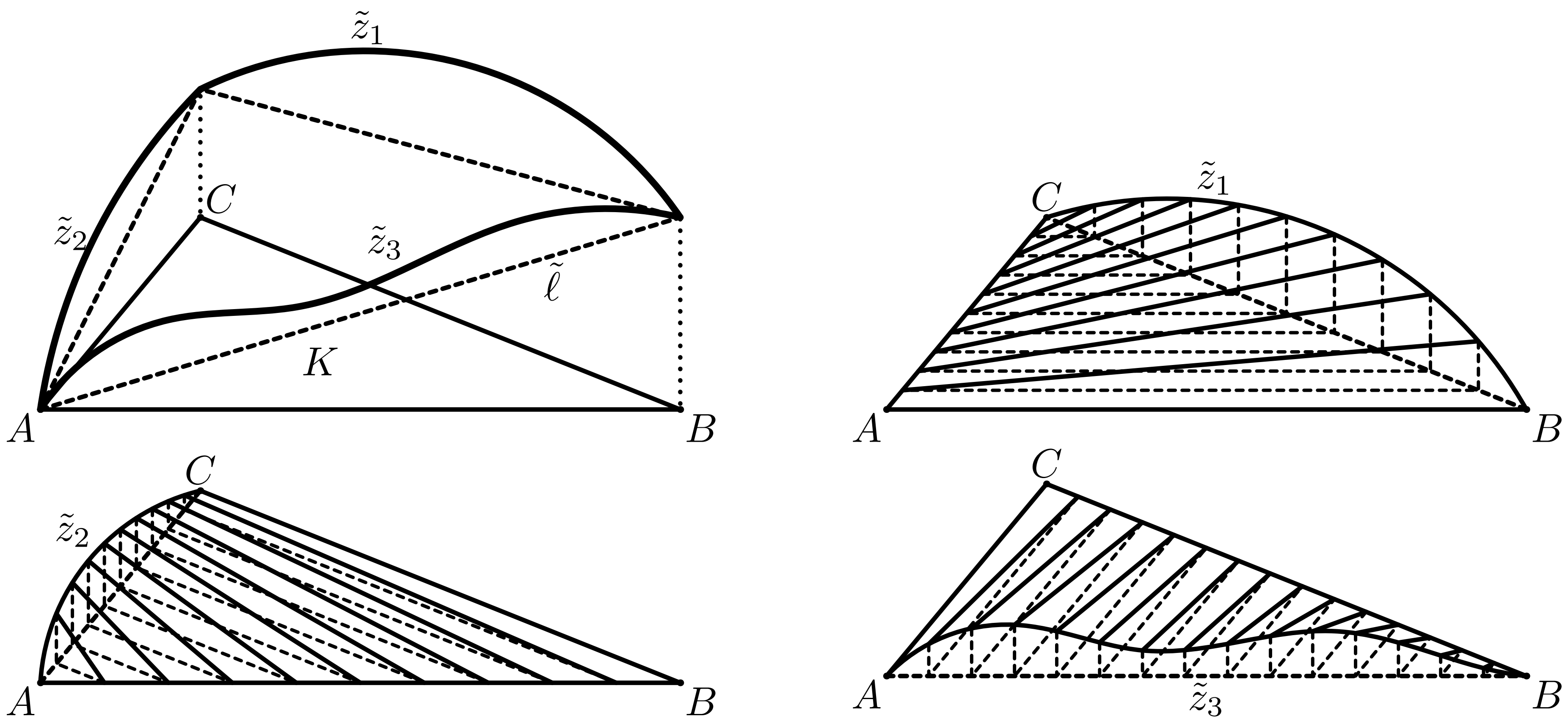
Notation: $\tau_K = (-\nu_{K,2}, \nu_{K,1})^\top$.

The combined method – construction of w_K

$$\frac{\partial w_K}{\partial \tau_K} = g_K - \mathbf{F}\cdot\nu_K \quad \text{on } \partial K \setminus \Gamma_D \quad \Rightarrow$$

values of w_K are given by primitive function to $g_K - \mathbf{F}\cdot\nu_K$.

Prolongation of these values into the interior of K:



The combined method – properties of the extension

Notation: $P^p(\Theta)$ – polynomials of degree p defined on the set Θ.

- If $\omega \in C^0(\partial K)$ and $\omega|_\gamma \in P^p(\gamma)$ for all edges $\gamma \subset \partial K$ then the prolongation $\tilde{\omega} \in P^p(K)$.

- Derivatives of the prolonged w_K – explicitly computable.

- If $u_h \in V_h$ is exact and $\mathcal{A}$ is constant then $\bar{\mathbf{p}}_K + \mathbf{curl}\, y_K = 0$ (estimator is exact).

The combined method – summary

- Compute boundary fluxes g_K using residual equilibration method.
- Construct for all triangles K in T_h vector

$$\bar{\mathbf{p}}_K = \mathbf{F} + \mathbf{curl}\, w_K - \mathcal{A}\nabla \bar{u}_h,$$

  where construction of w_K employs the prolongation shown above.

- Find solution $y_{Kh} \in W_h(K)$ of the finite dimensional local problem

$$\left(\mathcal{A}^{-1}\,\mathbf{curl}\, y_{Kh}, \mathbf{curl}\, v\right)_K = -\left(\mathcal{A}^{-1}\bar{\mathbf{p}}_K, \mathbf{curl}\, v\right)_K \quad \forall v \in W_h(K).$$

- Evaluate estimate

$$\|e\|^2 \leq \sum_{K \in T_h} \|\bar{\mathbf{p}}_K + \mathbf{curl}\, y_K\|^2_{\mathcal{A}^{-1},K}\,.$$

NUMERICAL EXPERIMENTS

- Finite element method:

  V_h ... continuous and piecewise quadratic functions with zero on Γ_{D}.

- Equilibrated residual method:

  $V_h(K)$ ... degree three polynomials with zero on Γ_{D}.

- The method of hypercircle:

  W_h ... continuous and piecewise quadratic functions with zero on Γ_{N}.

- The combined methods:

  $W_h(K)$ ... degree three polynomials with zero on $\partial K \setminus \Gamma_{\mathrm{D}}$.

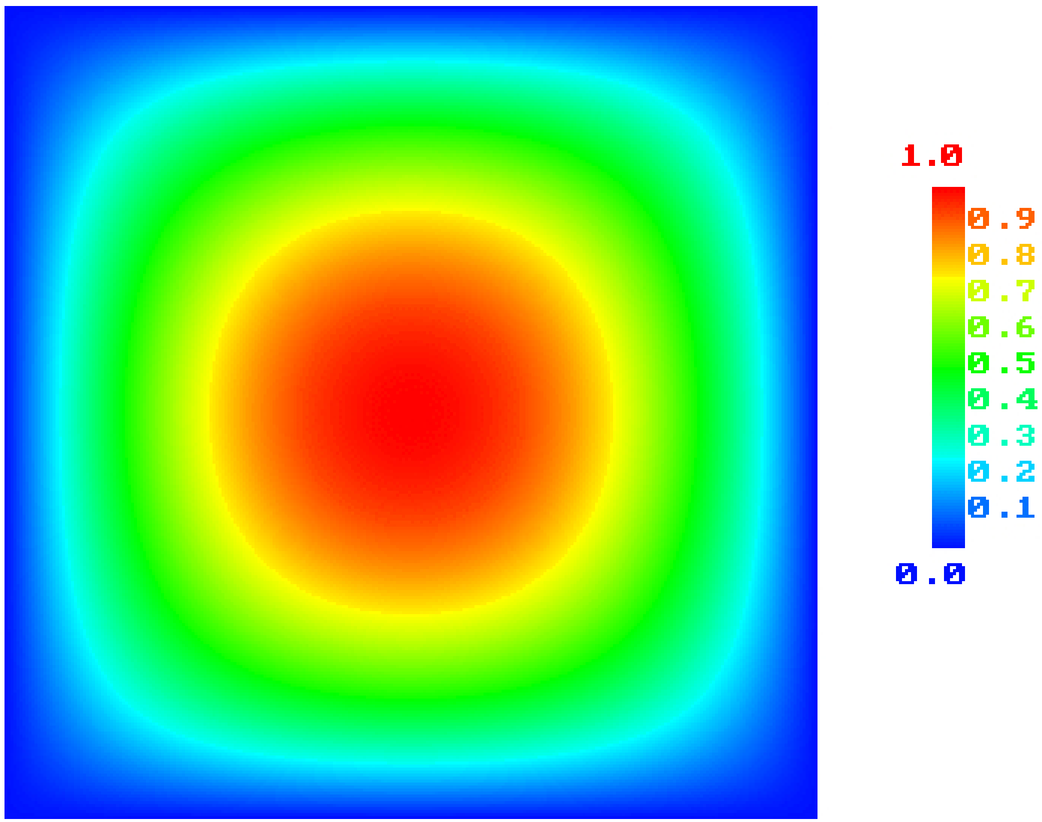
Remark: If we consider interior element K, then $\dim V_h(K) = 10$ and $\dim W_h(K) = 1$. Thus, the combined method performs faster.

Example 1

Data:

$$\Omega = [-1, 1]^2,$$
$$\Gamma_{\mathsf{D}} = \partial\Omega,$$
$$\Gamma_{\mathsf{N}} = \emptyset,$$
$$\mathcal{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$g_{\mathsf{D}} = 0,$$
$$f(x_1, x_2) = 2(2 - x_1^2 - x_2^2),$$
$$u(x_1, x_2) = (x_1^2 - 1)(x_2^2 - 1).$$

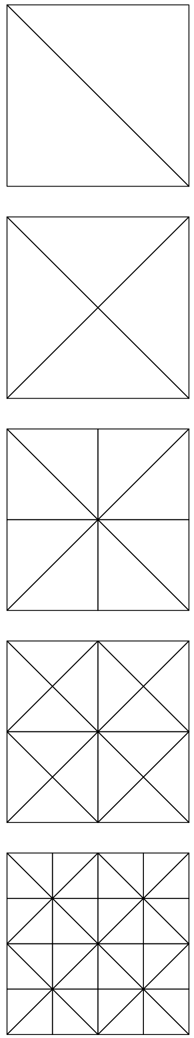
Exact solution:



Comparison of effectivity indices

N_{tri}	equilibrated residua	method of hypercircle	combined method
2	1.43	1.11	1.06
4	1.23	1.25	1.01
8	1.34	1.20	1.00
16	1.30	1.30	1.16
32	1.39	1.23	1.29
64	1.32	1.32	1.27
128	1.41	1.25	1.52
256	1.33	1.34	1.33
512	1.41	1.25	1.64
1024	1.33	1.34	1.36
2048	1.41	1.26	1.71
4096	1.33	1.34	1.38
8192	1.41	1.26	1.74
16384	1.33	1.34	1.38
32768	1.41	1.26	1.75

First five meshes:

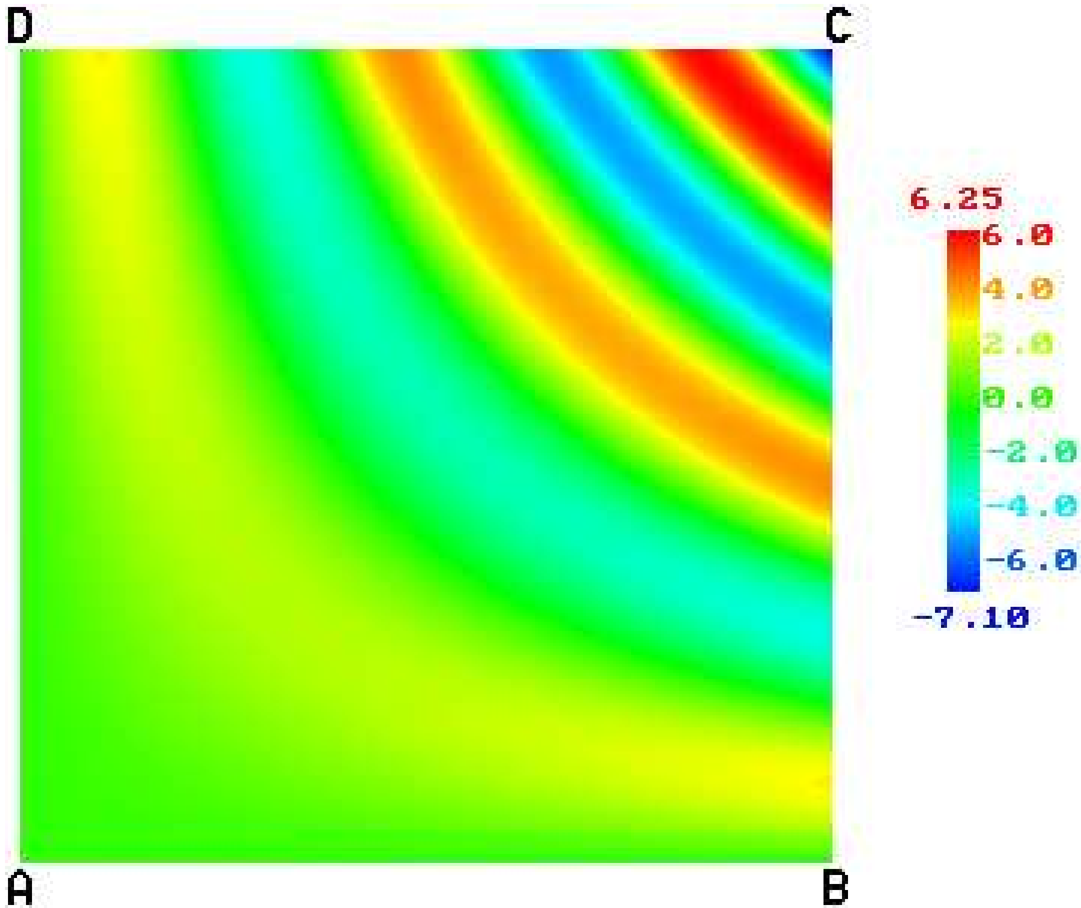


Example 2

Data:

$$\Omega = [0,1]^2,$$
$$\Gamma_\mathrm{D} = DA \cup AB,$$
$$\Gamma_\mathrm{N} = BC \cup CD,$$
$$\mathcal{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$g_\mathrm{D} = 0,$$
$$g_\mathrm{N}, f = \text{ such that}$$
$$u(x_1, x_2) = \sin(17 x_1 x_2) \mathrm{e}^{x_1 + x_2}$$

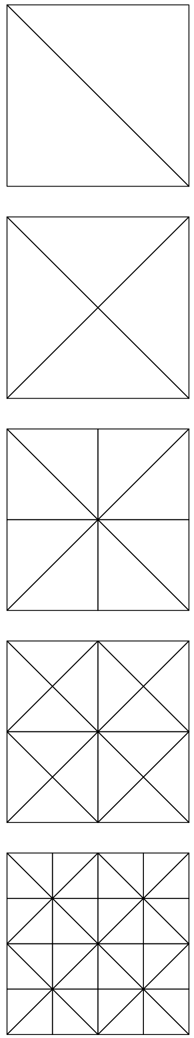
Exact solution:



Comparison of effectivity indices

N_{tri}	equilibrated residua	method of hypercircle	combined method
2	0.37	1.93	1.86
4	0.29	1.70	1.71
8	0.49	1.65	1.69
16	0.94	1.51	1.19
32	1.11	1.56	1.37
64	1.06	1.51	1.29
128	1.14	1.68	1.37
256	1.17	1.48	1.28
512	1.30	1.52	1.58
1024	1.22	1.49	1.35
2048	1.34	1.50	1.67
4096	1.24	1.50	1.38
8192	1.35	1.49	1.70
16384	1.25	1.51	1.39
32768	1.35	1.49	1.71

First five meshes:



Thank you for your attention.