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Guaranteed and locally computable a posteriori error estimator

Combination of the equilibrated residual method and the method of hypercircle

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INTRODUCTION – a posteriori error estimates

u	... exact solution of an elliptic problem
u_h	... finite element solution of the elliptic problem
$e = u - u_h$	... the error
$\mathcal{E} \approx \|e\|$	... a posteriori error estimator

- Computable from u_h and input data.
- Computation of $\mathcal{E}$ should be fast.
- Guaranteed upper bound

$$\|e\| \leq \mathcal{E}.$$

- Global error estimator – solution of global problem.
- Local error estimator – series of local problems.

INTRODUCTION – methods

- *The equilibrated residual method*
  – locally computable, not guaranteed upper bound.

  Origin of this method: Ladeveze and Leguillon (1983), Kelly (1984), Bank and Weiser (1985).

- *The method of hypercircle*
  – guaranteed upper bound, not locally computable.

  Fundamental book: Synge (1957).

- *The combined method*
  – guaranteed upper bound, locally computable.

  Published by Ladeveze and Leguillon (1983), but they consider piecewise constant data and their estimator is not completely computable in 2D.

Linear elliptic model problem – classical formulation

$$
\begin{aligned}
-\nabla\cdot(\mathcal{A}\nabla\bar{u}) &= f && \text{in } \Omega,\\
\bar{u} &= g_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}},\\
(\mathcal{A}\nabla\bar{u})\cdot\nu &= g_{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}}.
\end{aligned}
$$

Notation:

$\Omega \subset \mathbb{R}^2$ ... polygonal domain,

$\nu = \nu(x_1, x_2)$... unite outer normal to $\partial\Omega$,

$\Gamma_{\mathrm{D}} \subset \partial\Omega$ ... Dirichlet part of $\partial\Omega$,

$\Gamma_{\mathrm{N}} \subset \partial\Omega$ ... Neumann part of $\partial\Omega$.

Linear elliptic model problem – weak formulation

Find $\bar{u} \in H^1(\Omega)$ such that $\bar{u} = u + g_{\mathsf{D}}$ and $u \in V$ satisfies

$$(\mathcal{A}\nabla u, \nabla v) = (f, v) - (\mathcal{A}\nabla g_{\mathsf{D}}, \nabla v) + \langle g_{\mathsf{N}}, v\rangle \quad \forall v \in V,$$

where

$$V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_{\mathsf{D}}\},$$

$\mathcal{A} \in [L^\infty(\Omega)]^{2\times 2}$ ... symmetric, uniformly positive definite matrix,

$g_{\mathsf{D}} \in H^1(\Omega)$ ... extension of values on $\partial\Gamma_{\mathsf{D}}$ into interior of Ω,

$f \in L^2(\Omega)$ ... the right-hand side,

$g_{\mathsf{N}} \in L^2(\Gamma_N)$ ... the Neumann boundary condition,

$\Gamma_{\mathsf{N}} \subset \partial\Omega$ ... Neumann part of $\partial\Omega$.

$$(\mathcal{A}\nabla u, \nabla v) = \int_\Omega (\mathcal{A}\nabla u) \cdot \nabla v \, \mathrm{d}x, \quad \forall u, v \in V,$$

$$(f, v) = \int_\Omega f v \, \mathrm{d}x \quad \forall f, v \in L^2(\Omega),$$

$$\langle g_{\mathsf{N}}, v\rangle = \int_{\Gamma_{\mathsf{N}}} g_{\mathsf{N}} v \, \mathrm{d}s \quad \forall g_{\mathsf{N}}, v \in L^2(\Gamma_{\mathsf{N}}).$$

FINITE ELEMENT METHOD

Finite element solution:
$\bar{u}_h = u_h + g_{\mathsf{D}}$, where $\bar{u}_h \in H^1(\Omega)$ and $u_h \in V_h$ satisfies

$$(\mathcal{A}\nabla u_h, \nabla v_h) = (f, v_h) - (\mathcal{A}\nabla g_{\mathsf{D}}, \nabla v_h) + \langle g_{\mathsf{N}}, v_h\rangle \quad \forall v_h \in V_h.$$

T_h ... triangulation of Ω,

$V_h \subset V$ ... finite element space based on T_h consists of continuous and piecewise polynomial functions of degree p.

Residual equation:

$$(\mathcal{A}\nabla e, \nabla v) = \mathcal{R}(v) \quad \forall v \in V,$$

where

$$e = u - u_h = \bar{u} - \bar{u}_h \quad \text{... the error,}$$

$$\mathcal{R}(v) = (f, v) - (\mathcal{A}\nabla \bar{u}_h, \nabla v) + \langle g_{\mathsf{N}}, v\rangle \quad \text{... residuum,} \quad v \in V.$$

Notation: L^2-norm: $\|v\|_{0,\Omega}^2 = (v, v)$, energy norm: $\|v\|^2 = (\mathcal{A}\nabla v, \nabla v)$.

THE EQUILIBRATED RESIDUAL METHOD*

Split residuum $\mathcal{R}(v)$ into contribution from individual elements:

$$\mathcal{R}(v) = \sum_{K \in T_h} \mathcal{R}_K^{\mathsf{EQ}}(v|_K) \quad \forall v \in V,$$

where

$$\mathcal{R}_K^{\mathsf{EQ}}(v) = (f, v)_K - (\mathcal{A}\nabla \bar{u}_h, \nabla v)_K + \langle g_K, v \rangle_{\partial K} \quad \forall v \in V(K),$$

and $V(K) = \left\{ v \in H^1(K) : v = 0 \text{ on } \Gamma_{\mathsf{D}} \right\}$.

Notation:

$$\begin{aligned} (\mathcal{A}\nabla u, \nabla v)_K &= \int_K (\mathcal{A}\nabla u) \cdot \nabla v \, \mathrm{d}x, && \forall u, v \in H^1(K), \\ (f, v)_K &= \int_K f v \, \mathrm{d}x && \forall f, v \in L^2(K), \\ \langle g_K, v \rangle_{\partial K} &= \int_{\partial K} g_K v \, \mathrm{d}s && \forall g_K, v \in L^2(\partial K). \end{aligned}$$

*According to the book Ainsworth and Oden (2000).

The equilibrated residual method – boundary fluxes g_K

- $g_K \in P^p(\gamma)$ ... polynomials of degree p on edges γ of elements K.
- Approximate the actual fluxes of the true solution on the elements boundaries:

$$g_K \approx \nabla u \cdot \nu_K, \text{ on } \partial K.$$

- If K and K^* denote two adjacent elements then

$$\left.\begin{aligned} g_K + g_{K^*} &= 0 \qquad \text{on } \partial K \cap \partial K^*, \\ g_K &= g_{\mathsf{N}} \qquad \text{on } \partial K \cap \partial \Gamma_{\mathsf{N}}, \end{aligned}\right\} \implies \mathcal{R}(v) = \sum_{K \in T_h} \mathcal{R}_K^{\mathsf{EQ}}(v).$$

- Satisfy p-th order equilibration condition:

$$\mathcal{R}_K^{\mathsf{EQ}}(\theta_K) = (f, \theta_K)_K - (\mathcal{A}\nabla \bar{u}_h, \nabla \theta_K)_K + \langle g_K, \theta_K \rangle_{\partial K} = 0$$

for all polynomials θ_K of degree p from $V(K)$.

- Boundary fluxes g_K can be computed quickly.

The equilibrated residual method – local problem

Define the solution $\Phi_K \in V(K)$ of the local residual problem

$$(\mathcal{A}\nabla\Phi_K, \nabla v)_K = (f, v)_K - (\mathcal{A}\nabla\bar{u}_h, \nabla v)_K + \langle g_K, v\rangle_{\partial K} \quad \forall v \in V(K),$$

i.e.,

$$(\mathcal{A}\nabla\Phi_K, \nabla v)_K = \mathcal{R}_K^{\mathsf{EQ}}(v) \qquad \forall v \in V(K).$$

This is local elliptic problem with Neumann boundary conditions given by g_K. The existence of solution Φ_K is guaranteed by the equilibration condition:

$$(f, 1)_K + \langle g_K, 1\rangle_{\partial K} = 0.$$

Nonuniqueness of Φ_K is not important since we are interested in $\nabla\Phi_K$.

The equilibrated residual method – guaranteed upper bound

Notation: $\|v\|_K^2 = (\mathcal{A}\nabla v, \nabla v)_K$.

Residual equation:

$$(\mathcal{A}\nabla e, \nabla v) = \sum_{K \in T_h} \mathcal{R}_K^{\mathsf{EQ}}(v) = \sum_{K \in T_h} (\mathcal{A}\nabla \Phi_K, \nabla v)_K \quad \forall v \in V.$$

Two times Cauchy-Schwarz inequality:

$$|(\mathcal{A}\nabla e, \nabla v)| \leq \sum_{K \in T_h} \|\Phi_K\|_K \|v\|_K \leq \left(\sum_{K \in T_h} \|\Phi_K\|_K^2 \right)^{1/2} \|v\|,$$

Finally:

$$\|e\| = \sup_{0 \neq v \in V} \frac{|(\mathcal{A}\nabla e, \nabla v)|}{\|v\|} \leq \left(\sum_{K \in T_h} \|\Phi_K\|_K^2 \right)^{1/2}.$$

Thus, the local solutions Φ_K provide guaranteed upper bound.

Trouble:

Φ_K as solutions of infinitely dimensional problems are not computable.

The equilibrated residual method – summary

- Compute boundary fluxes g_K – fast algorithm.

- Find approximate solutions to the local residual problems

$$(\mathcal{A}\nabla\Phi_K, \nabla v)_K = (f, v)_K - (\mathcal{A}\nabla\bar{u}_h, \nabla v)_K + \langle g_K, v\rangle_{\partial K} \quad \forall v \in V(K).$$

- Evaluate the estimator

$$\|e\|^2 \leq \sum_{K\in T_h} \|\Phi_K\|_K^2 .$$

This a posteriori error estimator is locally computable, but it is not guaranteed upper bound.

THE METHOD OF HYPERCIRCLE

Notation: $\|\mathbf{q}\|^2_{\mathcal{A}^{-1},\Omega} = (\mathcal{A}^{-1}\mathbf{q},\mathbf{q})$; $\quad H^1(\mathrm{div},\Omega) \subset \left[L^2(\Omega)\right]^2$

Substituting $v = e = \bar{u} - \bar{u}_h$ into the weak formulation we get:

$$-\left(\mathcal{A}\bar{u},\nabla e\right) = -\left(f,e\right) - \langle g_{\mathrm{N}},e\rangle\,.$$

Let us compute for any $\mathbf{q} \in H^1(\mathrm{div},\Omega)$:

$$\begin{aligned}
&\|\mathbf{q}-\mathcal{A}\nabla\bar{u}_h\|^2_{\mathcal{A}^{-1},\Omega}\\
&\quad = \left(\mathcal{A}^{-1}\mathbf{q}-\nabla\bar{u}-\nabla\bar{u}_h+\nabla\bar{u}\,,\,\mathbf{q}-\mathcal{A}\nabla\bar{u}-\mathcal{A}\nabla\bar{u}_h+\mathcal{A}\nabla\bar{u}\right)\\
&\quad = \left\|\mathbf{q}-\mathcal{A}\nabla\bar{u}\right\|^2_{\mathcal{A}^{-1},\Omega} + 2\left(\mathbf{q}-\mathcal{A}\nabla\bar{u},\nabla\bar{u}-\nabla\bar{u}_h\right) + \left\|\bar{u}-\bar{u}_h\right\|^2\\
&\quad = \left\|\mathbf{q}-\mathcal{A}\nabla\bar{u}\right\|^2_{\mathcal{A}^{-1},\Omega} + 2\left(\mathbf{q},\nabla e\right) - 2\left(f,e\right) - 2\left\langle g_{\mathrm{N}},e\right\rangle + \left\|\bar{u}-\bar{u}_h\right\|^2.
\end{aligned}$$

$$Q(f,g_{\mathrm{N}}) = \left\{\mathbf{q}\in H^1(\mathrm{div},\Omega) : (\mathbf{q},\nabla v) = (f,v) + \langle g_{\mathrm{N}},v\rangle \quad \forall v\in V\right\}$$

$$\Downarrow$$

$$\|\mathbf{q}-\mathcal{A}\nabla\bar{u}_h\|^2_{\mathcal{A}^{-1},\Omega} = \|\mathbf{q}-\mathcal{A}\nabla\bar{u}\|^2_{\mathcal{A}^{-1},\Omega} + \|\bar{u}-\bar{u}_h\|^2, \quad \forall\mathbf{q}\in Q(f,g_{\mathrm{N}}).$$

The method of hypercircle – guaranteed upper bound

Thus,

$$\|e\|^2 = \|\bar{u} - \bar{u}_h\|^2 \leq \|\mathbf{q} - \mathcal{A}\nabla\bar{u}_h\|^2_{\mathcal{A}^{-1},\Omega} \quad \forall \mathbf{q} \in Q(f, g_{\mathsf{N}}).$$

This estimator is exact if $\mathbf{q} = \nabla u$, but it is unreachable.

How to find suitable function $\mathbf{p}_h \in Q(f, g_{\mathsf{N}})$, which would produce tight upper bound?

The crucial ingredient:
the structure of $Q(f, g_{\mathsf{N}})$, described by Křížek (1983).

The method of hypercircle – structure of $Q(f, g_{\mathsf{N}})$

Let $\bar{\mathbf{p}} \in Q(f, g_{\mathsf{N}})$ be arbitrary but fixed, then

$$Q(f, g_{\mathsf{N}}) = \bar{\mathbf{p}} + Q(0,0),$$

where

$$Q(0,0) = \left\{ \mathbf{q} \in H^1(\mathrm{div}, \Omega) : (\mathbf{q}, \nabla v) = 0 \quad \forall v \in V \right\}.$$

It is

$$Q(0,0) = \mathbf{curl}\, W,$$

where

$$W = \left\{ w \in H^1(\Omega) : w = 0 \text{ on } \Gamma_{\mathsf{N}} \right\}.$$

Definition: $\mathbf{curl} = (\partial/\partial x_2, -\partial/\partial x_1)^\top$.

How to construct $\bar{\mathbf{p}} \in Q(f, g_{\mathsf{N}})$?

The method of hypercircle – construction of $\bar{\mathbf{p}} \in Q(f, g_{\mathsf{N}})$

Any $\bar{\mathbf{p}} \in Q(f, g_{\mathsf{N}})$ have to satisfy: $-\operatorname{div} \bar{\mathbf{p}} = f$ and $\bar{\mathbf{p}} \cdot \nu = g_{\mathsf{N}}$. Therefore,

$$\bar{\mathbf{p}} = \mathbf{F} + \mathbf{curl}\, w,$$

where

$$\mathbf{F}(x_1, x_2) = \left(-\int_0^{x_1} f(s, x_2)\, \mathrm{d}s, 0 \right)^\top$$

and $w \in H^1(\Omega)$ is an arbitrary function satisfying

$$\mathbf{curl}\, w \cdot \nu = \nabla w \cdot \tau = g_{\mathsf{N}} - F \cdot \nu \qquad \text{on } \Gamma_{\mathsf{N}},$$

Notation: $\tau = (-\nu_2, \nu_1)$ . . . a unit tangent vector to Γ_{N}.

Remark:
the values of w on ∂K are given by primitive function to $g_{\mathsf{N}} - F \cdot \nu$.

The method of hypercircle – conclusion

$$\|e\| \le \|\mathbf{q} - \mathcal{A}\nabla \bar{u}_h\|_{\mathcal{A}^{-1},\Omega} \qquad \forall \mathbf{q} \in Q(f, g_{\mathsf{N}})$$

$$\Updownarrow$$

$$\|e\| \le \|\bar{\mathbf{p}} + \mathbf{curl}\, y - \mathcal{A}\nabla \bar{u}_h\|_{\mathcal{A}^{-1},\Omega} \qquad \forall y \in W$$

To obtain computable estimate we replace W by a finite dimensional subspace $W_h \subset W$.
The optimal choice $y_h \in W_h$ minimizes the estimator over W_h:

$$(\mathcal{A}^{-1}\,\mathbf{curl}\, y_h, \mathbf{curl}\, v_h) = (\nabla u_h - \mathcal{A}^{-1}\bar{\mathbf{p}}, \mathbf{curl}\, v_h) \quad \forall v_h \in W_h.$$

$\|\bar{\mathbf{p}} + \mathbf{curl}\, y_h - \mathcal{A}\nabla \bar{u}_h\|_{\mathcal{A}^{-1},\Omega}$ ... computable guaranteed upper bound.

Trouble: evaluation of this estimator involves solution of a global problem, i.e., this estimator is not local.

THE COMBINED METHOD

To obtain locally computable guaranteed upper bound, we combine the equilibrated residual method with the hypercircle method.

- Compute boundary fluxes g_K by the equilibrated residual method.

- Apply the method of hypercircle to the local residual problem

$$(\mathcal{A}\nabla\Phi_K, \nabla v)_K = (f, v)_K - (\mathcal{A}\nabla\bar{u}_h, \nabla v)_K + \langle g_K, v\rangle_{\partial K} \quad \forall v \in V(K).$$

The combined method – error expression

Local residual problem with $v = \Phi_K$:

$$-(\mathcal{A}\nabla\Phi_K, \nabla\Phi_K)_K = -(f, \Phi_K)_K + (\mathcal{A}\nabla\bar{u}_h, \nabla\Phi_K)_K - \langle g_K, \Phi_K\rangle_{\partial K}$$

Let us compute for any $\mathbf{q} \in H^1(\mathrm{div}, K)$:

$$\begin{aligned}
&\|\mathbf{q}\|^2_{\mathcal{A}^{-1},K} \\
&= \|\mathbf{q} - \mathcal{A}\nabla\Phi_K\|^2_{\mathcal{A}^{-1},K} + 2\left(\mathbf{q} - \mathcal{A}\nabla\Phi_K, \nabla\Phi_K\right)_K + \|\Phi_K\|^2_K \\
&= 2\,(\mathbf{q}, \nabla\Phi_K)_K - 2\,(f, \Phi_K)_K + 2\,(\mathcal{A}\nabla\bar{u}_h, \nabla\Phi_K)_K - 2\,\langle g_K, \Phi_K\rangle_{\partial K} \\
&\quad + \|\mathbf{q} - \mathcal{A}\nabla\Phi_K\|^2_{\mathcal{A}^{-1},K} + \|\Phi_K\|^2_K\,.
\end{aligned}$$

$$Q_K(f, g_K, \bar{u}_h) = \Big\{\mathbf{q} \in H^1(\mathrm{div}, K) :$$
$$(\mathbf{q}, \nabla v)_K = (f, v)_K - (\mathcal{A}\nabla\bar{u}_h, \nabla v)_K + \langle g_K, v\rangle_{\partial K} \quad \forall v \in V(K)\Big\}$$

$$\Downarrow$$

$$\|\mathbf{q}\|^2_{\mathcal{A}^{-1},K} = \|\mathbf{q} - \mathcal{A}\nabla\Phi_K\|^2_{\mathcal{A}^{-1},K} + \|\Phi_K\|^2_K \quad \forall \mathbf{q} \in Q_K(f, g_K, \bar{u}_h)$$

The combined method – structure of $Q_K(f, g_K, \bar{u}_h)$

$\bar{\mathbf{p}}_K \in Q_K(f, g_K, \bar{u}_h)$ is arbitrary but fixed.

$$\begin{aligned} Q_K(f, g_K, \bar{u}_h) &= \bar{\mathbf{p}}_K + Q_K(0, 0, 0) \\ &= \bar{\mathbf{p}}_K + \mathbf{curl}\, W(K) \end{aligned}$$

$$W(K) = \left\{ v \in H^1(K) : v = 0 \text{ on } \partial K \setminus \Gamma_D \right\}$$

$$Q_K(0, 0, 0) = \left\{ \mathbf{q} \in H^1(\text{div}, K) : (\mathbf{q}, \nabla v)_K = 0 \quad \forall v \in V(K) \right\}$$

The combined method – guaranteed upper bound

From the error expression we have

$$\|\Phi_K\|_K \le \|\mathbf{q}\|_{\mathcal{A}^{-1},K} \qquad \forall \mathbf{q} \in Q_K(f, g_K, \bar{u}_h),$$

$$\Updownarrow$$

$$\|\Phi_K\|_K \le \|\bar{\mathbf{p}}_K + \mathbf{curl}\, y_K\|_{\mathcal{A}^{-1},K} \quad \forall y_K \in W(K).$$

We conclude that

$$\|e\|^2 \le \sum_{K \in T_h} \|\Phi_K\|_K^2 \le \sum_{K \in T_h} \|\bar{\mathbf{p}}_K + \mathbf{curl}\, y_K\|^2_{\mathcal{A}^{-1},K} \quad \forall y_K \in W(K).$$

Finite dimensional subspace: $W_h(K) \subset W(K)$.
For example: $W_h(K) = P^{p+1}(K) \subset W(K)$.

The optimal $y_{Kh} \in W_h(K)$, which minimizes the right-hand side over $W_h(K)$, satisfies

$$\left(\mathcal{A}^{-1}\,\mathbf{curl}\, y_{Kh}, \mathbf{curl}\, v\right)_K = -\left(\mathcal{A}^{-1}\bar{\mathbf{p}}_K, \mathbf{curl}\, v\right)_K \quad \forall v \in W_h(K).$$

The combined method – construction of $\bar{\mathbf{p}}_K \in Q_K(f, g_K, \bar{u}_h)$

How to find vector $\bar{\mathbf{p}}_K \in Q_K(f, g_K, \bar{u}_h)$ efficiently?

We have

$$\bar{\mathbf{p}}_K = \mathbf{F} + \mathbf{curl}\, w_K - \mathcal{A}\nabla\bar{u}_h,$$

where $\mathcal{A}\nabla\bar{u}_h$ is known,

$$\mathbf{F}(x_1, x_2) = \left(-\int_0^{x_1} f(s, x_2)\,\mathrm{d}s, 0 \right)^\top$$

and $w_K \in H^1(K)$ has to satisfy

$$\mathbf{curl}\, w_K \cdot \nu_K = \frac{\partial w_K}{\partial \tau_K} = g_K - \mathbf{F} \cdot \nu_K \quad \text{on } \partial K \setminus \Gamma_\mathrm{D}.$$

Notation: $\tau_K = (-\nu_{K,2}, \nu_{K,1})^\top$.

Notice that the values of w_K on the boundary ∂K are given by a primitive function to $g_K - \mathbf{F} \cdot \nu_K$.

The combined method – primitive function to $g_K - \mathbf{F}\cdot\nu_K$

Consider triangle K with vertices A, B, C. Then

$$w_K(x) = \begin{cases} w_K(A) + \int_A^x (g_K - \mathbf{F}\cdot\nu_K)\,\mathrm{d}s, & \text{for } x \in \overline{AB}, \\ w_K(B) + \int_B^x (g_K - \mathbf{F}\cdot\nu_K)\,\mathrm{d}s, & \text{for } x \in \overline{BC}, \\ w_K(C) + \int_C^x (g_K - \mathbf{F}\cdot\nu_K)\,\mathrm{d}s, & \text{for } x \in \overline{CA}, \end{cases}$$

where $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ denote the edges of triangle and

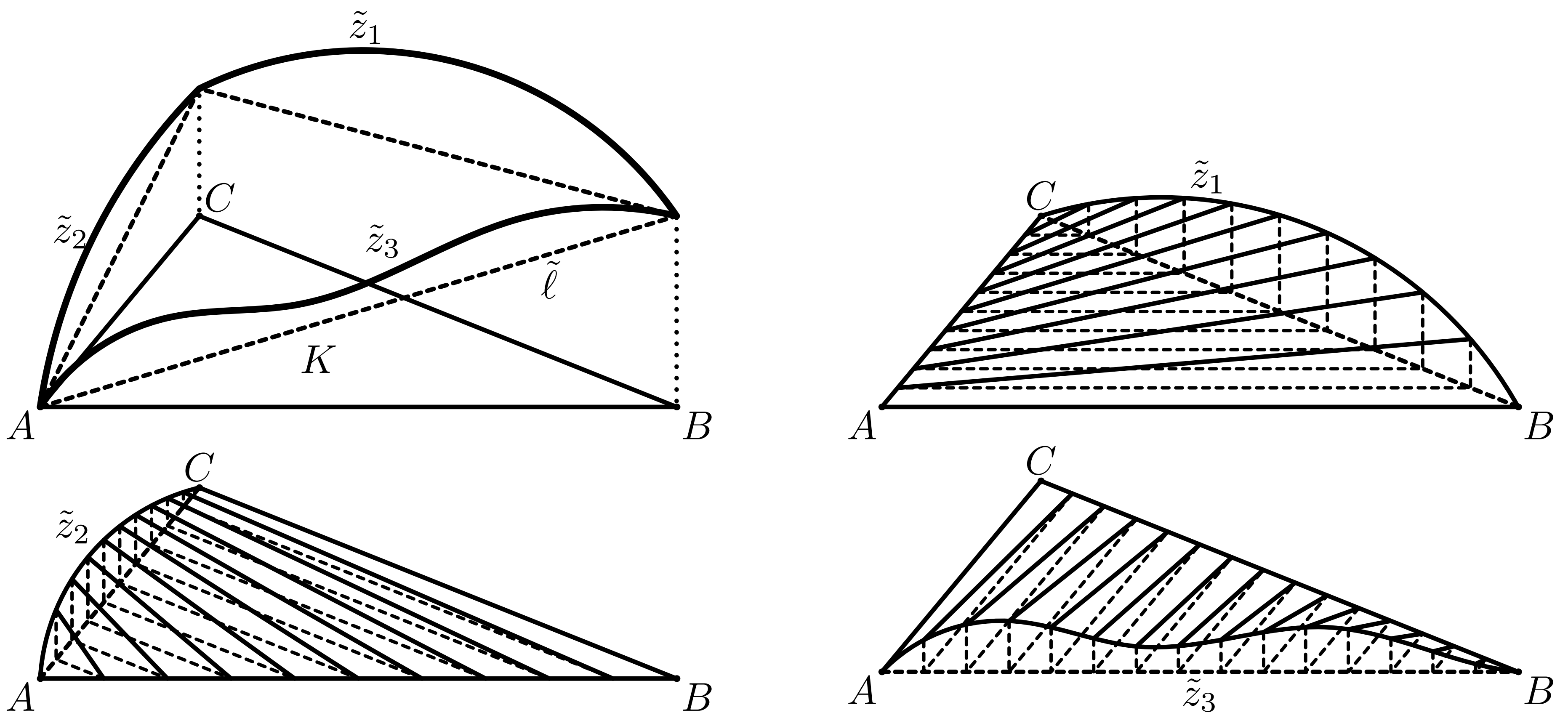
$$\begin{aligned} w_K(A) &\in \mathbb{R} \text{ is arbitrary,} \\ w_K(B) &= w_K(A) + \int_A^B (g_K - \mathbf{F}\cdot\nu_K)\,\mathrm{d}s, \\ w_K(C) &= w_K(B) + \int_B^C (g_K - \mathbf{F}\cdot\nu_K)\,\mathrm{d}s. \end{aligned}$$

The constants $w_K(B)$ and $w_K(C)$ are chosen such that w_K is continuous in points B and C.
The function w_K is continuous also in point A:

$$[w_K(A)] = \int_{\partial K} \frac{\partial w_K}{\partial \tau_K} = \int_{\partial K} (g_K - \mathbf{F}\cdot\nu_K)\,\mathrm{d}s = \int_{\partial K} g_K\,\mathrm{d}s + \int_K f\,\mathrm{d}x = 0.$$

The combined method – extension into interior of K

The function w_K is continuous on ∂K and it is possible to extend it into interior of K such that $w \in H^1(K)$. We suggest this extension:



The combined method – description of the extension

Consider triangle K with vertices A, B and C and $\omega \in C^0(\partial K)$. Let us define extension $\tilde{\omega} \in C^0(\overline{K})$ of ω into the interior of K by

$$\tilde{\omega}(X) = \tilde{\ell}(X) + \tilde{z}_1(X) + \tilde{z}_2(X) + \tilde{z}_3(X), \quad X \in \overline{K}.$$

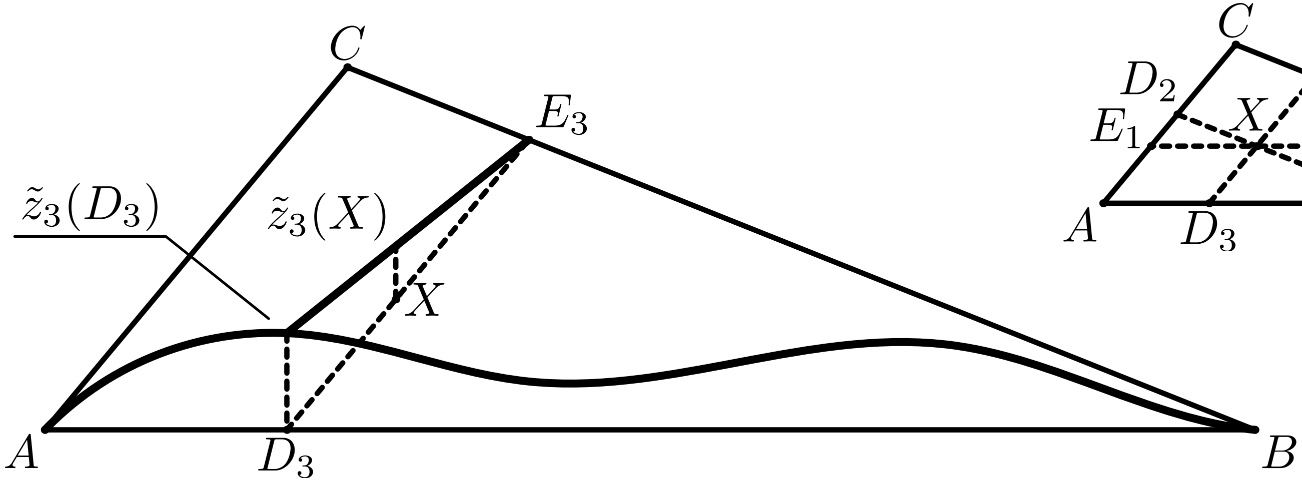
- Function $\tilde{\ell}$ is a linear function on $\overline{K}$ such that $\tilde{\ell}(A) = \omega(A)$, $\tilde{\ell}(B) = \omega(B)$, and $\tilde{\ell}(C) = \omega(C)$.

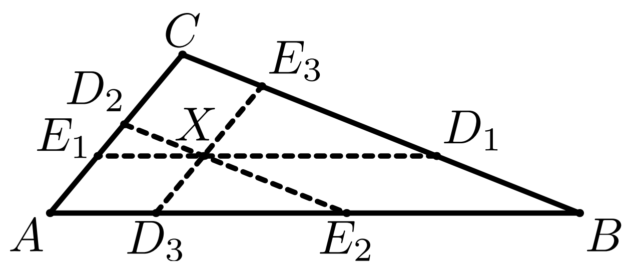
- Functions $\tilde{z}_1 \in C^0(\overline{K})$, which is zero on $\partial K \setminus \overline{BC}$, and $\tilde{z}_2 \in C^0(\overline{K})$, which is zero on $\partial K \setminus \overline{CA}$ are define in analogy with the definition of $\tilde{z}_3$.

- Function $\tilde{z}_3 \in C^0(\overline{K})$ is zero on $\partial K \setminus \overline{AB}$ and is defined by

$$\tilde{z}_3(X) = \omega(X) - \tilde{\ell}(X) \qquad \text{for } X \in \overline{AB},$$
$$\tilde{z}_3(X) = 0 \qquad \text{for } X \in \overline{BC} \cup \overline{CA},$$
$$\tilde{z}_3(X) = \Big(\omega(D_3) - \tilde{\ell}(D_3)\Big) \frac{|XE_3|}{|D_3E_3|} \qquad \text{for } X \in K,$$

where $|XE_3|$ denotes the distance between points X and E_3.





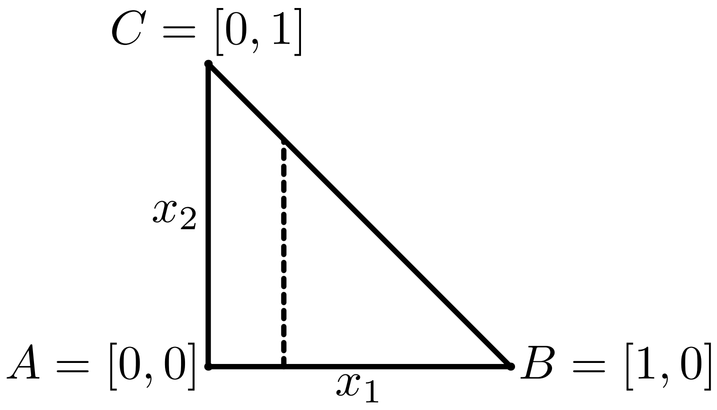
Notice that $\tilde{\omega}(X) = \omega(X)$ on ∂K.

The combined method – properties of the extension

Notation: $P^p(\Theta)$ – the space of polynomials of degree p defined on the set Θ.

Lemma 1. Consider a triangle K and $\omega \in C^0(\partial K)$. Moreover, let $\omega|_\gamma \in P^p(\gamma)$ for all edges γ of triangle K and for arbitrary $p \in \mathbb{N}$. Then the extension $\tilde{\omega}$ of function ω into interior of K described above is a polynomial of degree p in K, i.e., $\tilde{\omega} \in P^p(K)$.

The idea of proof.



The following functions form a basis of space $P_0^p([0,1])$ of all polynomials on $[0,1]$ with zeroes at 0 and 1:

$$\varphi_n^{1D}(x) = x^n(1-x), \quad n = 1, 2, \ldots, p-1.$$

Functions

$$\varphi_n^{2D}(x_1, x_2) = x_1^n(1 - x_1 - x_2), \quad n = 1, 2, \ldots, p-1,$$

are the standard finite element basis functions corresponding to the edge $\overline{AB}$ of the reference triangle.
Consider lines parallel with edge $\overline{CA}$, i.e., lines described by equality $x_1 = k$, $k \in \mathbb{R}$. All basis functions $\varphi_n^{2D}(x_1, x_2)$ are linear on these lines:

$$\varphi_n^{2D}(k, x_2) = k^n(1 - k - x_2), \quad n = 1, 2, \ldots, p-1.$$

$\square$

The combined method – properties of the extension

Lemma 2. Consider a triangle K with vertices A, B, and C, function $\omega \in C^0(\partial K)$ and its extension $\tilde{\omega} \in C^0(K)$ described above. If there exist finite tangent derivative $\partial\omega/\partial\tau_K$ on all edges of triangle K then the derivatives of function $\tilde{z}_3$ at any interior point $X = (x_1, x_2) \in K$ in the directions $\overrightarrow{ED}$ and $\overrightarrow{XB}$ are given by

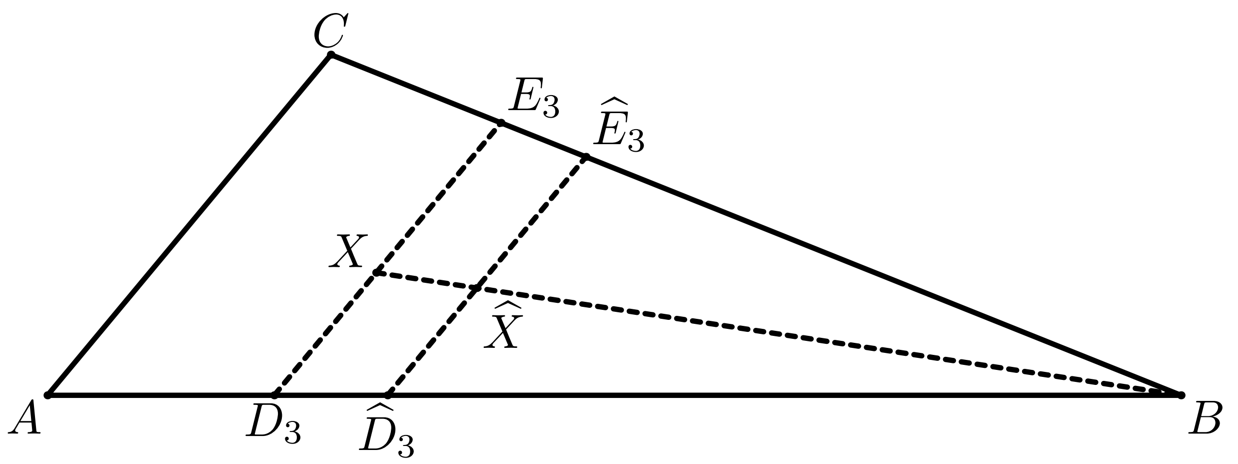
$$\frac{\partial \tilde{z}_3(X)}{\partial \overrightarrow{E_3D_3}} = \frac{\tilde{z}_3(D_3)}{|D_3E_3|},$$

$$\frac{\partial \tilde{z}_3(X)}{\partial \overrightarrow{XB}} = \frac{\partial \tilde{z}_3(D_3)}{\partial \overrightarrow{AB}} \frac{|AB|}{|XB|} \frac{|XE_3|}{|D_3E_3|}\alpha = \frac{\partial \tilde{z}_3(D_3)}{\partial \tau_K} \frac{|AB|}{|XB|} \frac{|XE_3|}{|D_3E_3|}\alpha,$$

where

$$\alpha = \frac{(B_1 - x_1)(A_1 - C_1) - (B_2 - x_2)(A_2 - C_2)}{(B_1 - A_1)(A_1 - C_1) - (B_2 - A_2)(A_2 - C_2)}.$$

Proof.



The derivative in the directions $\overrightarrow{XB}$ is given by

$$\lim_{r\to 0}\frac{\tilde{z}_3(\widehat{X})-\tilde{z}_3(X)}{r\,|BX|}=\lim_{r\to 0}\frac{\tilde{z}_3(\widehat{D}_3)-z_3(D_3)}{r\,|BX|}\frac{|XE_3|}{|D_3E_3|},$$

where $\widehat{X}=X+r(B-X)$,

$$\tilde{z}_3(X)=\tilde{z}_3(D_3)\frac{|XE_3|}{|D_3E_3|},\quad \tilde{z}_3(\widehat{X})=\tilde{z}_3(\widehat{D}_3)\frac{\left|\widehat{X}\hat{E}_3\right|}{\left|\widehat{D}_3\hat{E}_3\right|},\quad \frac{|XE_3|}{|D_3E_3|}=\frac{\left|\widehat{X}\hat{E}_3\right|}{\left|\widehat{D}_3\hat{E}_3\right|}.$$

The rest of proof is an exercise in analytical geometry. □

Remark: the derivatives of $\tilde{z}_1$ and $\tilde{z}_2$ can be evaluated analogically.

The combined method – summary

- Compute boundary fluxes g_K using residual equilibration method.
- Construct for all triangles K in T_h vector

$$\bar{\mathbf{p}}_K = \mathbf{F} + \mathbf{curl}\, w_K - \mathcal{A}\nabla \bar{u}_h,$$

  where construction of w_K employs the extension described above.

  Notice that the values of $\mathbf{curl}\, w_K$ are easily computable from values of w_K on ∂K and from $\partial w_K / \partial \tau_K = g_K - \mathbf{F} \cdot \nu$ on ∂K – see Lemma 2.

- Find solution $y_{Kh} \in W_h(K)$ of the finite dimensional local problem

$$\left(\mathcal{A}^{-1}\, \mathbf{curl}\, y_{Kh}, \mathbf{curl}\, v\right)_K = -\left(\mathcal{A}^{-1} \bar{\mathbf{p}}_K, \mathbf{curl}\, v\right)_K \quad \forall v \in W_h(K).$$

- Evaluate estimate

$$\|e\|^2 \leq \sum_{K \in T_h} \|\bar{\mathbf{p}}_K + \mathbf{curl}\, y_K\|^2_{\mathcal{A}^{-1},K}\, .$$

The combined method – exactness of the estimator

Lemma 3. Let the finite element solution $u_h \in V_h$ be exact, i.e., $u_h = u$ and let the matrix $\mathcal{A}$ be constant. If the vector $\bar{\mathbf{p}}_K \in Q_K(f, g_K, \bar{u}_h)$ is constructed as described above then the combined error estimator is exact, i.e., $\bar{\mathbf{p}}_K + \mathbf{curl}\, y_K = 0$.

Proof. From the equilibrated residual method follows that

$$g_K = \nabla \bar{u} \cdot \nu_K \text{ on } \partial K.$$

This implies that

$$\begin{aligned} Q_K(f, g_K, \bar{u}_h) &= \{\mathbf{q} \in H^1(\mathrm{div}, K) : (\mathbf{q}, \nabla v)_K = 0 \quad \forall v \in V(K)\} \\ &= \mathbf{curl}\, W(K). \end{aligned}$$

Moreover, the extension of w_K is polynomial, because f is polynomial, see Lemma 1. Therefore, $\bar{\mathbf{p}}_K$ is also polynomial and $\bar{\mathbf{p}}_K \in \mathbf{curl}\, W_h(K)$. Thus, the solution of the local problem

$$\left(\mathcal{A}^{-1}\mathbf{curl}\, y_K, \mathbf{curl}\, v\right)_K = -\left(\mathcal{A}^{-1}\bar{\mathbf{p}}_K, \mathbf{curl}\, v\right)_K \quad \forall v \in W_h(K)$$

satisfies $\mathbf{curl}\, y_k = -\bar{\mathbf{p}}_K$. □

NUMERICAL EXPERIMENTS

- Finite element method:

  V_h ... continuous and piecewise quadratic functions with zero on Γ_{D}.

- Equilibrated residual method:

  $V_h(K)$ ... degree three polynomials with zero on Γ_{D}.

- The method of hypercircle:

  W_h ... continuous and piecewise quadratic functions with zero on Γ_{N}.

- The combined methods:

  $W_h(K)$ ... degree three polynomials with zero on $\partial K \setminus \Gamma_{\mathrm{D}}$.

Remark: If we consider interior element K, then $\dim V_h(K) = 10$ and $\dim W_h(K) = 1$. Thus, the combined method performs faster.

Example 1

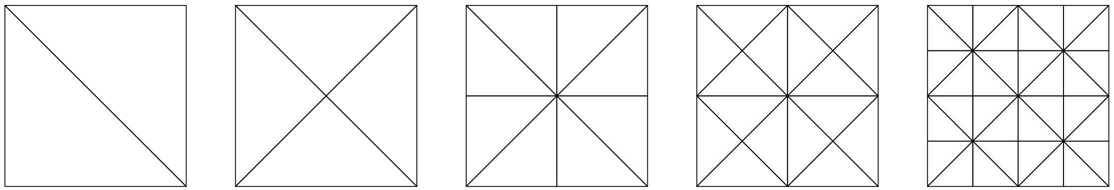
Consider the following data:

$$
\begin{aligned}
\Omega &= [-1, 1]^2, \\
\Gamma_{\mathrm{D}} &= \partial\Omega, \\
\Gamma_{\mathrm{N}} &= \emptyset, \\
\mathcal{A} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
g_1 &= 0, \\
f(x_1, x_2) &= 2(2 - x_1^2 - x_2^2), \\
u(x_1, x_2) &= (x_1^2 - 1)(x_2^2 - 1).
\end{aligned}
$$

Comparison of effectivity indices

N_{tri}	equilibrated residual method	method of hypercircle	combined method
2	1.43	1.11	1.06
4	1.23	1.25	1.01
8	1.34	1.20	1.00
16	1.30	1.30	1.16
32	1.39	1.23	1.29
64	1.32	1.32	1.27
128	1.41	1.25	1.52
256	1.33	1.34	1.33
512	1.41	1.25	1.64
1024	1.33	1.34	1.36

First five meshes:



Thank you for your attention.