Discrete Maximum Principle for Higher Order Finite Elements in 1D

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Maximum, Minimum, and Comparison Principles

- MaxP: $\mathcal{L}(u) \leq 0 \text{ in } \Omega \quad \Rightarrow \quad \max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$

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- CmpP:
$$\begin{aligned} \mathcal{L}(u_1) &= f_1 & \mathcal{L}(u_2) &= f_2 & f_1 &\leq f_2 \text{ in } \Omega, \\ u_1 &= g_1 & u_2 &= g_2 & g_1 &\leq g_2 \text{ on } \partial\Omega, \end{aligned}$$
$\Rightarrow\quad u_1 \leq u_2.$

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  $\Rightarrow \quad u_1 \leq u_2.$
- A: $\begin{array}{c} \mathcal{L}(u) = f \geq 0 \\ u = 0 \end{array}$ $\Rightarrow$ $u \geq 0$ B: $\begin{array}{c} \mathcal{L}(u) = 0 \\ u = g \geq 0 \end{array}$ $\Rightarrow$ $u \geq 0$

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- $\mathcal{L}$ linear & $\mathcal{L}(\mathrm{const}) = 0$ $\Rightarrow$

MaxP $\Leftrightarrow$ MinP $\Leftrightarrow$ CmpP $\Leftrightarrow$ (A & B)

1D Poisson Problem

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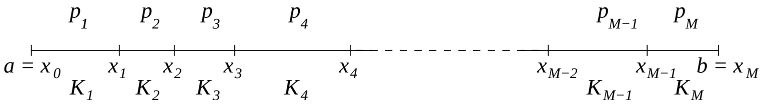

- $V_{hp} = \{v_{hp} \in H_0^1(\Omega) : v_{hp}|_{K_i} \in P^{p_i}(K_i)\}$
- $a(u, v) = \int_a^b u'v' \, \mathrm{d}x \qquad (u, v) = \int_a^b uv \, \mathrm{d}x$
- Find $u_{hp} \in V_{hp}$: $a(u_{hp}, v_{hp}) = (f, v_{hp}) \qquad$ for all $v_{hp} \in V_{hp}$

Discrete Maximum Principle (DMP)

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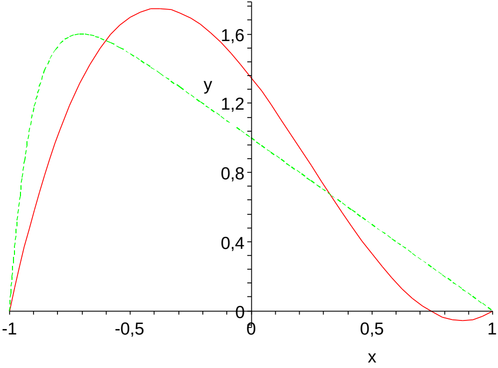
Example ($\Omega = (-1, 1)$, K_1, $p_1 = 3$, $f(x) = 200e^{-10(x+1)}$)



Discrete Green's Function (DGF)

Definition (DGF)

Let $z \in \Omega$. $G_{hp,z} \in V_{hp}$ uniquely given by

$$a(v_{hp}, G_{hp,z}) = v_{hp}(z) \quad \forall v_{hp} \in V_{hp}.$$

Notation: $G_{hp}(x,z) = G_{hp,z}(x)$.

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Properties

$$u_{hp}(z) = \int_\Omega G_{hp}(x,z) f(x) \, \mathrm{d}x \quad \forall z \in \Omega$$

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Properties

$$u_{hp}(z) = a(u_{hp}, G_{hp,z}) = (f, G_{hp,z}) = \int_\Omega G_{hp}(x,z) f(x)\,\mathrm{d}x$$

DGF – Properties

Lemma (1)

Let $\{\varphi_1, \varphi_2, \dots, \varphi_N\}$ *be a basis in* V_{hp}. *If* $A_{ij} = a(\varphi_j, \varphi_i)$ *then*

$$G_{hp}(x,z) = \sum_{j=1}^{N} \sum_{k=1}^{N} A^{-1}_{jk} \varphi_k(x) \varphi_j(z), \quad \textit{where} \sum_{j=1}^{N} A_{ij} A^{-1}_{jk} = \delta_{ik}.$$

Proof.

$$a(v_{hp}, G_{hp,z}) = v_{hp}(z)$$

$$G_{hp}(x,z) = \sum_{i=1}^{N} c_i(z) \varphi_i(x)$$

$$v_{hp} = \varphi_j$$

$$\sum_{i=1}^{N} c_i(z) \underbrace{a(\varphi_j, \varphi_i)}_{A_{ij}} = \varphi_j(z)$$

$$c_k(z) = \sum_{j=1}^{N} \varphi_j(z) A^{-1}_{jk}$$

□

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Corollary (2)

Let $\{l_1, l_2, \cdots, l_N\}$ *be a basis of* V_{hp} *such that* $a(l_i, l_j) = \delta_{ij}$. *Then*

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DGF – Properties

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Let $\{l_1, l_2, \cdots, l_N\}$ *be a basis of* V_{hp} *such that* $a(l_i, l_j) = \delta_{ij}$*. Then*

$$G_{hp}(x, z) = \sum_{i=1}^{N} l_i(x) l_i(z).$$

Lemma (3)

If there exists a basis $\{l_1, l_2, \cdots, l_N\}$ *of* V_{hp} *such that* $a(l_i, l_j) = \delta_{ij}$ *then* $G_{hp}(x, x) > 0$ *for all* $x \in \Omega$*.*

Proof.

Let $x \in \Omega$. $\exists k \in \{1, 2, \ldots, N\} : l_k(x) \neq 0$

$$G_{hp}(x, x) = \sum_{i=1}^{N} l_i^2(x) > 0.$$

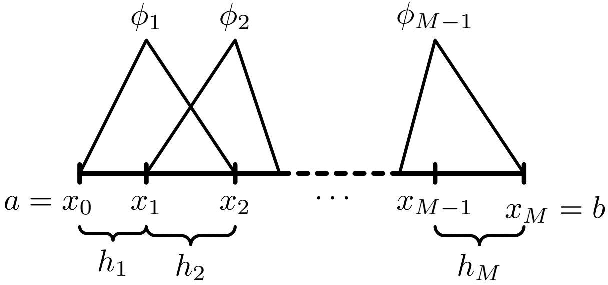
DGF – Summary

- $u_{hp}(z) = \int_\Omega G_{hp}(x,z) f(x)\,\mathrm{d}x$
- $G_{hp}(x,z) = \sum_{j=1}^{N} \sum_{k=1}^{N} A_{jk}^{-1} \varphi_k(x) \varphi_j(z)$

Theorem

The problem satisfies the DMP if and only if $G_{hp}(x,z) \geq 0$ *in* Ω^2.

DGF – Piecewise Linear Case

- $p_1 = p_2 = \ldots = p_M = 1$
- $\mathcal{B}^L = \{\phi_1, \phi_2, \ldots, \phi_{M-1}\}$ ... "hat functions"
- 

DGF – Piecewise Linear Case

$$A^L = \begin{pmatrix} \frac{1}{h_1} + \frac{1}{h_2} & -\frac{1}{h_2} & 0 & 0 & \dots \\ -\frac{1}{h_2} & \frac{1}{h_2} + \frac{1}{h_3} & -\frac{1}{h_3} & 0 & \dots \\ 0 & -\frac{1}{h_3} & \frac{1}{h_3} + \frac{1}{h_4} & -\frac{1}{h_4} & \dots \\ 0 & 0 & -\frac{1}{h_4} & \frac{1}{h_4} + \frac{1}{h_5} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$(A^L)^{-1} = \frac{1}{b-a}$$

$$\begin{pmatrix} (x_1-a)(b-x_1) & (x_1-a)(b-x_2) & (x_1-a)(b-x_3) & \dots \\ (x_1-a)(b-x_2) & (x_2-a)(b-x_2) & (x_2-a)(b-x_3) & \dots \\ (x_1-a)(b-x_3) & (x_2-a)(b-x_3) & (x_3-a)(b-x_3) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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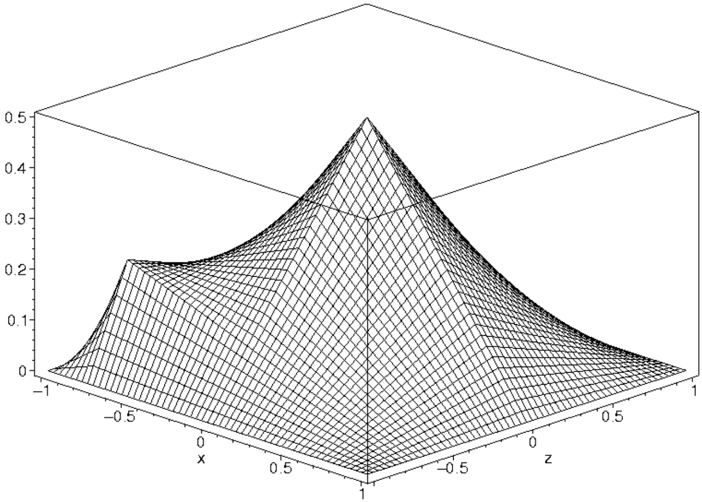
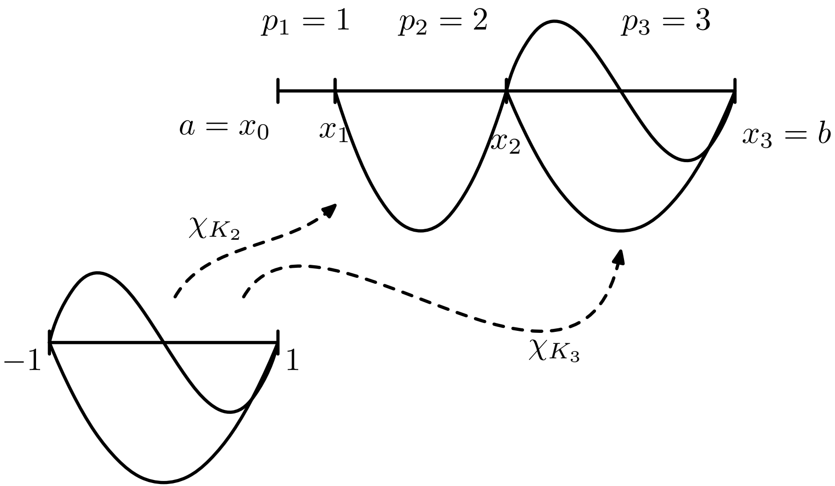
$$G^L_{hp}(x,z) = \frac{1}{b-a}\left(\sum_{i=1}^{M-1}(x_i-a)(b-x_i)\phi_i(x)\phi_i(z) \right.$$
$$\left. + \sum_{i=1}^{M-2}\sum_{j=i+1}^{M-1}(x_i-a)(b-x_j)[\phi_i(x)\phi_j(z)+\phi_j(x)\phi_i(z)]\right)$$
$$\geq 0 \quad \forall [x,z] \in \Omega^2.$$

DGF – Piecewise Linear Case



Higher-Order Case



Lobatto Shape Functions

- $\xi \in \hat{K} = [-1, 1]$
- $l_j(\xi) = \sqrt{\dfrac{2j-1}{2}} \displaystyle\int_{-1}^{\xi} P_{j-1}(x)\,\mathrm{d}x, \quad j = 2, 3, \ldots,$
- $P_j(x) = \dfrac{1}{2^j j!} \dfrac{\mathrm{d}^j}{\mathrm{d}x^j}(x^2-1)^j$ ... Legendre polynomials
- $l_{j+2}(\xi) = \dfrac{\sqrt{2j+1}\sqrt{2j+3}}{j+2} \xi l_{j+1}(\xi) - \dfrac{j-1}{j+2}\sqrt{\dfrac{2j+3}{2j-1}} l_j(\xi), \quad j = 2, 3, \ldots.$
- $\displaystyle\int_{-1}^{1} l_i'(\xi) l_j'(\xi)\,\mathrm{d}\xi = \delta_{ij}, \quad i, j = 2, 3, \ldots.$

Lobatto Shape Functions

$$\xi \in \hat{K} = [-1, 1]$$

$$l_0(\xi) = (1 - \xi)/2$$
$$l_1(\xi) = (1 + \xi)/2$$
$$l_j(\xi) = l_0(\xi) l_1(\xi) \kappa_j(\xi)$$
$$j = 2, 3, \dots$$

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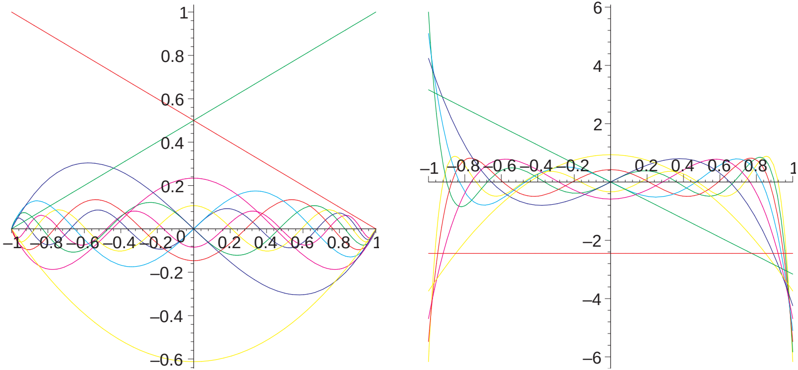
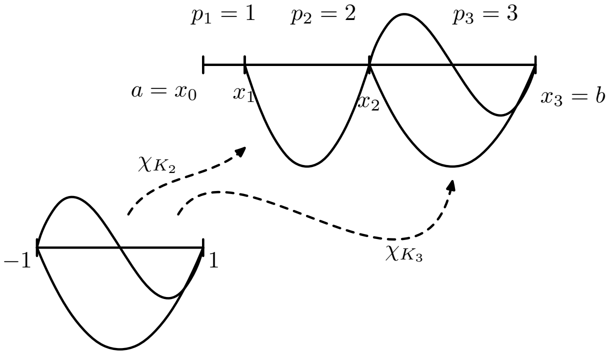
$$\kappa_2(\xi) = -\sqrt{6}$$
$$\kappa_3(\xi) = -\sqrt{10}\xi$$
$$\kappa_4(\xi) = -\frac{1}{4}\sqrt{14}(5\xi^2 - 1)$$
$$\kappa_5(\xi) = -\frac{3}{4}\sqrt{2}(7\xi^2 - 3)\xi$$
$$\kappa_6(\xi) = -\frac{1}{8}\sqrt{22}(21\xi^4 - 14\xi^2 + 1)$$
$$\kappa_7(\xi) = -\frac{1}{8}\sqrt{26}(33\xi^4 - 30\xi^2 + 5)\xi$$
$$\kappa_8(\xi) = -\frac{1}{64}\sqrt{30}(429\xi^6 - 495\xi^4 + 135\xi^2 - 5)$$
$$\kappa_9(\xi) = -\frac{1}{64}\sqrt{34}(715\xi^6 - 1001\xi^4 + 385\xi^2 - 35)\xi$$

Lobatto Shape Functions



Higher-Order Basis Functions



Map $l_2(\xi), l_3(\xi), \ldots, l_{p_i}(\xi)$ from $\hat{K}$ to K_i by

$$\chi_{K_i}(\xi) = \frac{(x_i - x_{i-1})\xi + (x_i + x_{i-1})}{2}$$

to define $\mathcal{B}^B = \{\phi_M, \phi_{M+1}, \ldots, \phi_N\}$.

Higher-Order Stiffness Matrix

Proposition

$$a(\phi^L, \phi^B) = 0 \quad \forall \phi^L \in \mathcal{B}^L, \forall \phi^B \in \mathcal{B}^B,$$
$$a(\phi^B, \psi^B) = 0 \quad \forall \phi^B \in \mathcal{B}^B, \forall \psi^B \in \mathcal{B}^B, \phi^B \neq \psi^B$$

$$A = \begin{pmatrix} A^L & 0 \\ 0 & D \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} (A^L)^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix}$$

$$D = \operatorname{diag}\Big(\underbrace{\frac{2}{h_1}, \dots, \frac{2}{h_1}}_{(p_1-1)\text{ times}}, \underbrace{\frac{2}{h_2}, \dots, \frac{2}{h_2}}_{(p_2-1)\text{ times}}, \dots, \underbrace{\frac{2}{h_M}, \dots, \frac{2}{h_M}}_{(p_M-1)\text{ times}}\Big)$$

DGF – Higher-Order Case

$$G_{hp}(x,z) = G^L_{hp}(x,z) + G^B_{hp}(x,z)$$

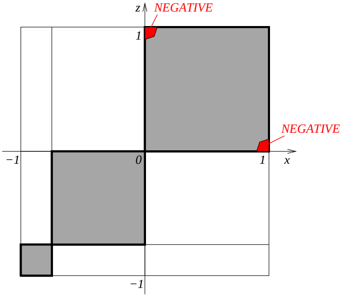
$$G^L_{hp}(x,z) = \frac{1}{b-a}\left(\sum_{i=1}^{M-1}(x_i - a)(b - x_i)\phi_i(x)\phi_i(z) \right.$$
$$\left. + \sum_{i=1}^{M-2}\sum_{j=i+1}^{M-1}(x_i - a)(b - x_j)[\phi_i(x)\phi_j(z) + \phi_j(x)\phi_i(z)]\right)$$

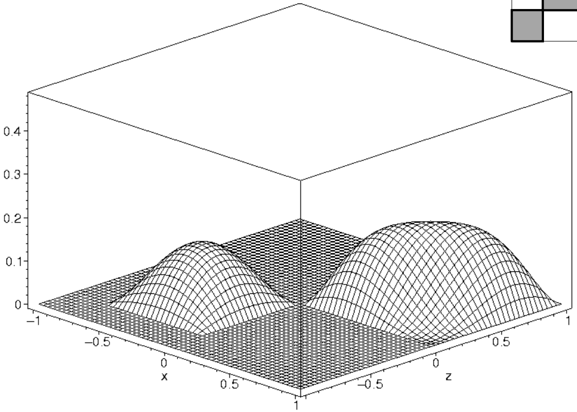
$$G^B_{hp}(x,z) = \sum_{k=M}^{N} D^{-1}_{kk}\phi_k(x)\phi_k(z), \quad \forall [x,z] \in \Omega^2$$

Lemma

$G_{hp}(x,z) \geq 0$ *in* $\Omega^2 \setminus \bigcup_{i=1}^{M} K_i^2$

DGF – Higher-Order Case





DGF on K_i^2

- $[x, z] \in K_i^2$, $1 \leq i \leq M$

$$G_{hp}(x,z)\big|_{K_i^2} = \frac{(x_{i-1}-a)(b-x_{i-1})}{b-a}\phi_{i-1}(x)\phi_{i-1}(z) + \frac{(x_i-a)(b-x_i)}{b-a}\phi_i(x)\phi_i(z) + \frac{(x_{i-1}-a)(b-x_i)}{b-a}\left[\phi_i(x)\phi_{i-1}(z) + \phi_{i-1}(x)\phi_i(z)\right] + \frac{x_i - x_{i-1}}{2} G_{hp}^B(x,z)\big|_{K_i^2}$$

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- $K_i = [x_{i-1}, x_i] = [L, R]$.

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- $K_i = [x_{i-1}, x_i] = [L, R]$.

$$G_{hp}(x,z)\big|_{K_i^2} = \frac{(L-a)(b-L)}{b-a}\phi_{i-1}(x)\phi_{i-1}(z) + \frac{(R-a)(b-R)}{b-a}\phi_i(x)\phi_i(z) + \frac{(L-a)(b-R)}{b-a}\left[\phi_i(x)\phi_{i-1}(z)+\phi_{i-1}(x)\phi_i(z)\right] + \frac{R-L}{2} G_{hp}^B(x,z)\big|_{K_i^2}$$

DGF on K_i^2

$$
\begin{aligned}
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&+ \frac{(L-a)(b-R)}{b-a}\left[\phi_i(x)\phi_{i-1}(z) + \phi_{i-1}(x)\phi_i(z)\right] + \frac{R-L}{2}G_{hp}^B(x,z)\big|_{K_i^2}
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DGF on K_i^2

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- Transform G_{hp} from K_i^2 to $\hat{K}^2 = [-1,1]^2 : x = \chi_{K_i}(\xi)$, $z = \chi_{K_i}(\eta)$.

DGF on K_i^2

$$G_{hp}(x,z)\big|_{K_i^2} = \frac{(L-a)(b-L)}{b-a}\phi_{i-1}(x)\phi_{i-1}(z) + \frac{(R-a)(b-R)}{b-a}\phi_i(x)\phi_i(z) + \frac{(L-a)(b-R)}{b-a}\left[\phi_i(x)\phi_{i-1}(z) + \phi_{i-1}(x)\phi_i(z)\right] + \frac{R-L}{2}G^B_{hp}(x,z)\big|_{K_i^2}$$

- Transform G_{hp} from K_i^2 to $\hat{K}^2 = [-1,1]^2$: $x = \chi_{K_i}(\xi)$, $z = \chi_{K_i}(\eta)$.

$$G_{hp}(x,z)\big|_{K_i^2} = \hat{G}_{hp}(\xi,\eta) = \frac{(L-a)(b-L)}{b-a}l_0(\xi)l_0(\eta) + \frac{(R-a)(b-R)}{b-a}l_1(\xi)l_1(\eta) + \frac{(L-a)(b-R)}{b-a}\left[l_1(\xi)l_0(\eta) + l_0(\xi)l_1(\eta)\right] + \frac{R-L}{2}\hat{G}^{p,B}_{hp}(\xi,\eta)$$

$$\hat{G}^{p,B}_{hp}(\xi,\eta) = \sum_{k=2}^{p} l_k(\xi)l_k(\eta) = l_0(\xi)l_0(\eta)l_1(\xi)l_1(\eta)\sum_{k=2}^{p}\kappa_k(\xi)\kappa_k(\eta)$$

DGF on K_i^2

$$G_{hp}(x,z)\big|_{K_i^2} = \hat{G}_{hp}(\xi,\eta) =$$

$$\frac{(L-a)(b-L)}{b-a} l_0(\xi) l_0(\eta) + \frac{(R-a)(b-R)}{b-a} l_1(\xi) l_1(\eta)$$

$$+ \frac{(L-a)(b-R)}{b-a} \left[l_1(\xi) l_0(\eta) + l_0(\xi) l_1(\eta) \right] + \frac{R-L}{2} \hat{G}_{hp}^{p,B}(\xi,\eta)$$

DGF on K_i^2

$$G_{hp}(x,z)\Big|_{K_i^2} = \hat{G}_{hp}(\xi,\eta) =$$
$$\frac{(L-a)(b-L)}{b-a} l_0(\xi) l_0(\eta) + \frac{(R-a)(b-R)}{b-a} l_1(\xi) l_1(\eta)$$
$$+ \frac{(L-a)(b-R)}{b-a} \left[l_1(\xi) l_0(\eta) + l_0(\xi) l_1(\eta) \right] + \frac{R-L}{2} \hat{G}_{hp}^{p,B}(\xi,\eta)$$

- Divide by $R - L > 0$.
- $\dfrac{(L-a)(b-L)}{(b-a)(R-L)} = \dfrac{(L-a)(b-R)}{(b-a)(R-L)} + \dfrac{L-a}{b-a}$
- $\dfrac{(R-a)(b-R)}{(b-a)(R-L)} = \dfrac{(L-a)(b-R)}{(b-a)(R-L)} + \dfrac{b-R}{b-a}$
- $l_0(\xi) l_0(\eta) + l_1(\xi) l_1(\eta) + l_0(\xi) l_1(\eta) + l_1(\xi) l_0(\eta) = 1 \quad \forall [\xi,\eta] \in \hat{K}^2$

DGF on K_i^2

$$G_{hp}(x,z)\big|_{K_i^2} = \hat{G}_{hp}(\xi,\eta) =$$
$$\frac{(L-a)(b-L)}{b-a} l_0(\xi) l_0(\eta) + \frac{(R-a)(b-R)}{b-a} l_1(\xi) l_1(\eta)$$
$$+ \frac{(L-a)(b-R)}{b-a} [l_1(\xi) l_0(\eta) + l_0(\xi) l_1(\eta)] + \frac{R-L}{2} \hat{G}_{hp}^{p,B}(\xi,\eta)$$

- Divide by $R - L > 0$.
- $\dfrac{(L-a)(b-L)}{(b-a)(R-L)} = \dfrac{(L-a)(b-R)}{(b-a)(R-L)} + \dfrac{L-a}{b-a}$
- $\dfrac{(R-a)(b-R)}{(b-a)(R-L)} = \dfrac{(L-a)(b-R)}{(b-a)(R-L)} + \dfrac{b-R}{b-a}$
- $l_0(\xi) l_0(\eta) + l_1(\xi) l_1(\eta) + l_0(\xi) l_1(\eta) + l_1(\xi) l_0(\eta) = 1 \quad \forall [\xi,\eta] \in \hat{K}^2$

$$\frac{\hat{G}_{hp}(\xi,\eta)}{R-L} = \frac{(L-a)(b-R)}{(b-a)(R-L)} + \frac{L-a}{b-a} l_0(\xi) l_0(\eta) + \frac{b-R}{b-a} l_1(\xi) l_1(\eta) + \frac{1}{2} \hat{G}_{hp}^{p,B}(\xi,\eta)$$

DGF on K_i^2

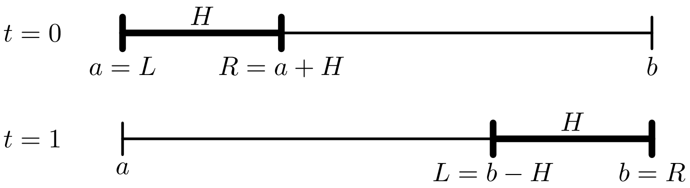
$$\frac{\hat{G}_{hp}(\xi,\eta)}{R-L} = \frac{(L-a)(b-R)}{(b-a)(R-L)} + \frac{L-a}{b-a} l_0(\xi) l_0(\eta) + \frac{b-R}{b-a} l_1(\xi) l_1(\eta) + \frac{1}{2} \hat{G}_{hp}^{p,B}(\xi,\eta)$$

DGF on K_i^2

$$\frac{\hat{G}_{hp}(\xi,\eta)}{R-L} = \frac{(L-a)(b-R)}{(b-a)(R-L)} + \frac{L-a}{b-a} l_0(\xi) l_0(\eta) + \frac{b-R}{b-a} l_1(\xi) l_1(\eta) + \frac{1}{2} \hat{G}_{hp}^{p,B}(\xi,\eta)$$

- Parametrization: $H = R - L$, $H_{\mathrm{rel}} = \dfrac{H}{b-a}$

$$L = (1-t)a + t(b-H), \quad R = (1-t)(a+H) + tb, \quad t \in [0,1]$$



DGF on K_i^2

$$\frac{\hat{G}_{hp}(\xi,\eta)}{R-L} = \frac{(L-a)(b-R)}{(b-a)(R-L)} + \frac{L-a}{b-a} l_0(\xi) l_0(\eta) + \frac{b-R}{b-a} l_1(\xi) l_1(\eta) + \frac{1}{2} \hat{G}_{hp}^{p,B}(\xi,\eta)$$

- Parametrization: $H = R - L,\ H_{\mathrm{rel}} = \dfrac{H}{b-a}$

$$L = (1-t)a + t(b-H), \quad R = (1-t)(a+H) + tb, \quad t \in [0,1]$$

- Compute:

$$\frac{L-a}{b-a} = \frac{t(b-a-H)}{b-a} = t(1-H_{\mathrm{rel}}),$$
$$\frac{b-R}{b-a} = \frac{(1-t)(b-a-H)}{b-a} = (1-t)(1-H_{\mathrm{rel}}),$$
$$\frac{(L-a)(b-R)}{(b-a)(R-L)} = \frac{t(1-t)(b-a-H)^2}{(b-a)H} = t(1-t)\frac{(1-H_{\mathrm{rel}})^2}{H_{\mathrm{rel}}}$$

DGF on K_i^2

$$\frac{\hat{G}_{hp}(\xi,\eta)}{R-L} = \frac{(L-a)(b-R)}{(b-a)(R-L)} + \frac{L-a}{b-a} l_0(\xi) l_0(\eta) + \frac{b-R}{b-a} l_1(\xi) l_1(\eta) + \frac{1}{2} \hat{G}_{hp}^{p,B}(\xi,\eta)$$

- Compute:

$$\frac{L-a}{b-a} = \frac{t(b-a-H)}{b-a} = t(1-H_{\mathrm{rel}}),$$

$$\frac{b-R}{b-a} = \frac{(1-t)(b-a-H)}{b-a} = (1-t)(1-H_{\mathrm{rel}}),$$

$$\frac{(L-a)(b-R)}{(b-a)(R-L)} = \frac{t(1-t)(b-a-H)^2}{(b-a)H} = t(1-t)\frac{(1-H_{\mathrm{rel}})^2}{H_{\mathrm{rel}}}$$

$$\frac{\hat{G}_{hp}(\xi,\eta)}{H} = t(1-t)\frac{(1-H_{\mathrm{rel}})^2}{H_{\mathrm{rel}}} + t(1-H_{\mathrm{rel}}) l_0(\xi) l_0(\eta) + (1-t)(1-H_{\mathrm{rel}}) l_1(\xi) l_1(\eta) + \frac{1}{2} \hat{G}_{hp}^{p,B}(\xi,\eta)$$

DGF on K_i^2

$$\frac{\hat{G}_{hp}(\xi,\eta)}{H} = t(1-t)\frac{(1-H_{\mathrm{rel}})^2}{H_{\mathrm{rel}}} + t(1-H_{\mathrm{rel}})l_0(\xi)l_0(\eta) + (1-t)(1-H_{\mathrm{rel}})l_1(\xi)l_1(\eta) + \frac{1}{2}\hat{G}_{hp}^{p,B}(\xi,\eta)$$

DGF on K_i^2

$$\frac{\hat{G}_{hp}(\xi,\eta)}{H} = t(1-t)\frac{(1-H_{\mathrm{rel}})^2}{H_{\mathrm{rel}}} + t(1-H_{\mathrm{rel}})l_0(\xi)l_0(\eta) + (1-t)(1-H_{\mathrm{rel}})l_1(\xi)l_1(\eta) + \frac{1}{2}\hat{G}_{hp}^{p,B}(\xi,\eta)$$

- Use

$$\hat{G}_{hp}^{p,B}(\xi,\eta) = t\,\hat{G}_{hp}^{p,B}(\xi,\eta) + (1-t)\,\hat{G}_{hp}^{p,B}(\xi,\eta)$$

$$\hat{G}_{hp}^{p,B}(\xi,\eta) = l_0(\xi)l_0(\eta)l_1(\xi)l_1(\eta)\sum_{k=2}^{p}\kappa_k(\xi)\kappa_k(\eta)$$

DGF on K_i^2

$$\frac{\hat{G}_{hp}(\xi,\eta)}{H} = t(1-t)\frac{(1-H_{\mathrm{rel}})^2}{H_{\mathrm{rel}}} + t(1-H_{\mathrm{rel}})l_0(\xi)l_0(\eta) + (1-t)(1-H_{\mathrm{rel}})l_1(\xi)l_1(\eta) + \frac{1}{2}\hat{G}_{hp}^{p,B}(\xi,\eta)$$

- Use
$$\hat{G}_{hp}^{p,B}(\xi,\eta) = t\,\hat{G}_{hp}^{p,B}(\xi,\eta) + (1-t)\,\hat{G}_{hp}^{p,B}(\xi,\eta)$$
$$\hat{G}_{hp}^{p,B}(\xi,\eta) = l_0(\xi)l_0(\eta)l_1(\xi)l_1(\eta)\sum_{k=2}^{p}\kappa_k(\xi)\kappa_k(\eta)$$

$$\frac{\hat{G}_{hp}(\xi,\eta)}{H} = t(1-t)\frac{(1-H_{\mathrm{rel}})^2}{H_{\mathrm{rel}}} + t l_0(\xi)l_0(\eta)\left[1-H_{\mathrm{rel}} + \frac{1}{2}l_1(\xi)l_1(\eta)\sum_{k=2}^{p}\kappa_k(\xi)\kappa_k(\eta)\right] + (1-t)l_1(\xi)l_1(\eta)\left[1-H_{\mathrm{rel}} + \frac{1}{2}l_0(\xi)l_0(\eta)\sum_{k=2}^{p}\kappa_k(\xi)\kappa_k(\eta)\right]$$

DGF on K_i^2

$$\frac{\hat{G}_{hp}(\xi,\eta)}{H} = t(1-t)\frac{(1-H_{\mathrm{rel}})^2}{H_{\mathrm{rel}}}$$
$$+ t l_0(\xi) l_0(\eta)\left[1 - H_{\mathrm{rel}} + \frac{1}{2} l_1(\xi) l_1(\eta) \sum_{k=2}^{p} \kappa_k(\xi)\kappa_k(\eta)\right]$$
$$+ (1-t) l_1(\xi) l_1(\eta)\left[1 - H_{\mathrm{rel}} + \frac{1}{2} l_0(\xi) l_0(\eta) \sum_{k=2}^{p} \kappa_k(\xi)\kappa_k(\eta)\right]$$

Lemma

$$\min_{[\xi,\eta]\in\hat{K}^2} l_0(\xi) l_0(\eta) \sum_{k=2}^{p} \kappa_k(\xi)\kappa_k(\eta) = \min_{[\xi,\eta]\in\hat{K}^2} l_1(\xi) l_1(\eta) \sum_{k=2}^{p} \kappa_k(\xi)\kappa_k(\eta)$$

DGF on K_i^2

Lemma

$$\min_{[\xi,\eta]\in\hat{K}^2} l_0(\xi)l_0(\eta)\sum_{k=2}^{p}\kappa_k(\xi)\kappa_k(\eta) = \min_{[\xi,\eta]\in\hat{K}^2} l_1(\xi)l_1(\eta)\sum_{k=2}^{p}\kappa_k(\xi)\kappa_k(\eta)$$

Proof.

$$\left.\begin{array}{ll}\kappa_k(\xi)=\kappa_k(-\xi) & \text{for } k \text{ even}\\ \kappa_k(\xi)=-\kappa_k(-\xi) & \text{for } k \text{ odd}\end{array}\right\} \Rightarrow \quad \kappa_k(\xi)\kappa_k(\eta)=\kappa_k(-\xi)\kappa_k(-\eta)$$

Moreover, $l_0(\xi)=l_1(-\xi)$:

$$\min_{[\xi,\eta]\in\hat{K}^2} l_0(\xi)l_0(\eta)\sum_{k=2}^{p}\kappa_k(\xi)\kappa_k(\eta) = \min_{[\xi,\eta]\in\hat{K}^2} l_1(-\xi)l_1(-\eta)\sum_{k=2}^{p}\kappa_k(-\xi)\kappa_k(-\eta)$$

$$= \min_{[\xi,\eta]\in\hat{K}^2} l_1(\xi)l_1(\eta)\sum_{k=2}^{p}\kappa_k(\xi)\kappa_k(\eta)$$

$\square$

Relative Critical Element Length

$$\frac{\hat{G}_{hp}(\xi,\eta)}{H} = t(1-t)\frac{(1-H_{\mathrm{rel}})^2}{H_{\mathrm{rel}}} + t l_0(\xi) l_0(\eta)\left[1 - H_{\mathrm{rel}} + \frac{1}{2} l_1(\xi) l_1(\eta) \sum_{k=2}^{p} \kappa_k(\xi)\kappa_k(\eta)\right] + (1-t) l_1(\xi) l_1(\eta)\left[1 - H_{\mathrm{rel}} + \frac{1}{2} l_0(\xi) l_0(\eta) \sum_{k=2}^{p} \kappa_k(\xi)\kappa_k(\eta)\right]$$

Definition

$$H^*_{\mathrm{rel}}(1) = 1$$

$$H^*_{\mathrm{rel}}(p) = 1 + \frac{1}{2} \min_{(\xi,\eta)\in\hat{K}^2} l_0(\xi) l_0(\eta) \sum_{k=2}^{p} \kappa_k(\xi)\kappa_k(\eta) = 1 + \frac{1}{2} \min_{(\xi,\eta)\in\hat{K}^2} l_1(\xi) l_1(\eta) \sum_{k=2}^{p} \kappa_k(\xi)\kappa_k(\eta) \quad \text{for } p \geq 2$$

Subcritical Elments $\Rightarrow G_{hp} \geq 0$

Theorem

If $a \leq L < R \leq b$ *and* $\dfrac{R-L}{b-a} \leq H^*_{\mathrm{rel}}(p)$,
then $\hat{G}_{hp}(\xi,\eta) \geq 0$ *for all* $[\xi,\eta] \in \hat{K}^2 = [-1,1]^2$.

Proof.

$$\frac{\hat{G}_{hp}(\xi,\eta)}{H} = t(1-t)\frac{(1-H_{\mathrm{rel}})^2}{H_{\mathrm{rel}}} + t l_0(\xi) l_0(\eta)\left[1 - H_{\mathrm{rel}} + \frac{1}{2} l_1(\xi) l_1(\eta) \sum_{k=2}^{p} \kappa_k(\xi)\kappa_k(\eta)\right] + (1-t) l_1(\xi) l_1(\eta) \underbrace{\left[1 - H_{\mathrm{rel}} + \frac{1}{2} l_0(\xi) l_0(\eta) \sum_{k=2}^{p} \kappa_k(\xi)\kappa_k(\eta)\right]}_{\geq 1 - H^*_{\mathrm{rel}}(p) + \frac{1}{2} l_0(\xi) l_0(\eta) \sum_{k=2}^{p} \kappa_k(\xi)\kappa_k(\eta) \geq 0}$$

$\square$

Main Result

Theorem

If the partition $a = x_0 < x_1 < \ldots < x_M = b$ *of the domain* $\Omega = (a, b)$ *satisfies the condition*

$$\frac{x_i - x_{i-1}}{b - a} \leq H^*_{\mathrm{rel}}(p_i) \quad \text{for all } i = 1, 2, \ldots, M,$$

where $p_i \geq 1$ *is the polynomial degree assigned to the element* $K_i = [x_{i-1}, x_i]$, *then the problem satisfies the discrete maximum principle (i.e.,* $u_{hp} \geq 0$ *in* Ω *for arbitrary* $f \in L^2(\Omega)$ *which is nonnegative a.e. in* Ω*).*

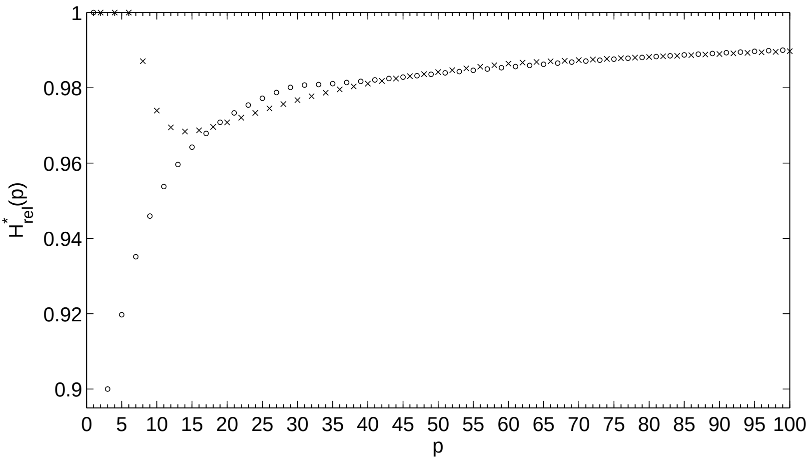
Computation of $H^*_{\mathrm{rel}}(p)$

p	$H^*_{\mathrm{rel}}(p)$
1	1
2	1
3	9/10
4	1
5	0.919731
6	1
7	0.935127
8	0.987060
9	0.945933
10	0.973952

p	$H^*_{\mathrm{rel}}(p)$
11	0.953759
12	0.969485
13	0.959646
14	0.968378
15	0.964221
16	0.968695
17	0.967874
18	0.969629
19	0.970855
20	0.970814

$$H^*_{\mathrm{rel}}(p) = 1 + \frac{1}{2} \min_{(\xi,\eta)\in\hat{K}^2} l_0(\xi) l_0(\eta) \sum_{k=2}^{p} \kappa_k(\xi)\kappa_k(\eta)$$

Computation of $H^*_{\mathrm{rel}}(p)$



$$H^*_{\mathrm{rel}}(p) = 1 + \frac{1}{2} \min_{(\xi,\eta)\in\hat{K}^2} l_0(\xi) l_0(\eta) \sum_{k=2}^{p} \kappa_k(\xi)\kappa_k(\eta)$$

Thak you for your attention

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