Higher Order Approximations and Discrete Maximum Principles

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Outline

- Model problem
- *hp*-FEM
- (Discrete) Maximum Principle (DMP)
- Classical results about DMP
- New result about DMP for higher-order approximations

Model Problem

$$
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

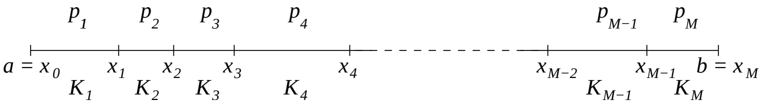
Model Problem

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

Weak formulation: $V = H_0^1(\Omega)$

$$u \in V: \quad \underbrace{a(u,v)}_{\int_\Omega \nabla u \cdot \nabla v} = \underbrace{F(v)}_{\int_\Omega fv} \quad \forall v \in V.$$

hp-FEM:



$$V_{hp} = \{v_{hp} \in V : v_{hp}|_{K_i} \in P^{p_i}(K_i)\}$$

$$u_{hp} \in V_{hp} : \quad a(u_{hp}, v_{hp}) = F(v_{hp}) \quad \forall v_{hp} \in V_{hp}$$

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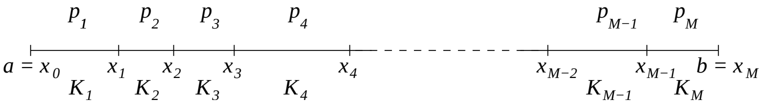

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$\varphi_1, \varphi_2, \ldots, \varphi_N$ – basis in V_{hp}

$$u_{hp}(x) = \sum_{j=1}^{N} c_j \varphi_j(x) \quad \Rightarrow \quad Ac = b, \quad \text{where} \quad \begin{array}{l} A_{ij} = a(\varphi_j, \varphi_i) \\ b_i = F(\varphi_i) \end{array}$$

Maximum Principle

$$
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

Maximum Principle:

$$f \geq 0 \quad \Rightarrow \quad u \geq 0$$

Discrete Maximum principle (DMP):

$$f \geq 0 \quad \Rightarrow \quad u_{hp} \geq 0$$

Example

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad \Omega = (-1,1)^2,$$
$$f(x,y) = \exp(-50(x^2 + y^2))$$

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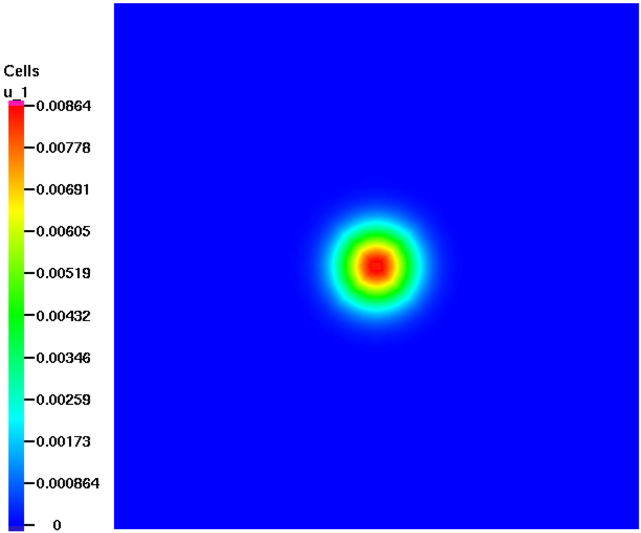

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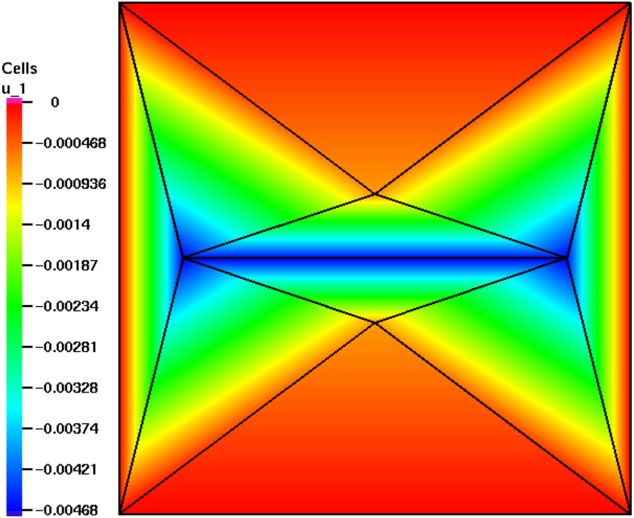

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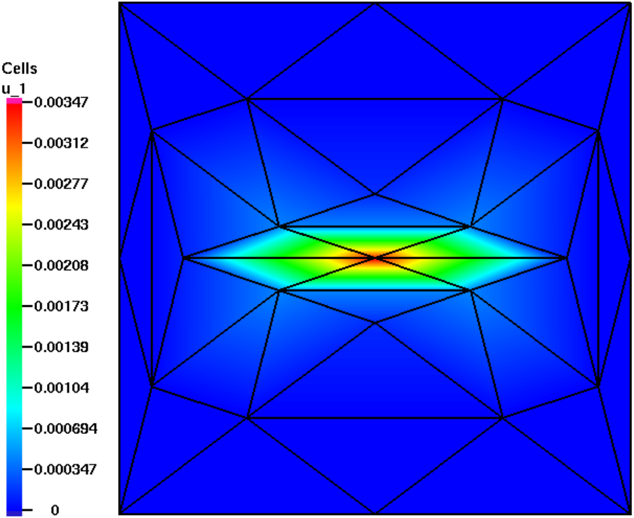

DMP – classical result

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$\Omega \subset \mathbb{R}^2$, piecewise linears on triangles

Theorem (Ciarlet ≈1970)

All angles in triangulation are non-obtuse $\Rightarrow$ *DMP*

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Theorem (Ciarlet ≈1970)

All angles in triangulation are non-obtuse ⇒ DMP

Proof.

Based on the fact that $A^{-1} \geq 0$.

$$A_{ij} = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \,\mathrm{d}x \qquad F_j = \int_\Omega f\varphi_j \,\mathrm{d}x$$

$$u_{hp}(x) = \sum_{i=1}^{N} c_i\varphi_i(x), \quad Ac = F, \quad c = \underbrace{A^{-1}}_{\geq 0}\underbrace{F}_{\geq 0} \geq 0$$

M-matrices

$$A = \begin{pmatrix} + & - & - & - \\ - & + & - & - \\ - & - & + & - \\ - & - & - & + \end{pmatrix} \quad \text{and} \quad A \text{ is s.p.d.} \quad \Rightarrow \quad A^{-1} \geq 0$$

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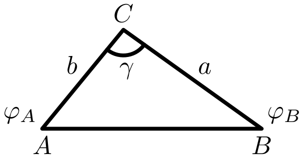
- $A_{ij} = \sum_k \int_{K_k} \nabla \varphi_i \cdot \nabla \varphi_j \, \mathrm{d}x$

$$\nabla \varphi_A \cdot \nabla \varphi_B = -\frac{ab}{(2 \operatorname{meas}_2 K)^2} \cos \gamma$$



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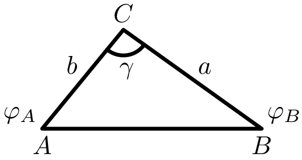
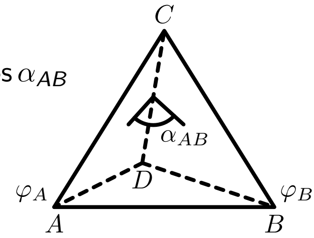


$$\nabla \varphi_A \cdot \nabla \varphi_B = -\frac{ab}{(2\,\mathrm{meas}_2\, K)^2} \cos \gamma$$

$$\nabla \varphi_A \cdot \nabla \varphi_B = -\frac{\mathrm{meas}_2(BCD)\,\mathrm{meas}_2(ADC)}{(3\,\mathrm{meas}_3\, K)^2} \cos \alpha_{AB}$$



Higher order approximation

Theorem (Ciarlet ≈1970)

All angles in triangulation are non-obtuse $\Rightarrow$ *DMP*

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$$A_{ij} = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \,\mathrm{d}x \qquad F_j = \int_\Omega f\varphi_j \,\mathrm{d}x$$

$$u_{hp}(x) = \sum_{i=1}^{N} c_i\varphi_i(x), \quad Ac = F, \quad c = \underbrace{A^{-1}}_{\geq 0}\underbrace{F}_{\geq 0} \geq 0$$

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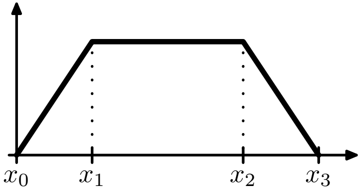
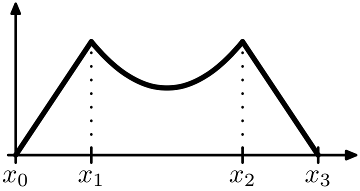
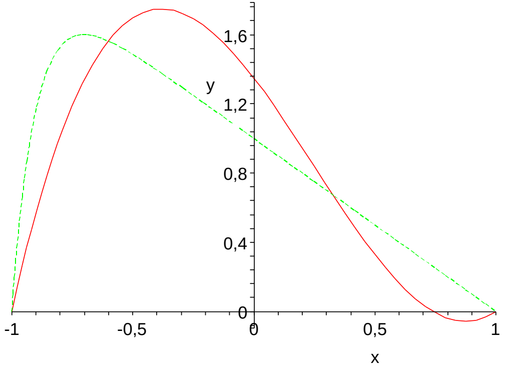




Counter Example

$$-u'' = f \text{ in } (-1,1), \quad u(\pm 1) = 0,$$

$$K_1 = (-1,1), \quad p_1 = 3, \quad f(x) = 200e^{-10(x+1)}$$



Green's function

- Find $u \in V : a(u,v) = F(v) \quad \forall v \in V$.
- Green's function

$$G_y \in V \;:\; a(w, G_y) = \underbrace{\delta_y(w)}_{w(y)} \quad \forall w \in V,\; y \in \Omega$$

- $$u(y) = a(u, G_y) = F(G_y)$$

Discrete Green's function (DGF)

- Find $u_{hp} \in V_{hp} \,:\, a(u_{hp}, v_{hp}) = F(v_{hp}) \quad \forall v_{hp} \in V_{hp}$
- Discrete Green's function

$$G_{hp,y} \in V_{hp} \;:\; a(w_{hp}, G_{hp,y}) = \underbrace{\delta_y(w_{hp})}_{w_{hp}(y)} \quad \forall w_{hp} \in V_{hp},\; y \in \Omega$$

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- $$u_{hp}(y) = a(u_{hp}, G_{hp,y}) = F(G_{hp,y})$$
- $G_{hp}(x, y) = G_{hp,y}(x)$
- If $a(\cdot, \cdot)$ is symmetric $\Rightarrow G_{hp}(x, y) = G_{hp}(y, x)$.
- $G_{hp}(x, y) = \displaystyle\sum_{i=1}^{N} \sum_{j=1}^{N} A^{-1}_{ij} \varphi_i(x) \varphi_j(y)$
- If A s.p.d. $\Rightarrow G_{hp}(x, x) > 0 \quad \forall x \in \Omega$.

Properties of DGF

Lemma

Let $\{\varphi_1, \varphi_2, \dots, \varphi_N\}$ *be a basis in* V_{hp}. *If* $A_{ij} = a(\varphi_j, \varphi_i)$ *then*

$$G_{hp}(x,y) = \sum_{j=1}^{N} \sum_{k=1}^{N} A_{jk}^{-1} \varphi_k(x) \varphi_j(y), \quad \textit{where} \sum_{j=1}^{N} A_{ij} A_{jk}^{-1} = \delta_{ik}.$$

Proof.

$$a(v_{hp}, G_{hp,y}) = v_{hp}(y)$$

$$G_{hp}(x,y) = \sum_{i=1}^{N} c_i(y) \varphi_i(x)$$

$$v_{hp} = \varphi_j$$

$$\sum_{i=1}^{N} c_i(y) \underbrace{a(\varphi_j, \varphi_i)}_{A_{ij}} = \varphi_j(y)$$

$$c_k(y) = \sum_{j=1}^{N} \varphi_j(y) A_{jk}^{-1}$$

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Corollary

If $a(\cdot,\cdot)$ *is symmetric then* $G_{hp}(x,y) = G_{hp}(y,x)$.

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Corollary

If $a(\cdot,\cdot)$ is symmetric then $G_{hp}(x,y) = G_{hp}(y,x)$.

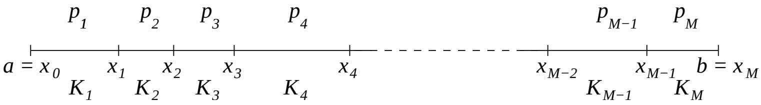
Corollary

Let $\{l_1, l_2, \cdots, l_N\}$ be a basis of V_{hp} such that $a(l_i, l_j) = \delta_{ij}$. Then

$$G_{hp}(x,y) = \sum_{i=1}^{N} l_i(x) l_i(y).$$

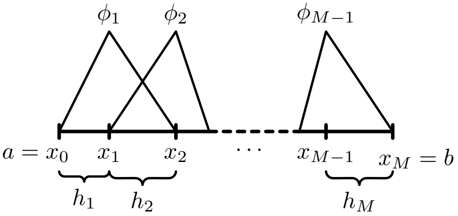
DGF and DMP

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$



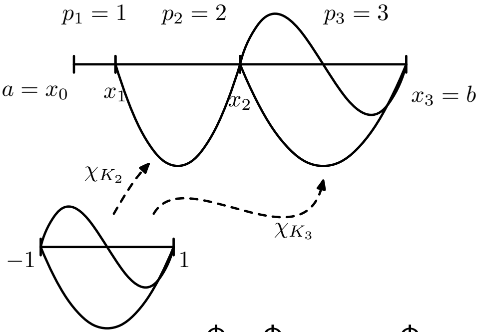
- $V_{hp} = \{v_{hp} \in H^1_0(\Omega) : v_{hp}|_{K_i} \in P^{p_i}(K_i)\}$
- $u_{hp} \in V_{hp} : \underbrace{\int_\Omega \nabla u_{hp} \cdot \nabla v_{hp} \,\mathrm{d}x}_{a(u_{hp}, v_{hp})} = \underbrace{\int_\Omega f v_{hp} \,\mathrm{d}x}_{F(v_{hp})} \quad \forall v_{hp} \in V_{hp}$
- $u_{hp}(y) = \int_\Omega G_{hp}(x, y) f(x) \,\mathrm{d}x$
- DMP $\Leftrightarrow G_{hp}(x, y) \geq 0 \quad \forall (x, y) \in \Omega^2.$

hp-FEM basis in 1D



$$-u'' = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$
$$\underbrace{\int_a^b u'v' \,\mathrm{d}x}_{a(u,v)} = \underbrace{\int_a^b fv \,\mathrm{d}x}_{(f,v)}$$

$$l_0(\xi) = (1-\xi)/2$$
$$l_1(\xi) = (1+\xi)/2$$
$$l_j(\xi) = \sqrt{\frac{2j-1}{2}} \int_{-1}^{\xi} P_{j-1}(x) \,\mathrm{d}x$$
$$\int_{-1}^{1} l_i'(\xi) l_j'(\xi) \,\mathrm{d}\xi = \delta_{ij}$$
$$l_j(\xi) = l_0(\xi) l_1(\xi) \kappa_j(\xi)$$

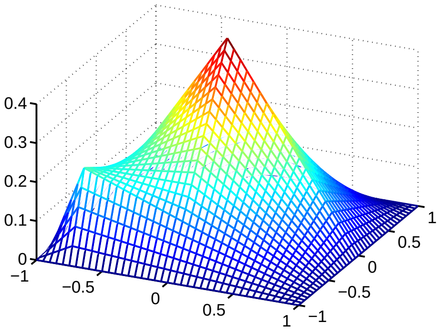


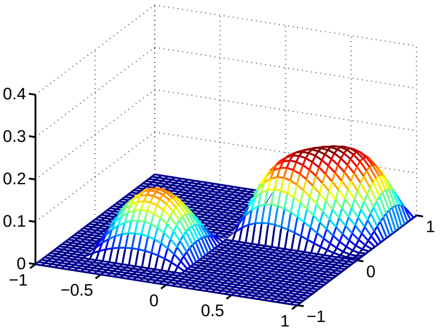
$$\Phi_M, \Phi_{M+1}, \ldots, \Phi_N$$

Explicit expression of DGF

$$A = \begin{pmatrix} A^L & 0 \\ 0 & D \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} (A^L)^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix}$$

$$G_{hp}(x,y) = \sum_{j=1}^{N} \sum_{k=1}^{N} A^{-1}_{jk} \Phi_k(x) \Phi_j(y) = \underbrace{G^L_{hp}(x,y)}_{\geq 0} + \underbrace{G^B_{hp}(x,y)}_{\not\geq 0}$$





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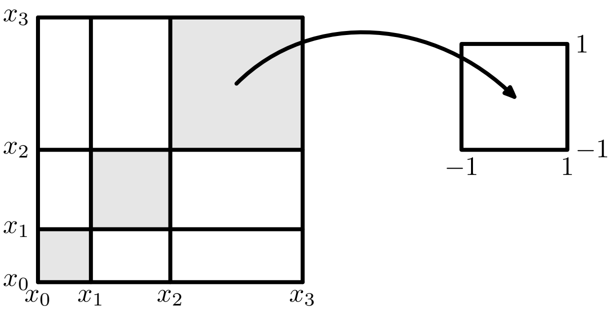

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$$K_i = [x_{i-1}, x_i] \qquad G_{hp}(x,y)|_{K_i\times K_i} \mapsto \hat{G}_{hp}(\xi,\eta)$$

$$H = x_i - x_{i-1} \quad H_{\mathrm{rel}} = H/(b-a) \quad t \in [0,1] \quad l_k(\xi) = l_0(\xi)l_1(\xi)\kappa_k(\xi)$$

$$\begin{aligned}\frac{\hat{G}_{hp}(\xi,\eta)}{H} = {} & t(1-t)\frac{(1-H_{\mathrm{rel}})^2}{H_{\mathrm{rel}}} \\ & + t l_0(\xi) l_0(\eta)\left[1 + \frac{1}{2} l_1(\xi) l_1(\eta)\sum_{k=2}^{p}\kappa_k(\xi)\kappa_k(\eta) - H_{\mathrm{rel}}\right] \\ & + (1-t) l_1(\xi) l_1(\eta)\left[1 + \frac{1}{2} l_0(\xi) l_0(\eta)\sum_{k=2}^{p}\kappa_k(\xi)\kappa_k(\eta) - H_{\mathrm{rel}}\right]\end{aligned}$$

Relative Critical Element Length

$$\frac{\hat{G}_{hp}(\xi,\eta)}{H} = t(1-t)\frac{(1-H_{\mathrm{rel}})^2}{H_{\mathrm{rel}}}$$
$$+t l_0(\xi) l_0(\eta)\left[1+\frac{1}{2}l_1(\xi)l_1(\eta)\sum_{k=2}^{p}\kappa_k(\xi)\kappa_k(\eta)-H_{\mathrm{rel}}\right]$$
$$+(1-t)l_1(\xi)l_1(\eta)\left[1+\frac{1}{2}l_0(\xi)l_0(\eta)\sum_{k=2}^{p}\kappa_k(\xi)\kappa_k(\eta)-H_{\mathrm{rel}}\right]$$

Definition

$$H^*_{\mathrm{rel}}(1) = 1$$

$$H^*_{\mathrm{rel}}(p) = 1 + \frac{1}{2}\min_{(\xi,\eta)\in\hat{K}^2} l_0(\xi)l_0(\eta)\sum_{k=2}^{p}\kappa_k(\xi)\kappa_k(\eta)$$
$$= 1 + \frac{1}{2}\min_{(\xi,\eta)\in\hat{K}^2} l_1(\xi)l_1(\eta)\sum_{k=2}^{p}\kappa_k(\xi)\kappa_k(\eta) \quad \text{for } p\geq 2$$

Main Result

Theorem

If the partition $a = x_0 < x_1 < \ldots < x_M = b$ of the domain $\Omega = (a, b)$ satisfies the condition

$$\frac{x_i - x_{i-1}}{b - a} \leq H^*_{\mathrm{rel}}(p_i) \quad \textit{for all } i = 1, 2, \ldots, M,$$

where $p_i \geq 1$ is the polynomial degree assigned to the element $K_i = [x_{i-1}, x_i]$, then the 1D Poisson problem satisfies the discrete maximum principle (i.e., $u_{hp} \geq 0$ in Ω for arbitrary $f \in L^2(\Omega)$ which is nonnegative a.e. in Ω).

Computation of $H^*_{\mathrm{rel}}(p)$

p	$H^*_{\mathrm{rel}}(p)$	p	$H^*_{\mathrm{rel}}(p)$
1	1	11	0.953759
2	1	12	0.969485
3	9/10	13	0.959646
4	1	14	0.968378
5	0.919731	15	0.964221
6	1	16	0.968695
7	0.935127	17	0.967874
8	0.987060	18	0.969629
9	0.945933	19	0.970855
10	0.973952	20	0.970814

$$H^*_{\mathrm{rel}}(p) = 1 + \frac{1}{2} \min_{(\xi,\eta)\in\hat{K}^2} l_0(\xi) l_0(\eta) \sum_{k=2}^{p} \kappa_k(\xi)\kappa_k(\eta)$$

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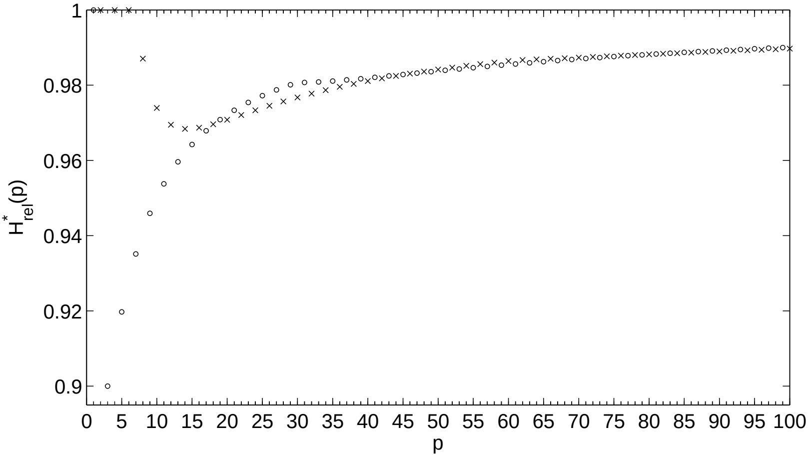

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Conclusions

DMP for 1D Poisson equaiton

- Dirichlet BC – if $H_{\mathrm{rel}} \le 9/10$.

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- mixed BC – valid on arbitrary *hp*-mesh.

Generalization

$$-(au')' = f, \quad a \text{ is piecewise constant.}$$

- Dirichlet BC – if $\widetilde{H}_{\mathrm{rel}} \leq 9/10$

$$\widetilde{H}_{\mathrm{rel}} = \frac{\tilde{h}_k}{\sum_{i=1}^{M} \tilde{h}_i} \qquad \tilde{h}_k = \frac{x_k - x_{k-1}}{a_k}$$

- mixed BC – valid on arbitrary *hp*-mesh.

Thank you for your attention

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