A Review of Discrete Maximum Principles for Higher-Order Finite Elements

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(Continuous) maximum principle

$$-\operatorname{div}(\mathcal{A}\nabla u) = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega$$

$$\text{MaxP}: \quad f \le 0 \quad \Rightarrow \quad \max_{\overline{\Omega}} u = \max_{\partial\Omega} u$$

$$\text{MinP}: \quad f \ge 0 \quad \Rightarrow \quad \min_{\overline{\Omega}} u = \min_{\partial\Omega} u$$

$$\text{ComP}: \quad f \ge 0 \ \&\ g \ge 0 \quad \Rightarrow \quad u \ge 0$$

$$\text{MaxP} \quad \Leftrightarrow \quad \text{MinP} \quad \Leftrightarrow \quad \text{ComP}$$

(Continuous) maximum principle

$$-\operatorname{div}(\mathcal{A}\nabla u) + \kappa^2 u = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega$$

$$\text{MaxP :} \qquad f \le 0 \quad \Rightarrow \quad \max_{\overline{\Omega}} u \le \max\{0, \max_{\partial\Omega} u\}$$

$$\text{MinP :} \qquad f \ge 0 \quad \Rightarrow \quad \min_{\overline{\Omega}} u \ge \min\{0, \min_{\partial\Omega} u\}$$

$$\text{ComP :} \quad f \ge 0 \ \& \ g \ge 0 \quad \Rightarrow \quad u \ge 0$$

$$\text{MaxP} \quad \Leftrightarrow \quad \text{MinP} \quad \Leftrightarrow \quad \text{ComP}$$

Discrete Maximum Principle (DMP)

$$\text{DMP}: \quad f \geq 0 \ \& \ g \geq 0 \quad \Rightarrow \quad u_{hp} \geq 0$$

Simplification: $g = 0$.

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- $u_{hp} \in V_{hp}: \quad a(u_{hp}, v_{hp}) = \mathcal{F}(v_{hp}) \quad \forall v_{hp} \in V_{hp}$

  $$a(u,v) = \int_\Omega \mathcal{A}\nabla u \cdot \nabla v + \kappa^2 uv \,\mathrm{d}x \qquad \mathcal{F}(v) = \int_\Omega fv \,\mathrm{d}x$$

- $G_{hp,y} \in V_{hp}: \quad a(w_{hp}, G_{hp,y}) = w_{hp}(y) \quad \forall w_{hp} \in V_{hp}, \;\; y \in \overline{\Omega}$

  $$G_{hp}(x,y) = G_{hp,y}(x)$$

- $u_{hp}(y) = a(u_{hp}, G_{hp,y}) = \mathcal{F}(G_{hp,y}) = \displaystyle\int_\Omega G_{hp}(x,y) f(x) \,\mathrm{d}x$

- $\text{DMP} \quad \Leftrightarrow \quad G_{hp} \geq 0 \quad \text{in } \overline{\Omega}^2$

- $G_{hp}(x,y) = \displaystyle\sum_{j=1}^{N} \sum_{k=1}^{N} A^{-1}_{jk} \varphi_k(x) \varphi_j(y), \quad (x,y) \in \overline{\Omega}^2$

Formula for DGF

Lemma

Let $\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$ be a basis in V_{hp}. If $A_{ij} = a(\varphi_j, \varphi_i)$ then

$$G_{hp}(x,y) = \sum_{j=1}^{N} \sum_{k=1}^{N} A^{-1}_{jk} \varphi_k(x) \varphi_j(y), \quad \text{where } \sum_{j=1}^{N} A_{ij} A^{-1}_{jk} = \delta_{ik}.$$

Proof.

$$a(v_{hp}, G_{hp,y}) = v_{hp}(y)$$

$$G_{hp}(x,y) = \sum_{i=1}^{N} c_i(y) \varphi_i(x)$$

$$v_{hp} = \varphi_j$$

$$\sum_{i=1}^{N} c_i(y) \underbrace{a(\varphi_j, \varphi_i)}_{A_{ij}} = \varphi_j(y)$$

$$c_k(y) = \sum_{j=1}^{N} \varphi_j(y) A^{-1}_{jk}$$

$\square$

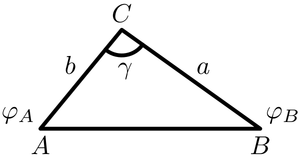
Classical result (2D Poisson, triangular FEM)

$$-\Delta u = f \quad \text{in } \Omega \subset \mathbb{R}^2, \quad u = 0 \quad \text{on } \partial\Omega.$$

u_h . . . p.w. linear FEM solution, triangular mesh

Theorem: If all angles $\gamma \leq \pi/2$ then DMP.

- $$u_h = \sum_{j=1}^{N} c_i \varphi_i, \quad Ac = F, \quad c = A^{-1}F$$
  $$A_{ij} = \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \,\mathrm{d}x = \sum_K \int_K \nabla\varphi_j \cdot \nabla\varphi_i \,\mathrm{d}x \quad F_i = \int_\Omega f\varphi_i \,\mathrm{d}x$$
- $i \neq j$:
  $$\nabla\varphi_A \cdot \nabla\varphi_B = -\frac{|a||b|}{(2\,\mathrm{meas}_2\, K)^2} \cos\gamma$$



- $$A = \begin{pmatrix} + & - & - \\ - & + & - \\ - & - & + \end{pmatrix} \quad \text{and} \quad A \text{ s.p.d.} \quad \Rightarrow \quad A^{-1} \geq 0$$
- $$G_{hp}(x,y) = \sum_{i=1}^{N} \sum_{j=1}^{N} A^{-1}_{ij} \varphi_i(x)\varphi_j(y)$$

Negative result

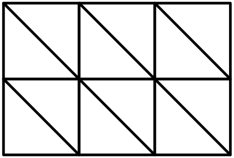
Höhn, Mittelmann: "Some remarks on the Discrete Maximum-Principle for Finite Elements of Higher Order", Computing 27, 145–154 (1981).

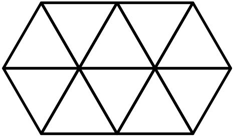
- "Strong" DMP:

$$\max_{\overline{\omega}_{x_k}} u_{hp} = \max_{\partial\omega_{x_k}} u_{hp},$$

  where ω_{x_k} ... patch of elements sharing vertex x_k.

- "Strong" DMP is not valid for $p \geq 2$ unless prohibitively strong restrictions on the mesh.



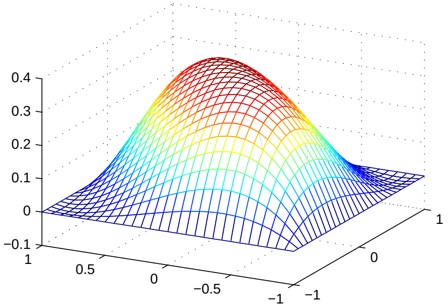


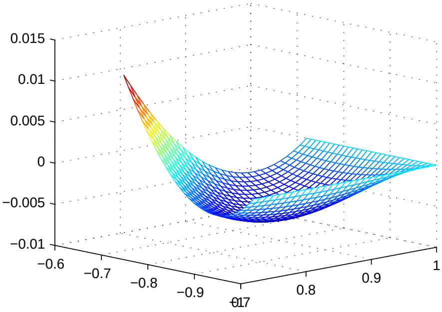
1D counter example

$$-u'' = f \text{ in } (\alpha, \beta), \quad u(\alpha) = u(\beta) = 0.$$

Piecewise linear FEM approximations satisfies DMP.

BUT: One element of degree $p = 3$.





Recent results

- $-u'' = f$ in (α, β) $\quad + \quad$ Dirichlet b.c.
- $-(au')' = f$ in (α, β) $\quad + \quad$ Dirichlet b.c.; $a(x)$ p.w. const.
- $-(au')' = f$ in (α, β) $\quad + \quad$ mixed b.c.; $a(x)$ p.w. const.

1D Poisson, Dirichlet b.c.

$$-u'' = f \text{ in } (\alpha, \beta) \quad + \quad \text{Dirichlet b.c.}$$

$$\frac{x_i - x_{i-1}}{\beta - \alpha} \le H^*_{\mathrm{rel}}(p_i) \quad \text{for } i = 1, 2, \dots, M \quad \Rightarrow \quad \text{DMP}$$

- $\alpha = x_0 < x_1 < \dots < x_M = \beta$ partition
- $p_i \ge 1$ ... polynomial degree of $K_i = [x_{i-1}, x_i]$
- $H^*_{\mathrm{rel}}(1) = 1$

  $$H^*_{\mathrm{rel}}(p) = 1 + \frac{1}{2} \min_{(\xi,\eta)\in[-1,1]^2} l_0(\xi) l_0(\eta) \sum_{k=0}^{p-2} \kappa_k(\xi)\kappa_k(\eta), \qquad p \ge 2$$

- $l_0(\xi) = (1-\xi)/2$,

  $$\kappa_k(\xi) = \frac{-2\sqrt{2(2k+3)}}{(k+1)(k+2)} P'_{k+1}(\xi),\ k = 0, 1, 2, \dots$$

  $$\int_{-1}^{1} \frac{(1-\xi^2)}{4} \kappa_k(\xi)\kappa_\ell(\xi)\,\mathrm{d}\xi = \begin{cases} 4/((k+1)(k+2)) & \text{for } k = \ell \\ 0 & \text{otherwise} \end{cases}$$

1D Poisson, Dirichlet b.c.

$$-u'' = f \text{ in } (\alpha, \beta) \quad + \quad \text{Dirichlet b.c.}$$

$$\frac{x_i - x_{i-1}}{\beta - \alpha} \leq H^*_{\mathrm{rel}}(p_i) \quad \text{for } i = 1, 2, \ldots, M \quad \Rightarrow \quad \text{DMP}$$

p	$H^*_{\mathrm{rel}}(p)$
1	1
2	1
3	9/10
4	1
5	0.919731
6	1
7	0.935127
8	0.987060
9	0.945933
10	0.973952

p	$H^*_{\mathrm{rel}}(p)$
11	0.953759
12	0.969485
13	0.959646
14	0.968378
15	0.964221
16	0.968695
17	0.967874
18	0.969629
19	0.970855
20	0.970814

1D Poisson, Dirichlet b.c.

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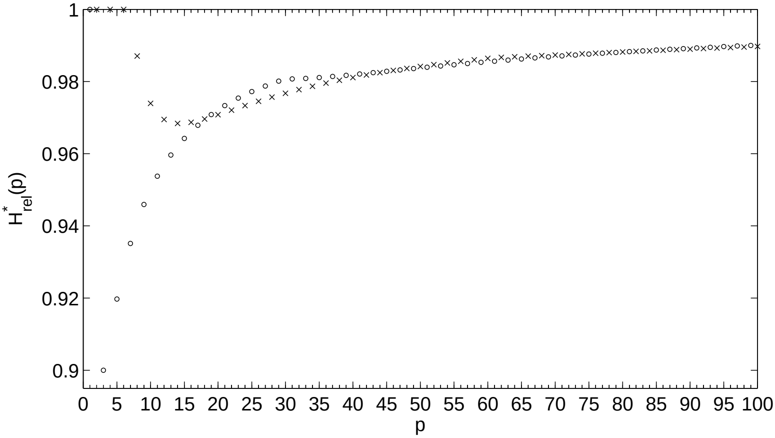

Piecewise constant coefficient

$$-(au')' = f \text{ in } (\alpha, \beta) \quad + \quad \text{Dirichlet b.c.}$$

$$a|_{K_i} = a_i = \text{const}$$

$$\frac{\widetilde{h}_i}{\sum_{k=1}^{M} \widetilde{h}_k} \le H^*_{\mathrm{rel}}(p_i) \quad \text{for } i = 1, 2, \dots, M \quad \Rightarrow \quad \text{DMP}$$

- $\alpha = x_0 < x_1 < \ldots < x_M = \beta$ partition
- $p_i \ge 1$ ...polynomial degree of $K_i = [x_{i-1}, x_i]$
- $\widetilde{h}_i = (x_i - x_{i-1})/a_i$

1D Poisson, mixed b.c.

$$-(au')' = f \text{ in } (\alpha,\beta) \quad + \quad \text{mixed (Dirichlet \& Neumann) b.c.}$$

$$a|_{K_i} = a_i = \mathrm{const}$$

$$0 \le H^*_{\mathrm{rel}}(p_i) \quad \text{for } i = 1,2,\dots,M \quad \Rightarrow \quad \text{DMP}$$

(checked for $p_i \le 100$)

Future work

- $-u'' + \kappa^2 u = f$, $\kappa = const.$
  - $p = 1: \ \kappa h_i \leq \sqrt{6} \approx 2.45 \quad \Leftrightarrow \quad \text{DMP}$
  - $p = 2: \ \kappa h_i \leq \sqrt{20/3} \approx 2.58 \quad \Rightarrow \quad \text{DMP}$
- 1D results give good hope for 2D
- 2D
- 3D
- etc.

Thank you for your attention

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