Higher Order Discrete Maximum Principle for a Problem with Absolute Term

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(Continuous) maximum principle

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

$$\text{MaxP}: \quad f \le 0 \quad \Rightarrow \quad \max_{\overline{\Omega}} u \le \max\{0, \max_{\partial\Omega} u\} = 0$$

$$\Updownarrow$$

$$\text{ComP}: \quad f \ge 0 \quad \Rightarrow \quad u \ge 0$$

$$\Updownarrow$$

$$G(x,y) \ge 0 \text{ in } \Omega^2$$

$$u(y) = \int_\Omega G(x,y) f(x)\,\mathrm{d}x$$

$$-\Delta G_y + \kappa^2 G_y = \delta_y \quad \text{in } \Omega$$
$$G_y = 0 \quad \text{on } \partial\Omega$$

$$G(x,y) = G_y(x)$$

Discretization

- Weak

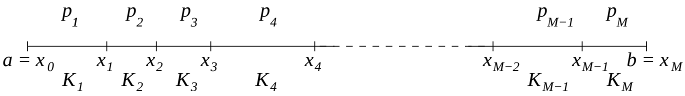
$$u \in V = H_0^1(\Omega): \qquad \underbrace{a(u,v)}_{\int_\Omega \nabla u \cdot \nabla v + \kappa^2 uv \,\mathrm{d}x} = \int_\Omega fv \,\mathrm{d}x \quad \forall v \in V$$

- *hp*-FEM

$$u_{hp} \in V_{hp} \subset V: \quad a(u_{hp}, v_{hp}) = \int_\Omega f v_{hp} \,\mathrm{d}x \quad \forall v_{hp} \in V_{hp}$$

- $V_{hp} = \{v_{hp} \in V : v_{hp}|_{K_i} \in P^{p_i}(K_i),\ K_i \in \mathcal{T}_{hp}\}$

  Triangulation $\mathcal{T}_{hp}$ of Ω



Discrete Maximum Principle (DMP)

Definition (DMP)

Characterize such triangulations $\mathcal{T}_{hp}$ that
any $f \geq 0 \quad \Rightarrow \quad u_{hp} \geq 0$ in Ω.

Theorem

$DMP \quad \Leftrightarrow \quad G_{hp} \geq 0$ *in* Ω^2

Proof.

$$G_{hp,y} \in V_{hp} \;:\; a(v_{hp}, G_{hp,y}) = \underbrace{\delta_y(v_{hp})}_{v_{hp}(y)} \quad \forall v_{hp} \in V_{hp},\; y \in \Omega$$

$$u_{hp}(y) = a(u_{hp}, G_{hp,y}) = \int_\Omega G_{hp}(x,y) f(x)\,\mathrm{d}x$$

$G_{hp}(x,y) = G_{hp,y}(x)$ $\square$

Discrete Maximum Principle (DMP)

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Theorem

Let $\varphi_1, \varphi_2, \dots, \varphi_N$ *be a basis of* V_{hp} *then*

$$G_{hp}(x,y) = \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbb{A}_{ij}^{-1} \varphi_i(x) \varphi_j(y), \quad \text{where } \mathbb{A}_{ij} = a(\varphi_i, \varphi_j).$$

Remark: $\Omega \subset \mathbb{R}^1$, $\kappa = 0$, $\quad -u'' = f$

$h \leq 0.9|\Omega| \quad \Rightarrow \quad$ DMP (Vejchodský, Šolín, Math. Comp. 2007)

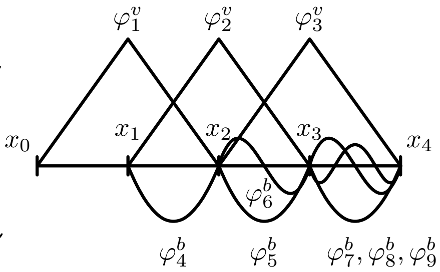
Scheme of the proof

- Standard basis

$$\underbrace{\varphi_1^v, \varphi_2^v, \dots, \varphi_M^v}_{\text{vertex part}}, \underbrace{\varphi_{M+1}^b, \dots, \varphi_N^b}_{\text{bubble part}}$$

- New basis

$$\underbrace{\psi_1^v, \psi_2^v, \dots, \psi_M^v}_{\text{vertex part}}, \underbrace{\psi_{M+1}^b, \dots, \psi_N^b}_{\text{bubble part}}$$



$$\psi_i^b = \varphi_i^b$$

$$\psi_i^v = \varphi_i^v - \sum_{j=1}^{M} c_{ij}\varphi_{M+j}^b \quad \text{such that} \quad a(\psi_i^v, \varphi_j^b) = 0 \quad \forall j$$

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Stiffness matrices

$$\mathbb{A} = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$$

$$\widetilde{\mathbb{A}} = \begin{pmatrix} S & 0 \\ 0 & D \end{pmatrix}$$

$$S = A - BD^{-1}B^T$$

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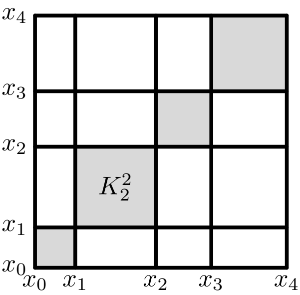
$$G_{hp}(x,y) = \sum_{i=1}^{N} \sum_{j=1}^{N} \widetilde{\mathbb{A}}_{ij}^{-1} \psi_i(x) \psi_j(y)$$

$$= \underbrace{\sum_{i=1}^{N} \sum_{j=1}^{N} S_{ij}^{-1} \psi_i^v(x) \psi_j^v(y)}_{G_{hp}^v(x,y)} + \underbrace{\sum_{i=1}^{N-M} \sum_{j=1}^{N-M} D_{ij}^{-1} \psi_{M+i}^b(x) \psi_{M+j}^b(y)}_{G_{hp}^b(x,y)}$$

Scheme of the proof



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Scheme of the proof

To prove $G_{hp}(x,y) \geq 0$ we require

(a) $\psi_i^v \geq 0$

(b) $a(\psi_i^v, \psi_j^v) \leq 0$ for $i \neq j$

(c) $G_{hp}(x,y)|_{K_k^2} = G_{hp}^v(x,y)|_{K_k^2} + G_{hp}^b(x,y)|_{K_k^2} \geq 0$

Stiffness matrices

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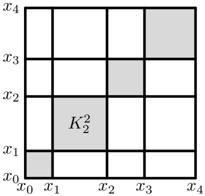

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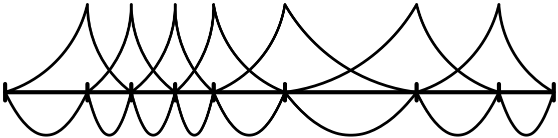
Verification of (a) and (b)

1D $\quad -u'' + \kappa^2 u = f$

p	(a)	(b)	(c)	DMP
1	always	$\kappa^2 h^2 \leq 6$	$G_{hp}^b = 0$	O.K.
2	$\kappa^2 h^2 \leq 20/3$	always	$G_{hp}^b \geq 0$	O.K.
3	$\kappa^2 h^2 \leq 38.61$	$\kappa^2 h^2 \leq 25.89$	$G_{hp}^b \not\geq 0$	?
4	$\kappa^2 h^2 \leq 18.91$	always	$\kappa^2 h^2 \leq 3.611$	O.K.
5	$\kappa^2 h^2 \leq 49.44$	$\kappa^2 h^2 \leq 59.82$	$G_{hp}^b \not\geq 0$	?
6	$\kappa^2 h^2 \leq 37.56$	always	$\kappa^2 h^2 \leq 0.887$	O.K.
7	$\kappa^2 h^2 \leq 72.82$	$\kappa^2 h^2 \leq 107.81$	$G_{hp}^b \not\geq 0$	?
8	$\kappa^2 h^2 \leq 62.62$	always	$G_{hp}^b \not\geq 0$	?
9	$\kappa^2 h^2 \leq 104.09$	$\kappa^2 h^2 \leq 169.85$	$G_{hp}^b \not\geq 0$	?
10	$\kappa^2 h^2 \leq 94.107$	always	$G_{hp}^b \not\geq 0$	?

Verification of (c)

$V_{hp} = \text{span}\{\psi_1^v, \ldots, \psi_M^v, \psi_{M+1}^b, \ldots, \psi_N^b\}$
$G_{hp} = G_{hp}^v + G_{hp}^b$



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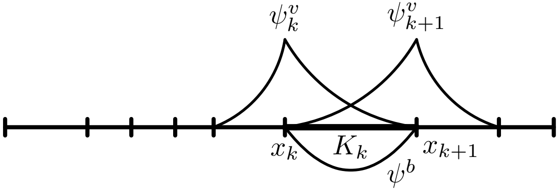

$\widehat{V}_{hp} = \text{span}\{\psi_i : \text{supp}(\psi_i) \cap K_k \neq \emptyset\} = \{\psi_k^v, \psi_{k+1}^v, \psi_{\dots}^b\}$
$\widehat{G}_{hp} = \widehat{G}_{hp}^v + \widehat{G}_{hp}^b$

Lemma

If (a) and (b) then $G_{hp}^v \geq \widehat{G}_{hp}^v$ and $G_{hp}^b = \widehat{G}_{hp}^b$ on K_k^2.

Corollary

$\widehat{G}_{hp}^v + \widehat{G}_{hp}^b \geq 0$ on all K_k^2, $K_k \in \mathcal{T}_{hp}$ $\Rightarrow$ DMP

Proof.

$G_{hp} = G_{hp}^v + G_{hp}^b \geq \widehat{G}_{hp}^v + \widehat{G}_{hp}^b \geq 0$ on K_k^2 □

Verification of (c)

$V_{hp} = \text{span}\{\psi_1^v, \dots, \psi_M^v, \psi_{M+1}^b, \dots, \psi_N^b\}$
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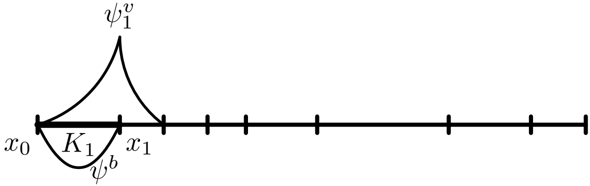

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Lemma

If (a) and (b) then $G_{hp}^v \geq \widehat{G}_{hp}^v$ *and* $G_{hp}^b = \widehat{G}_{hp}^b$ *on* K_k^2.

Corollary

$\widehat{G}_{hp}^v + \widehat{G}_{hp}^b \geq 0$ *on all* K_k^2, $K_k \in \mathcal{T}_{hp} \quad \Rightarrow \quad$ *DMP*

Proof.

$G_{hp} = G_{hp}^v + G_{hp}^b \geq \widehat{G}_{hp}^v + \widehat{G}_{hp}^b \geq 0 \quad \text{on } K_k^2$ □

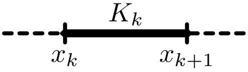
Proof of Lemma

Lemma

If (a) and (b) then $G^v_{hp} \geq \widehat{G}^v_{hp}$ *and* $G^b_{hp} = \widehat{G}^b_{hp}$ *on* K^2_k.

Proof.

$$\widetilde{\mathbb{A}} = \begin{pmatrix} S^A & S^B & 0 \\ (S^B)^T & S^D & \\ 0 & & D \end{pmatrix} \qquad S^A = \begin{pmatrix} a(\psi^v_k, \psi^v_k) & a(\psi^v_{k+1}, \psi^v_k) \\ a(\psi^v_{k+1}, \psi^v_k) & a(\psi^v_{k+1}, \psi^v_{k+1}) \end{pmatrix}$$



$$\widehat{G}^v_{hp}(x,y) = \sum_{i=1}^{2} \sum_{j=1}^{2} (S^A)^{-1}_{ij} \psi^v_{k+i-1}(x) \psi^v_{k+j-1}(y)$$

$$G^v_{hp}(x,y) = \sum_{i=1}^{2} \sum_{j=1}^{2} R^{-1}_{ij} \psi^v_{k+i-1}(x) \psi^v_{k+j-1}(y) \quad \text{on } K^2_k,$$

where

$$R = S^A - \underbrace{S^B (S^D)^{-1} (S^B)^T}_{\geq 0} \leq S^A$$

$$R^{-1} \geq (S^A)^{-1} \quad \ldots \text{ M-matrices}$$

Conditions for DMP

For $p = 3$ the DMP holds if for every element

- $\kappa^2 h^2 \leq 3.21$
- at the boundary $\quad h_{\text{neighbour}} \geq h$

- in the interior $\quad h_{\text{neighbour1}} \geq \frac{1}{6} h \leq h_{\text{neighbour2}}$

OR if for every element

- $\kappa^2 h^2 \leq 3.21$
- all neighbours $\quad h_{\text{neighbour1}} \geq \frac{1}{6} h \leq h_{\text{neighbour2}}$

- at the boundary if $\; h_{\text{neighbour}} \leq h \;$ then $\kappa^2 h^2 \leq 0.845$

Remark 1D $\quad -u'' + \kappa^2 u = f$

increase of poly. degrees $\Rightarrow$ increase of $G_{hp}(x_i, x_j) = \mathbb{A}_{ij}^{-1} = S_{ij}^{-1}$
(for fine enough meshes)

$$\overbrace{\mathbb{A} = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}}^{\text{Standard basis}} \qquad \overbrace{\widetilde{\mathbb{A}} = \begin{pmatrix} S & 0 \\ 0 & D \end{pmatrix}}^{\text{New basis}} \qquad S = A - \underbrace{B^T D^{-1} B}_{\geq 0}$$

$$D = \text{blockdiag}(D^1, D^2, \dots, D^M)$$

$$D^k = \frac{1}{h_k} \begin{pmatrix} 2 + \frac{1}{5}\kappa^2 h_k^2 & 0 & -\frac{\sqrt{84}}{420}\kappa^2 h_k^2 & 0 & \cdots \\ 0 & 2 + \frac{1}{21}\kappa^2 h_k^2 & 0 & -\frac{\sqrt{20}}{420}\kappa^2 h_k^2 & \cdots \\ -\frac{\sqrt{84}}{420}\kappa^2 h_k^2 & 0 & 2 + \frac{1}{45}\kappa^2 h_k^2 & 0 & \cdots \\ 0 & -\frac{\sqrt{20}}{420}\kappa^2 h_k^2 & 0 & 2 + \frac{1}{77}\kappa^2 h_k^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Remark 1D $\quad -u'' + \kappa^2 u = f$

increase of poly. degrees $\Rightarrow$ increase of $G_{hp}(x_i, x_j) = \mathbb{A}^{-1}_{ij} = S^{-1}_{ij}$
(for fine enough meshes)

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$$S = A - \underbrace{B^T D^{-1} B}_{\geq 0} \leq A$$

S and A are M-matrices

$$S^{-1} \geq A^{-1}$$

Thank you for your attention

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