A posteriori error estimates in the finite element method

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Poisson problem

- Classical formulation: find $u \in C^2(\Omega) \cap C(\overline{\Omega})$:

$$-\Delta u = f \quad \text{in } \Omega \qquad u = 0 \quad \text{on } \partial\Omega$$

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$$u \in H_0^1(\Omega) \; : \quad \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x \quad \forall v \in H_0^1(\Omega)$$

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- Galerkin method $\qquad V_h \subset H_0^1(\Omega) \quad \dim V_h < \infty$

$$u_h \in V_h \;:\quad \int_\Omega \nabla u_h \cdot \nabla v_h \, \mathrm{d}x = \int_\Omega f v_h \, \mathrm{d}x \quad \forall v_h \in V_h \quad \Leftrightarrow \quad Ay = F$$

$$u_h(x) = \sum_{j=1}^N y_j \varphi_j(x) \qquad \sum_{j=1}^N y_j \underbrace{\int_\Omega \nabla \varphi_j \cdot \nabla \varphi_i \, \mathrm{d}x}_{A_{ij}} = \underbrace{\int_\Omega f \varphi_i \, \mathrm{d}x}_{F_i}$$

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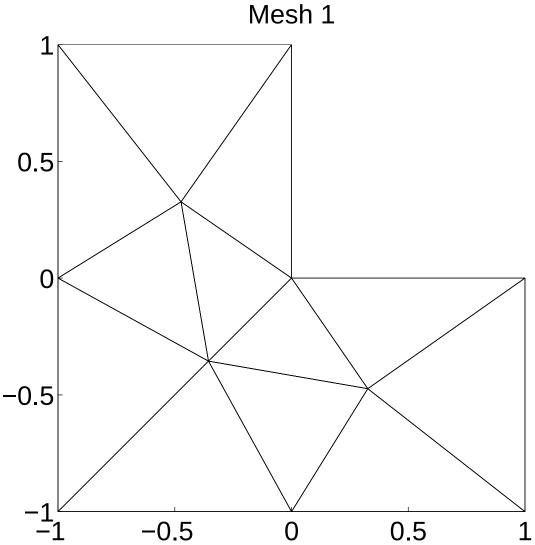
- FEM

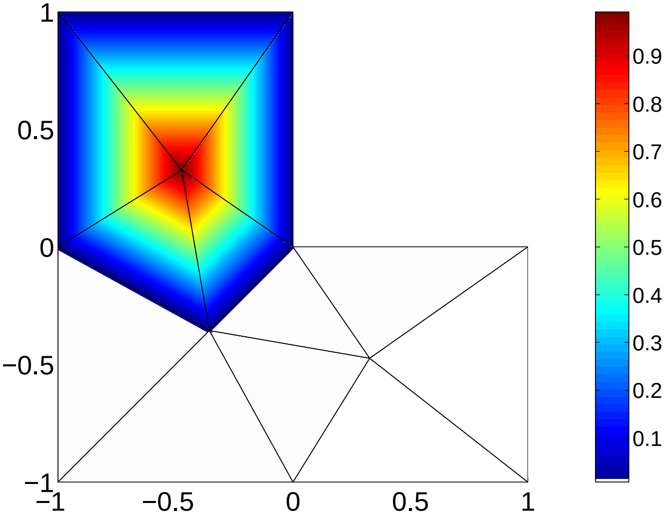
$$V_h = \{ v_h \in H_0^1(\Omega) : v_h|_K \in P^1(K) \ \forall K \in \mathcal{T}_h \}$$

$\varphi_1, \dots, \varphi_N$ ... FEM basis functions $\quad \varphi_i(x_j) = \delta_{ij}$

FEM



FEM



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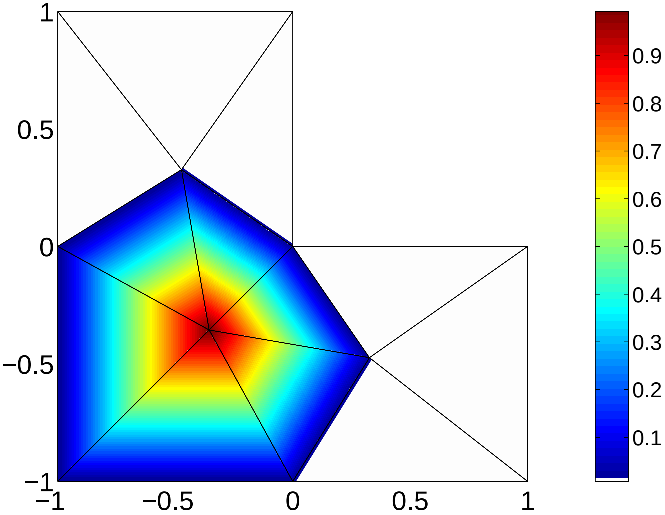

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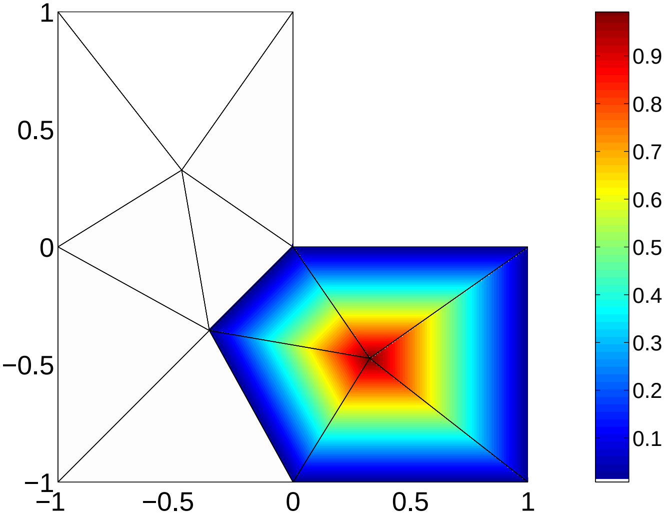

A priori error estimates

- Discretization error: $e = u - u_h$
- Energy norm: $\|u\|^2 = \int_\Omega \nabla u \cdot \nabla u \,\mathrm{d}x = |u|^2_{H^1(\Omega)}$
- $$u \in H^2(\Omega) \quad \Rightarrow \quad \|e\| \leq C\, h\, |u|_{H^2(\Omega)}$$

A posteriori error estimates

- $\|e\| \approx \eta$, where $\eta = \eta(u_h)$
- Efficient and realiable $C_1\eta \leq \|e\| \leq C_2\eta$
- Local $\eta^2 = \sum_{K\in\mathcal{T}_h} \eta_K^2$
- Guaranteed upper (lower) bound $\|e\| \leq \eta$ $(\eta \leq \|e\|)$
- Asymptotic exactness $\lim_{h\to 0} \frac{\eta}{\|e\|} = 1$

A posteriori error estimates

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x - \int_\Omega \nabla u_h \cdot \nabla v \,\mathrm{d}x}_{\displaystyle\int_\Omega \nabla e \cdot \nabla v \,\mathrm{d}x} = \underbrace{\int_\Omega f v \,\mathrm{d}x - \int_\Omega \nabla u_h \cdot \nabla v \,\mathrm{d}x}_{\mathcal{R}(v)}$$

- Explicit residual $\quad \|e\| = \|\mathcal{R}\|_*$

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$$e_K \in H^1(K): \quad \int_K \nabla e_K \cdot \nabla v \,\mathrm{d}x = \mathcal{R}_K(v) \quad \forall v \in H^1(K)$$

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- Error majorant (complementary energy)
  $$\|e\|^2 \leq \|\nabla u_h - y^*\|_0^2 + C_\Omega^2 \|f + \operatorname{div} y^*\|_0^2 \quad \forall y^* \in \mathbf{H}(\operatorname{div}, \Omega)$$

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- Postprocessing $\quad \nabla u_h \rightsquigarrow \mathcal{G}(u_h) \qquad \nabla e \approx \mathcal{G}(u_h) - \nabla u_h$

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- Quantity of interest $\quad Q(e) \approx \eta$

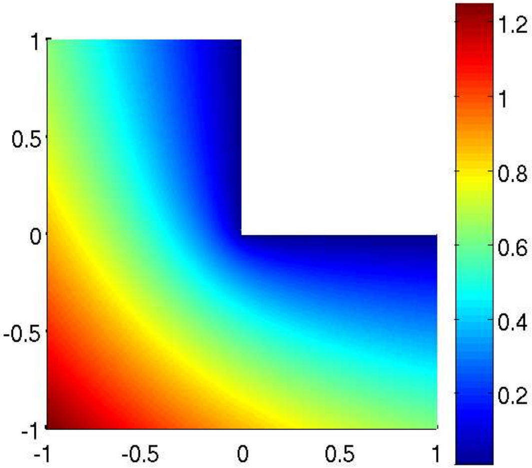
Adaptive algorithm

1. Construct the initial mesh $\mathcal{T}_h$.
2. Find u_h on $\mathcal{T}_h$.
3. Compute η_K for all $K \in \mathcal{T}_h$.
4. If $\displaystyle\sum_{K\in\mathcal{T}_h} \eta_K^2 \leq \mathrm{TOL}^2 \quad \Rightarrow \quad$ STOP.
5. If $\eta_K \geq \Theta\eta_{K,\max} \quad \Rightarrow \quad$ mark K. $\qquad 0 < \Theta < 1$
6. Refine marked elements and build the new mesh $\mathcal{T}_h$.
7. GO TO 2.

Example

$$-\Delta u = 0 \quad \text{in } \Omega$$
$$u = g_D \quad \text{on } \partial\Omega$$

$$u = r^{\frac{2}{3}} \sin \frac{2\theta - \pi}{3}$$



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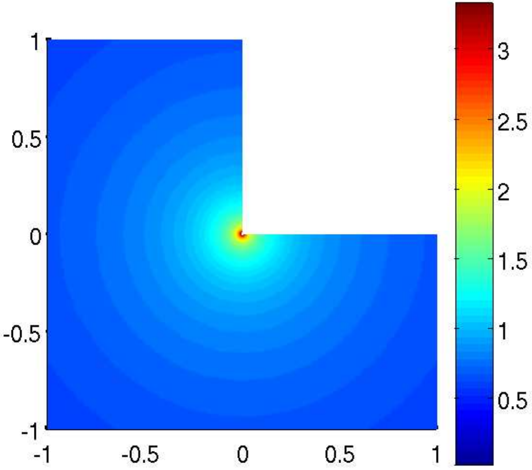

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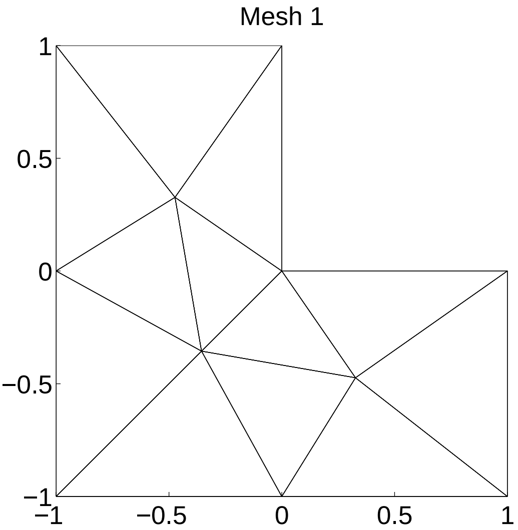

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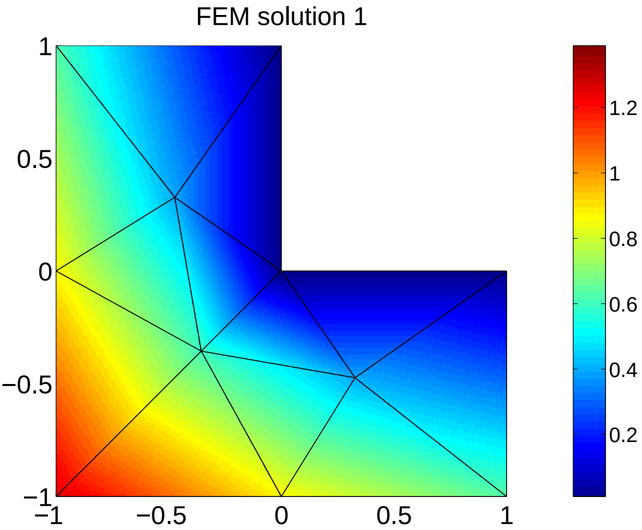

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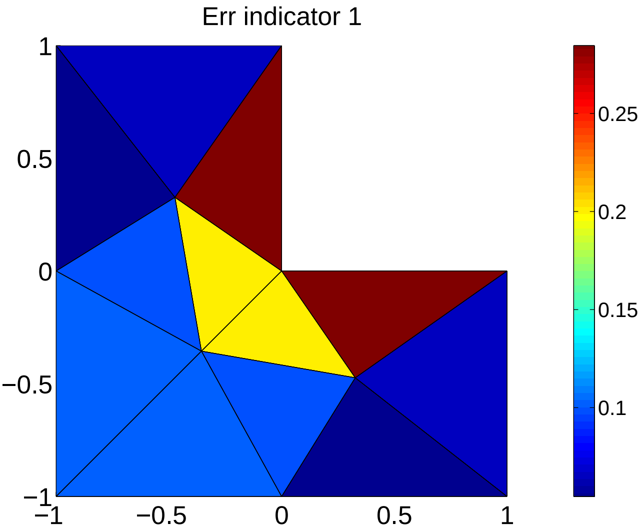
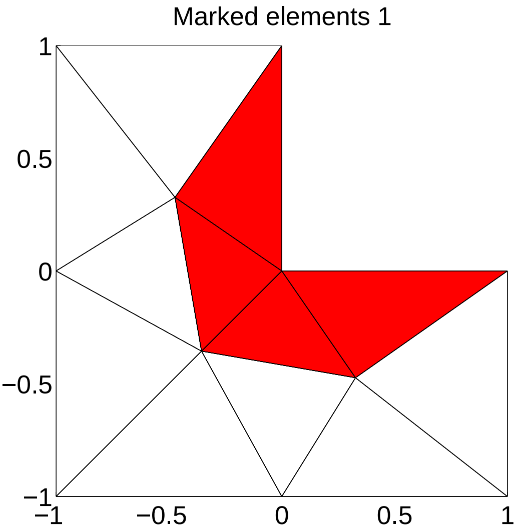


$$\eta_{\mathrm{rel}} = 38.8\,\%$$

$$e_{\mathrm{rel}} = 22.4\,\%$$

$$I_{\mathrm{eff}} = 1.73$$

Adaptive algorithm



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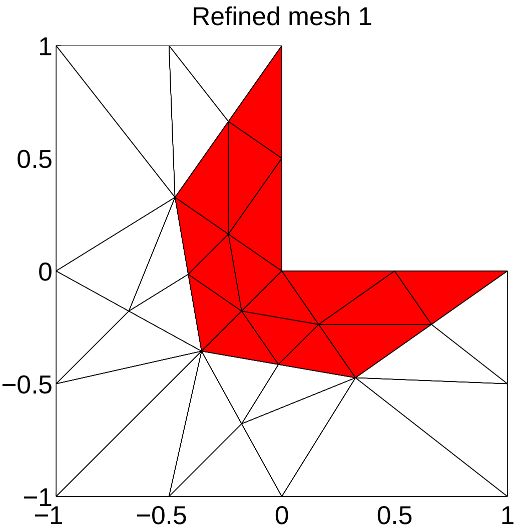

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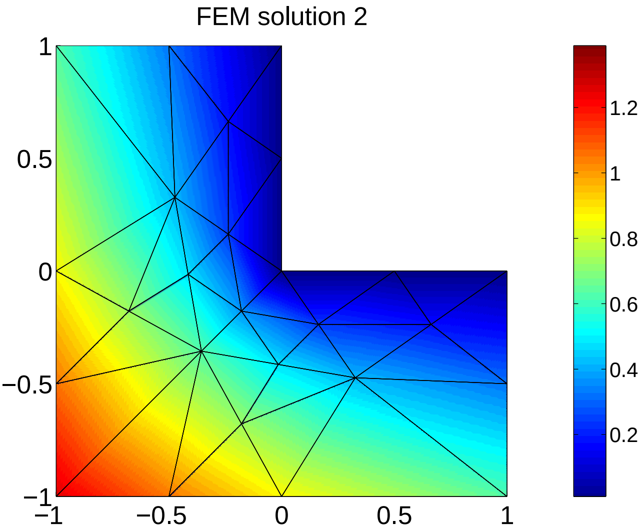

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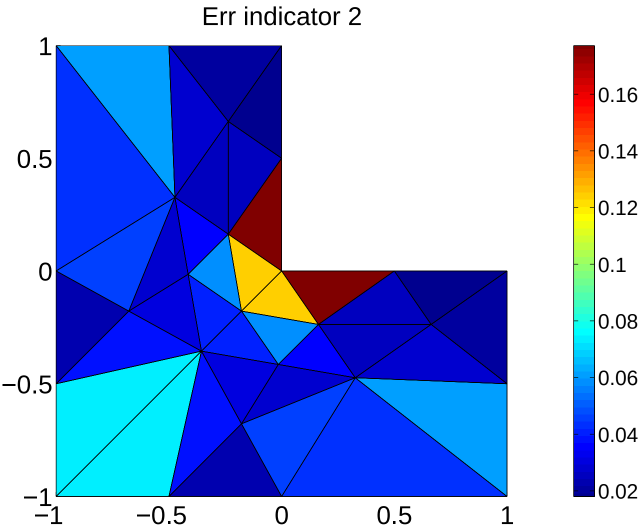

$\eta_{\text{rel}} = 27.5\,\%$

$e_{\text{rel}} = 15.9\,\%$

$I_{\text{eff}} = 1.73$

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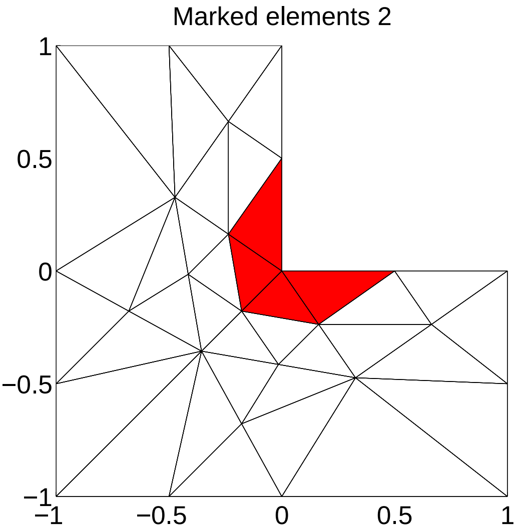

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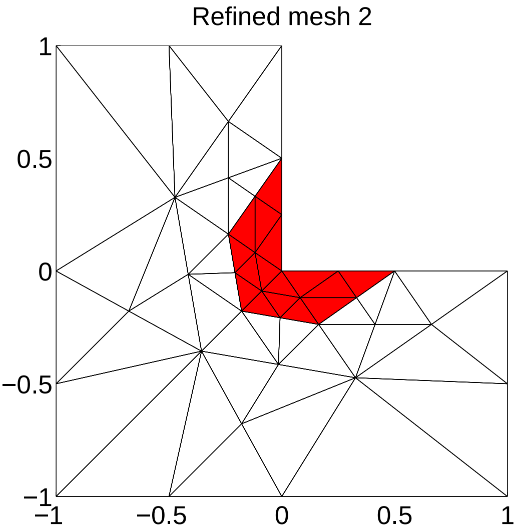

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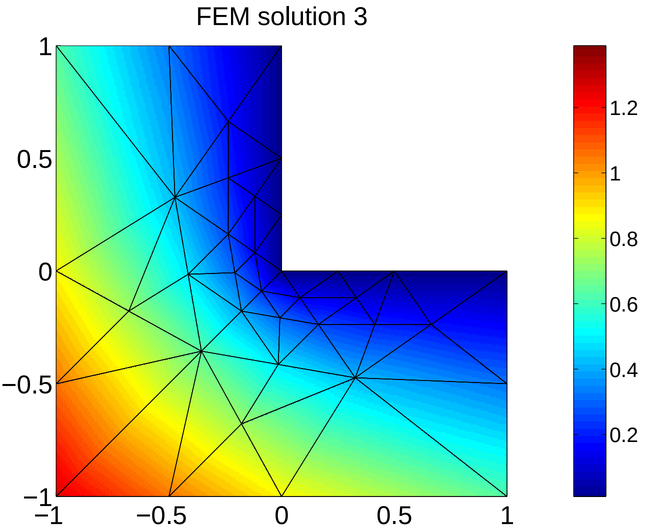

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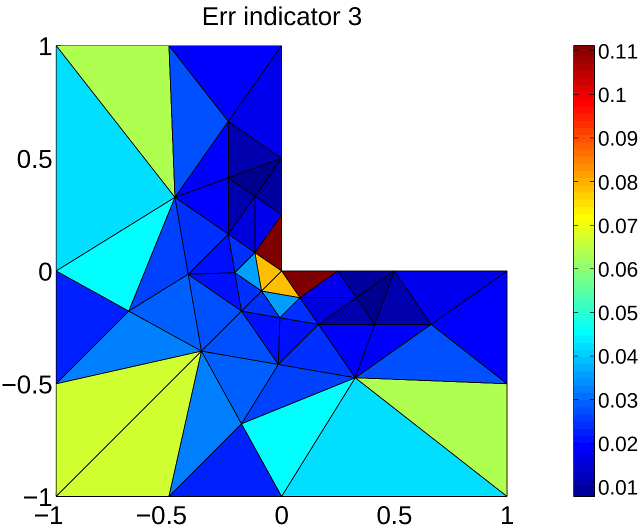
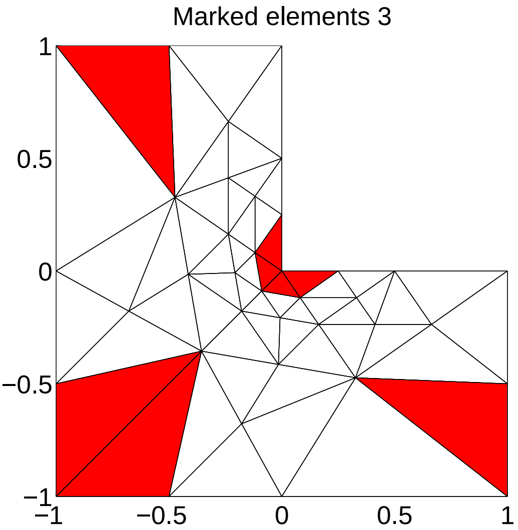


$\eta_{\text{rel}} = 20.9\,\%$

$e_{\text{rel}} = 12.7\,\%$

$I_{\text{eff}} = 1.65$

Adaptive algorithm



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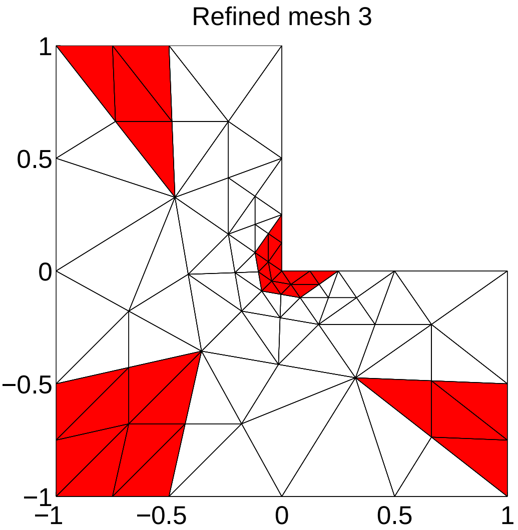

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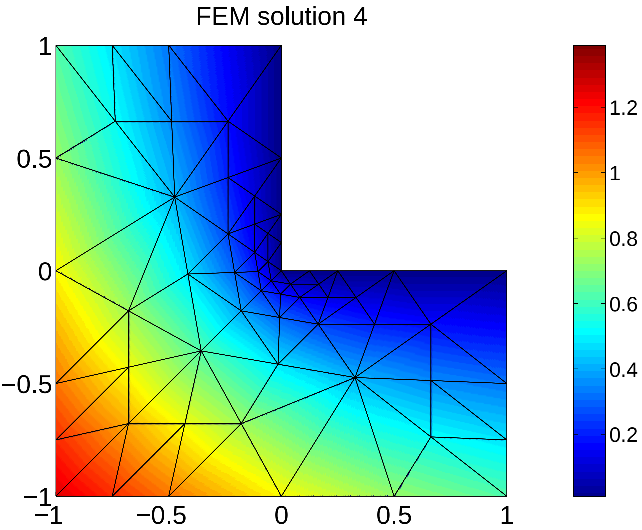

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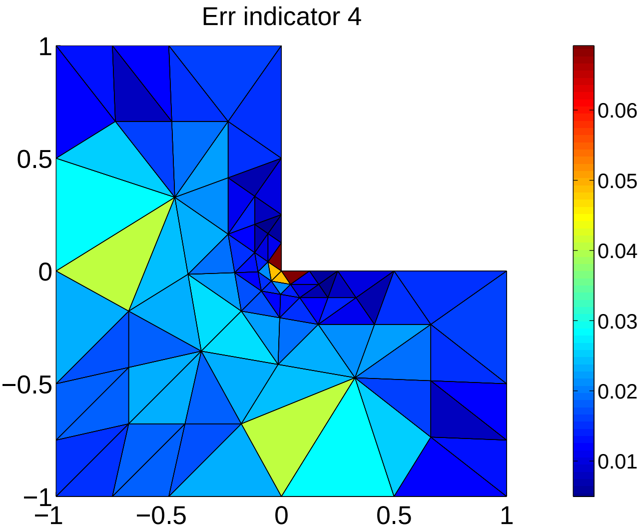
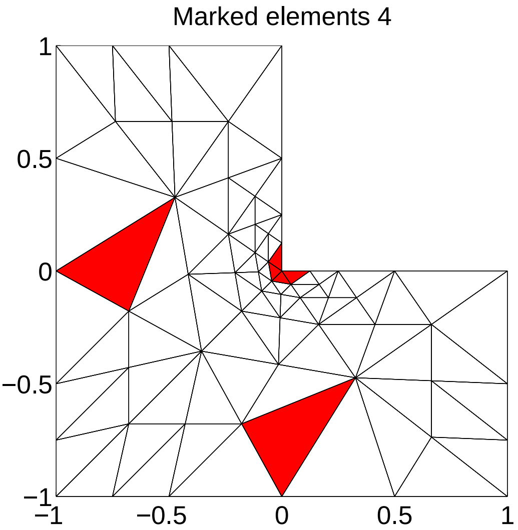


$\eta_{\text{rel}} = 15.4\,\%$

$e_{\text{rel}} = 9.55\,\%$

$I_{\text{eff}} = 1.61$

Adaptive algorithm



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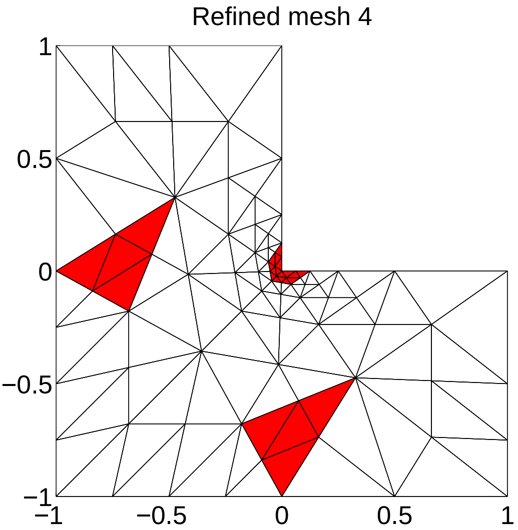

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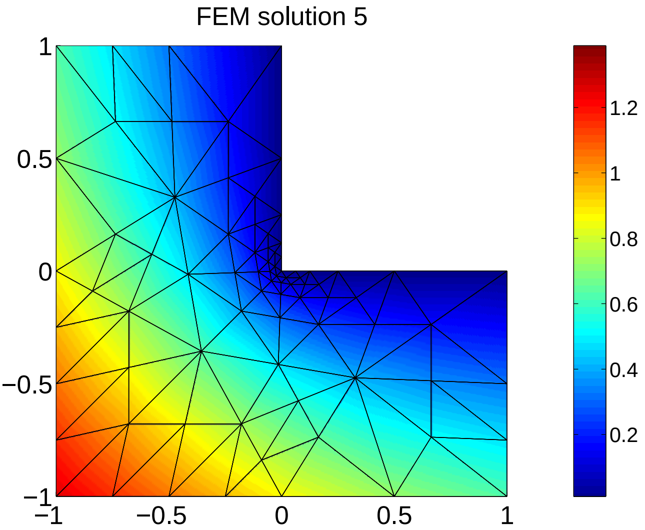

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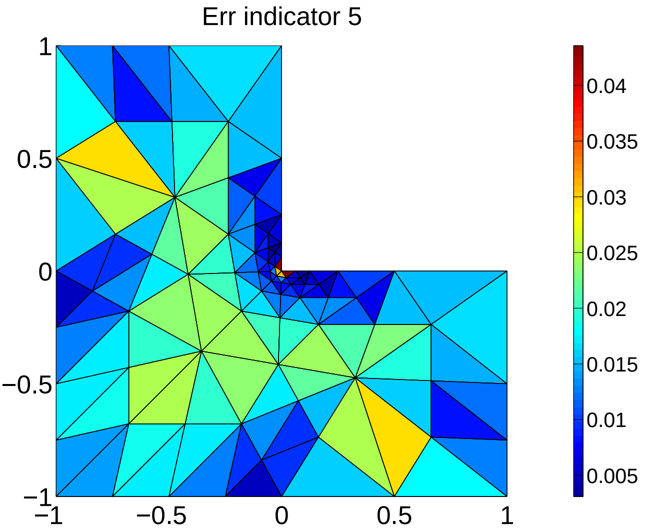
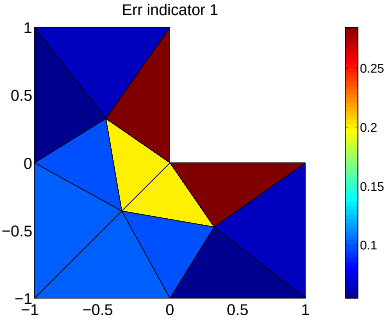


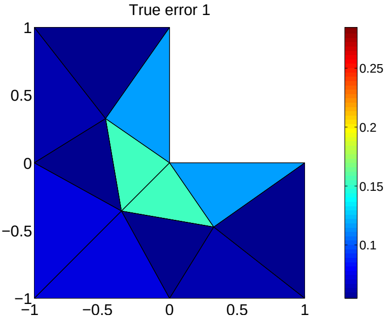
$\eta_{\mathrm{rel}} = 13.4\,\%$

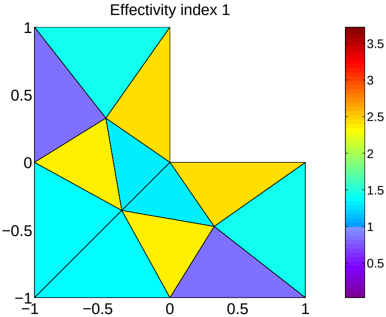
$e_{\mathrm{rel}} = 8.32\,\%$

$I_{\mathrm{eff}} = 1.61$

Effectivity index







$$I_{\text{eff}} = \frac{\eta}{\|e\|} = 1.73$$

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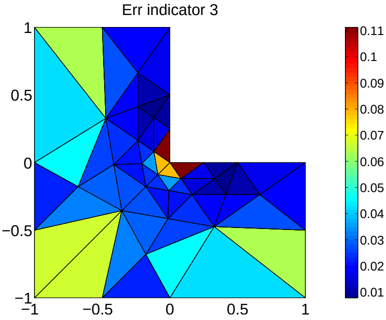

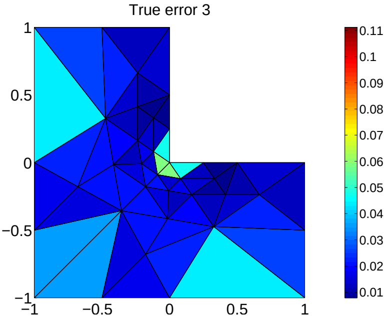

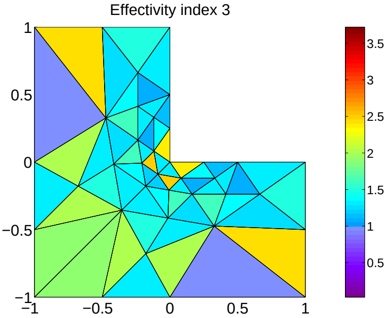

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Estimator by M. Ainsworth and T. Vejchodský

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega \subset \mathbb{R}^2$$
$$u = 0 \quad \text{on } \partial\Omega$$

$$\|e\|^2 \leq \sum_{K \in \mathcal{T}_h} \underbrace{\left(\left\| \frac{1}{\kappa} (r + \operatorname{div} \boldsymbol{\sigma}_K) \right\|_{0,K}^2 + \|\boldsymbol{\sigma}_K\|_{0,K}^2 \right)}_{\eta_K^2}$$

- $r = f + \Delta u_h - \kappa^2 u_h$
- $\sigma_K \in \mathbf{H}(\operatorname{div}, K)$
- $\sigma_K \cdot n_K = g_K - \dfrac{\partial u_h}{\partial n_K} \quad \text{on } \partial K$
- $g_K \approx \dfrac{\partial u}{\partial n_K} \quad \text{on } \partial K$

Thank you for your attention

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