Improving Conditioning of *hp*-FEM

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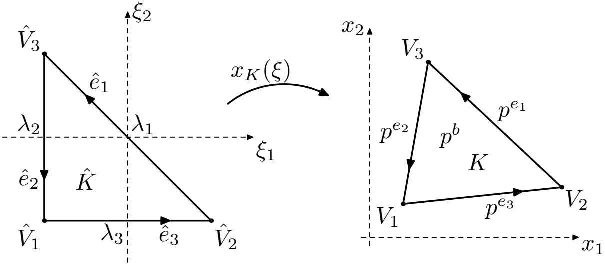
Introduction

- *hp*-FEM
- 

- What are the optimal shape functions?

Model problem

$$
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

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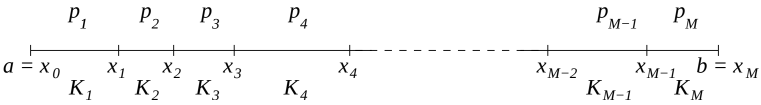
Weak formulation: $V = H^1_0(\Omega)$

$$u \in V: \quad \underbrace{a(u,v)}_{\int_\Omega \nabla u \cdot \nabla v} = \underbrace{F(v)}_{\int_\Omega fv} \quad \forall v \in V.$$

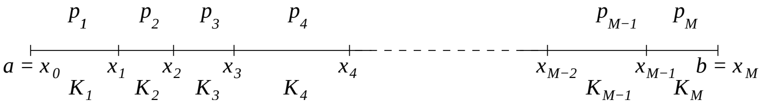
hp-FEM:



$$V_{hp} = \{v_{hp} \in V : v_{hp}|_{K_i} \in P^{p_i}(K_i)\}$$

$$u_{hp} \in V_{hp} : \quad a(u_{hp}, v_{hp}) = F(v_{hp}) \quad \forall v_{hp} \in V_{hp}$$

hp-FEM:



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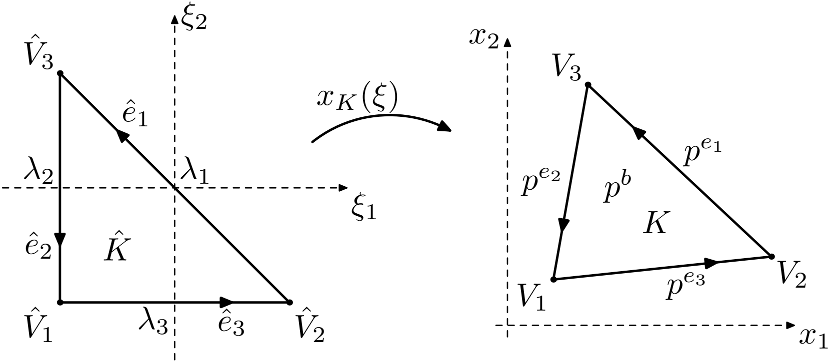
$\varphi_1, \varphi_2, \ldots, \varphi_N$ – basis in V_{hp}

$$u_{hp}(x) = \sum_{j=1}^{N} y_j \varphi_j(x) \quad \Rightarrow \quad Ay = b, \quad \text{where} \quad \begin{array}{l} A_{ij} = a(\varphi_j, \varphi_i) \\ b_i = F(\varphi_i) \end{array}$$

Basis functions



Shape functions
$\hat{\varphi}(\xi)$ on $\hat{K}$
$P^{p^b}(\hat{K})$

$x = \mathbf{x}_K(\xi)$
$\longmapsto$

Basis functions
$\varphi(x) := \hat{\varphi}\left(\mathbf{x}_K^{-1}(x)\right)$ on K
$P^{p^b}(K)$

$$\mathbf{x}_K(\xi) = \sum_{i=1}^{3} V_i \lambda_i(\xi)$$

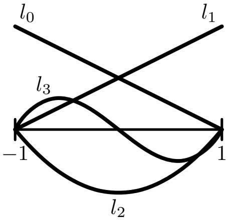
Kernel functions

$$l_0(\xi) = (1-\xi)/2 \quad \xi \in \hat{K} = [-1,1]$$

$$l_1(\xi) = (1+\xi)/2$$

$$l_j(\xi) = \sqrt{\frac{2j-1}{2}} \int_{-1}^{\xi} P_{j-1}(x)\,\mathrm{d}x$$

$$\int_{-1}^{1} l_i'(\xi) l_j'(\xi)\,\mathrm{d}\xi = \delta_{ij} \quad i,j = 2,3,\dots$$



$$l_j(\xi) = l_0(\xi) l_1(\xi) \kappa_{j-2}(\xi)$$

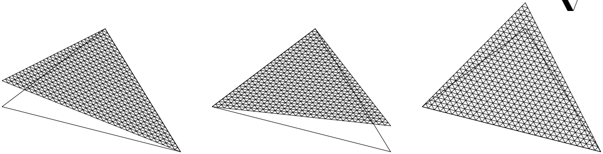
$$\kappa_k(\xi) = -\sqrt{\frac{2k+3}{2}} \frac{4}{(k+2)(k+1)} P_{k+1}'(\xi) \quad k = 0,1,\dots$$

$$\int_{-1}^{1} \frac{(1-\xi^2)}{4} \kappa_\ell(\xi)\kappa_k(\xi)\,\mathrm{d}\xi = \left\{ \begin{array}{l} 0, \;\; \ell \neq k \\ \frac{4}{(k+2)(k+1)}, \;\; \ell = k \end{array} \right.$$

Shape functions

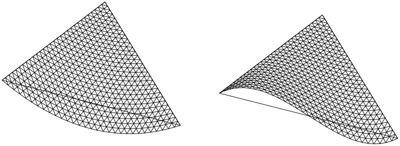
Vertex, 3:

$\hat{\varphi}^{v_1}(\xi) = \lambda_1(\xi)$
$\hat{\varphi}^{v_2}(\xi) = \lambda_2(\xi)$
$\hat{\varphi}^{v_3}(\xi) = \lambda_3(\xi)$



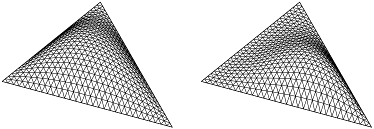
Edge, $p^{e_i} - 1$:

$\hat{\varphi}_k^{e_1} = \lambda_2\lambda_3\kappa_{k-2}(\lambda_3 - \lambda_2)$
$\hat{\varphi}_k^{e_2} = \lambda_3\lambda_1\kappa_{k-2}(\lambda_1 - \lambda_3)$
$\hat{\varphi}_k^{e_3} = \lambda_1\lambda_2\kappa_{k-2}(\lambda_2 - \lambda_1)$
$k = 2, 3, \ldots, p^{e_i}$



Bubble, $(p^b - 1)(p^b - 2)/2$:

$\hat{\varphi}_{n,m}^{b,l} = \lambda_1\lambda_2^n\lambda_3^m$
$m + n + 1 \leq p^b;\ m, n \geq 1$



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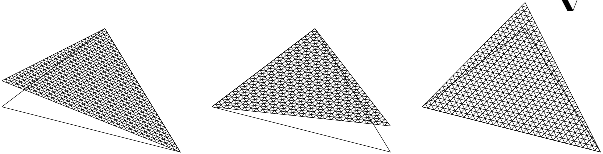

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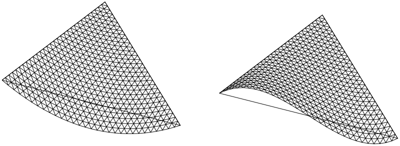
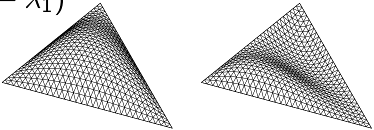


Bubble, $(p^b - 1)(p^b - 2)/2$:

$\hat{\varphi}^{b,II}_{n,m} = \lambda_1\lambda_2\lambda_3\kappa_{n-1}(\lambda_3 - \lambda_2)\kappa_{m-1}(\lambda_2 - \lambda_1)$
$m + n + 1 \leq p^b;\ m, n \geq 1$



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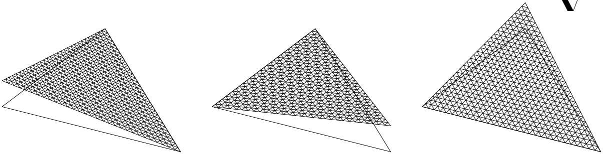

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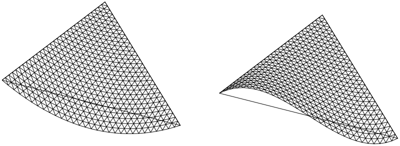

Bubble, $(p^b - 1)(p^b - 2)/2$:

$\hat{\varphi}^{b,III}_{n,m} = \ldots$ orthonormal by Gram-Schmidt process

$m + n + 1 \leq p^b;\ m, n \geq 1$

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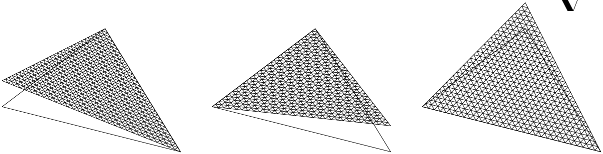

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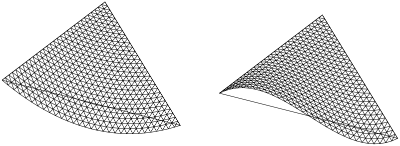


Bubble, $(p^b - 1)(p^b - 2)/2$:

$\hat{\varphi}_{n,m}^{b,IV} = \ldots$ generalized eigenfunctions of discrete Laplacian

$$m + n + 1 \leq p^b;\ m, n \geq 1$$

Bubble shape functions

I. (monomial): $\hat{\varphi}^{b,I}_{n,m} = \lambda_1 \lambda_2^n \lambda_3^m$

II. (Legendre): $\hat{\varphi}^{b,II}_{n,m} = \lambda_1 \lambda_2 \lambda_3 \kappa_{n-1}(\lambda_3 - \lambda_2)\kappa_{m-1}(\lambda_2 - \lambda_1)$

III. (orthonormal): $\hat{\varphi}^{b,III}_{n,m} = \hat{\psi}^b_r$

$$\hat{\psi}^b_r = \hat{\varphi}^{b,II}_r - \sum_{s=1}^{r-1} a_{\hat{K}}(\hat{\varphi}^{b,II}_r, \hat{\psi}^b_s)\hat{\psi}^b_s, \quad r = 1, 2, \dots, \frac{(p^b - 1)(p^b - 2)}{2}$$

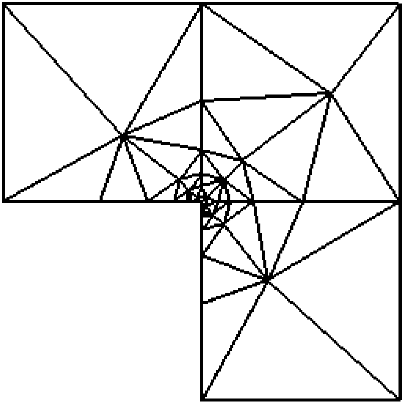
$$\hat{\psi}^b_r = \hat{\psi}^b_r / \|\hat{\psi}^b_r\|_{\hat{K}}$$

IV. (eigen): $\hat{\varphi}^{b,IV}_{n,m} \in P_0^{p^b}(\hat{K})$:

$$a_{\hat{K}}(\hat{\varphi}^{b,IV}_{n,m}, \hat{v}) = \mu_{n,m}(\hat{\varphi}^{b,IV}_{n,m}, \hat{v})_{\hat{K}} \quad \forall \hat{v} \in P_0^{p^b}(\hat{K})$$

$$n + m + 1 \le p^b; \; n, m \ge 1$$

Stiffness matrix conditioning



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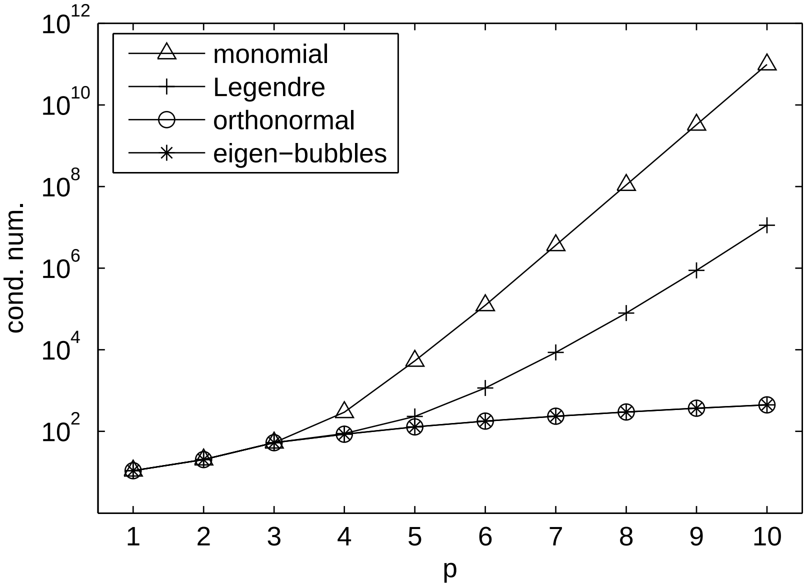
Stiffness matrix conditioning



Lemma

- $X = V \oplus W, \quad \dim X = N, \quad \dim V = M, \quad \dim W = N - M$
- $$\begin{array}{cc} \overbrace{\varphi_1, \dots, \varphi_M,}^{\text{basis of } V} & \overbrace{\varphi_{M+1}, \dots, \varphi_N}^{\text{basis of } W} \\ \| \qquad\qquad \| & \nparallel \qquad\qquad \nparallel \\ \underbrace{\psi_1, \dots, \psi_M,}_{\text{basis of } V} & \underbrace{\psi_{M+1}, \dots, \psi_N}_{\text{basis of } W} \end{array}$$
- $\mathcal{B} : W \times W \mapsto \mathbb{R}$ ... symmetric bilinear form:
  $\mathcal{B}(\varphi_j, \varphi_i) = \mathcal{B}(\psi_j, \psi_i) = \delta_{ij}$ for $i, j = M+1, M+2, \dots, N$
- $a : X \times X \mapsto \mathbb{R}$ ... symmetric bilinear form

Then

$$\left.\begin{array}{l} A_\varphi = \{a(\varphi_j, \varphi_i)\}_{i,j=1}^N \\ A_\psi = \{a(\psi_j, \psi_i)\}_{i,j=1}^N \end{array}\right. \quad \text{have identical eigenvalues.}$$

Lemma

- $X = V \oplus W, \quad \dim X = N, \quad \dim V = M, \quad \dim W = N - M$
- $$\begin{matrix} \overbrace{\varphi_1, \dots, \varphi_M,}^{\text{basis of } V} & \overbrace{\varphi_{M+1}, \dots, \varphi_N}^{\text{basis of } W} \\ \| \qquad\qquad \| & \not\parallel \qquad\qquad \not\parallel \\ \underbrace{\psi_1, \dots, \psi_M,}_{\text{basis of } V} & \underbrace{\psi_{M+1}, \dots, \psi_N}_{\text{basis of } W} \end{matrix}$$
- $\mathcal{B} : W \times W \mapsto \mathbb{R}$ . . . symmetric bilinear form:
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- $a : X \times X \mapsto \mathbb{R}$ . . . symmetric bilinear form

Then

$$\left.\begin{matrix} A_\varphi = \{a(\varphi_j, \varphi_i)\}_{i,j=1}^N \\ A_\psi = \{a(\psi_j, \psi_i)\}_{i,j=1}^N \end{matrix}\right. \quad \text{have identical eigenvalues.}$$

$$\mathcal{B}(u, v) = \begin{cases} 0 \text{ if } \operatorname{supp} u \neq \operatorname{supp} v, \\ \displaystyle\int_K \left(\frac{\mathrm{D}\mathbf{x}_K}{\mathrm{D}\xi}\right)^{\mathrm{T}} \nabla u \cdot \left(\frac{\mathrm{D}\mathbf{x}_K}{\mathrm{D}\xi}\right)^{\mathrm{T}} \nabla v \det \left(\frac{\mathrm{D}\mathbf{x}_K}{\mathrm{D}\xi}\right)^{-1} \mathrm{d}x \end{cases}$$

Proof

$$\psi_i = \sum_{k=1}^{N} c_{ki}\varphi_k \qquad C = \begin{pmatrix} I & 0 \\ 0 & C^{(2,2)} \end{pmatrix}$$

$$(A_\psi)_{ij} = a(\psi_j, \psi_i) = a\left(\sum_{\ell=1}^{N} c_{\ell j}\varphi_\ell, \sum_{k=1}^{N} c_{ki}\varphi_k\right) = \sum_{\ell=1}^{N}\sum_{k=1}^{N} c_{ki} A_{\varphi,k\ell} c_{\ell j}$$

$$I_{ij} = \mathcal{B}(\psi_j, \psi_i)$$

$$= \mathcal{B}\left(\sum_{\ell=M+1}^{N} c_{\ell j}^{(2,2)}\varphi_\ell, \sum_{k=M+1}^{N} c_{ki}^{(2,2)}\varphi_k\right) = \sum_{\ell=M+1}^{N}\sum_{k=M+1}^{N} c_{ki}^{(2,2)}\delta_{k\ell}c_{\ell j}^{(2,2)}$$

$$A_\psi = C^T A_\varphi C, \quad C^T C = I$$

□

Conclusions

- Condition number if very sensitive to the choice of bubble functions.
- Best results – orthonormal bubbles.
- All orthonormal bubbles have the same conditioning.
- Eigen bubbles – stable w.r.t. the reference mapping.

Thank you for your attention

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