Recent results about the discrete maximum principle for higher-order finite elements

Devoted to the 80th birthday of professor Richard S. Varga

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(Continuous) maximum principle

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

$$\text{MaxP}: \quad f \le 0 \quad \Rightarrow \quad \max_{\overline{\Omega}} u \le \max\{0, \max_{\partial\Omega} u\} = 0$$

$$\Updownarrow$$

$$\text{ComP}: \quad f \ge 0 \quad \Rightarrow \quad u \ge 0$$

$$\Updownarrow$$

$$G(x,y) \ge 0 \text{ in } \Omega^2$$

$$u(y) = \int_\Omega G(x,y) f(x)\,\mathrm{d}x$$

$$-\Delta G_y + \kappa^2 G_y = \delta_y \quad \text{in } \Omega$$

$$G_y = 0 \quad \text{on } \partial\Omega$$

$$G(x,y) = G_y(x)$$

Discrete Maximum Principle (DMP)

$$f \geq 0 \text{ in } \Omega \quad \Rightarrow \quad u_h \geq 0 \text{ in } \Omega$$

- Varga (1966): $u_h \ldots$ finite differences
- etc.
- etc.
- higher-order FEM
  - Höhn, Mittelmann (1981)
    no "strengthened" DMP for $p = 2, 3$ in 2D
  - Vejchodský, Šolín (2007)
    DMP for $-u'' = f$, $1 \leq p \leq 100$ in 1D

Discretization

- Classical

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

- Weak

$$u \in V = H_0^1(\Omega): \qquad \underbrace{\mathcal{B}(u,v)}_{\int_\Omega \nabla u \cdot \nabla v + \kappa^2 uv \,\mathrm{d}x} = \underbrace{(f,v)}_{\int_\Omega fv \,\mathrm{d}x} \qquad \forall v \in V$$

- *hp*-FEM

$$u_{hp} \in V_{hp} \subset V: \quad \mathcal{B}(u_{hp}, v_{hp}) = (f, v_{hp}) \quad \forall v_{hp} \in V_{hp}$$

Discrete Green's Function (DGF)

Definition For all $y \in \Omega$ define

$$G_{hp,y} \in V_{hp} : \quad \mathcal{B}(v_{hp}, G_{hp,y}) = v_{hp}(y) \quad \forall v_{hp} \in V_{hp}$$

Notation

$$G_{hp}(x,y) = G_{hp,y}(x) \quad \text{for } (x,y) \in \Omega^2$$

Properties

- $u_{hp}(y) = \int_\Omega G_{hp}(x,y) f(x) \, \mathrm{d}x$
- $G_{hp}(x,y) = \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbb{A}_{ij}^{-1} \varphi_i(x) \varphi_j(y)$
  - φ_i, $i = 1, 2, \dots, N$ ... any basis in V_{hp}
  - $\mathbb{A}_{ij} = \mathcal{B}(\varphi_j, \varphi_i)$ ... the stiffness matrix

Theorem

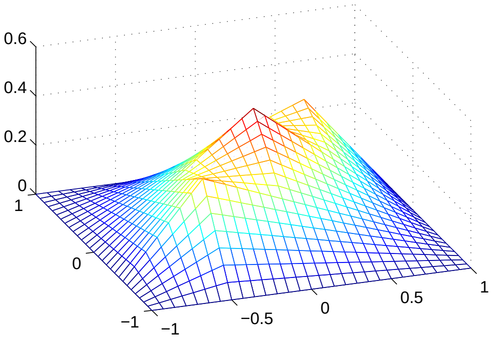
$$\text{DMP} \quad \Leftrightarrow \quad G_{hp}(x,y) \geq 0 \quad \forall (x,y) \in \Omega^2$$

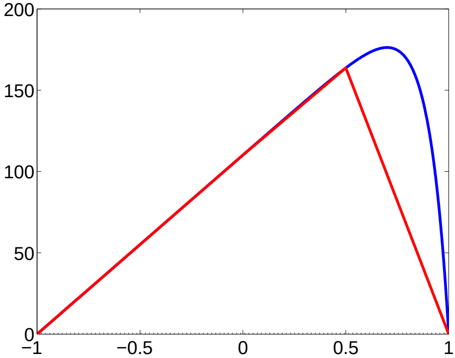
Example 1

$$-u'' = f \text{ in } (0,1) \qquad u(0) = u(1) = 0$$

Linear FEM $\Rightarrow$ DMP O.K.

$$f(x) = \exp(10x)$$





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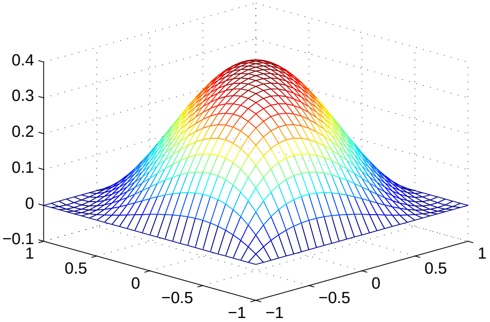
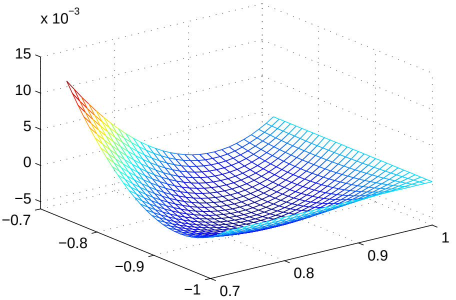
$$-u'' = f \text{ in } (0,1) \qquad u(0) = u(1) = 0$$

One element of degree 3 $\Rightarrow$ no DMP





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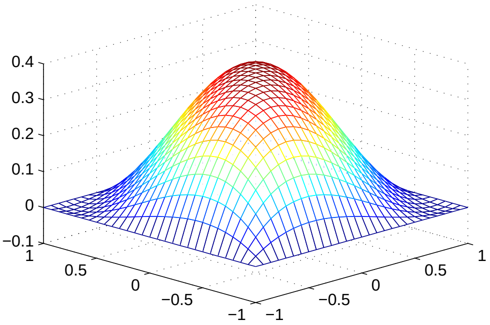
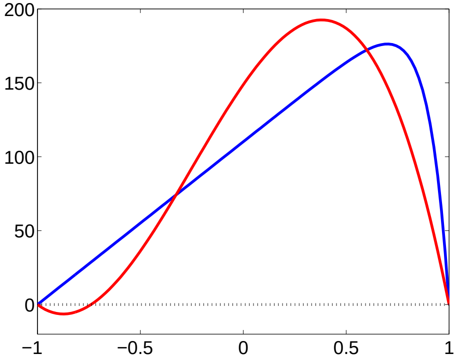


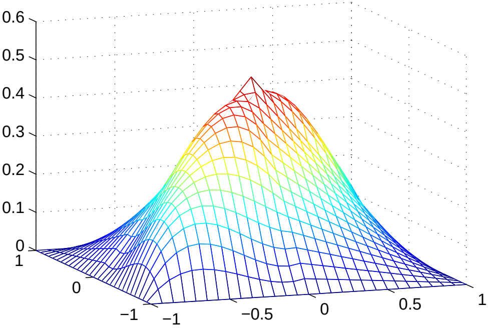


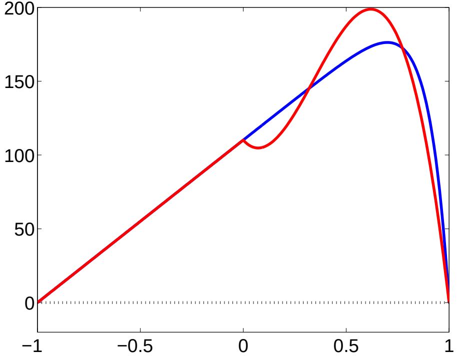
Example 1

$$-u'' = f \text{ in } (0,1) \qquad u(0) = u(1) = 0$$

Two (and more) elements of degree 3 $\Rightarrow$ DMP O.K.

$$f(x) = \exp(10x)$$





Example 2

$$-u'' + \kappa^2 u = f \text{ in } (0,1) \qquad u(0) = u(1) = 0$$

- $p = 1$: $\kappa^2 h^2 \le 6 \quad \Leftrightarrow$ DMP
- $p = 2$: $\kappa^2 h^2 \le 20/3 \Rightarrow$ DMP
- $p \ge 3$: more complicated

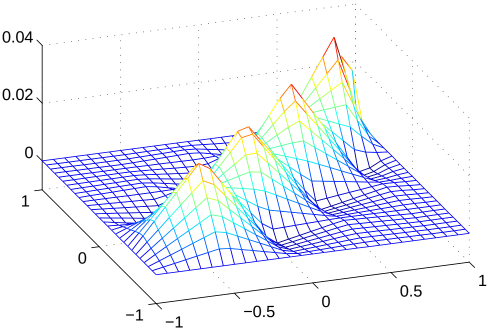
Example 2

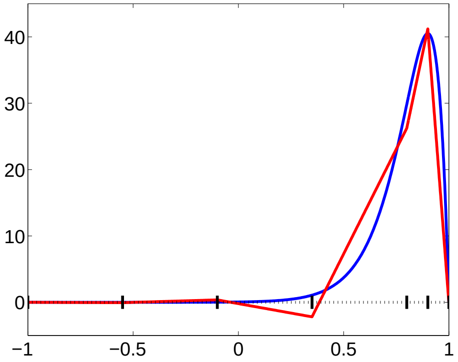
$$-u'' + \kappa^2 u = f \text{ in } (0,1) \qquad u(0) = u(1) = 0$$

Example:

- 6 linear elements
- $\kappa = 10$

$$f(x) = \exp(10x)$$





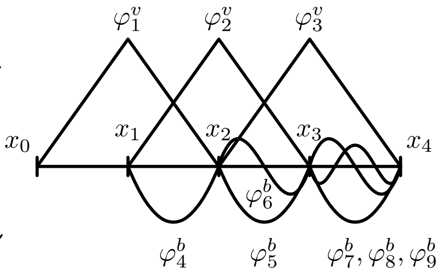
Vertex and higher-order basis functions

- Standard basis

$$\underbrace{\varphi_1^v, \varphi_2^v, \dots, \varphi_M^v}_{\text{vertex funs.}}, \underbrace{\varphi_{M+1}^b, \dots, \varphi_N^b}_{\text{higher-order funs.}}$$

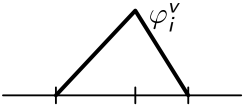
- New basis

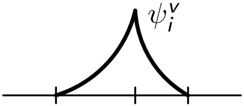
$$\underbrace{\psi_1^v, \psi_2^v, \dots, \psi_M^v}_{\text{vertex funs.}}, \underbrace{\psi_{M+1}^b, \dots, \psi_N^b}_{\text{higher-order funs.}}$$



$$\psi_i^b = \varphi_i^b$$

$$\psi_i^v = \varphi_i^v - \sum_{j=1}^{M} c_{ij} \varphi_{M+j}^b \quad \text{such that} \quad \mathcal{B}(\psi_i^v, \varphi_j^b) = 0 \quad \forall j$$





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Stiffness matrices

$$\mathbb{A} = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$$

$$\widetilde{\mathbb{A}} = \begin{pmatrix} S & 0 \\ 0 & D \end{pmatrix}$$

$$S = A - BD^{-1}B^T$$

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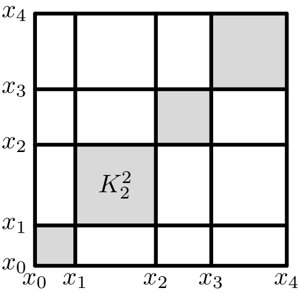
$$G_{hp}(x,y) = \sum_{i=1}^{N} \sum_{j=1}^{N} \widetilde{\mathbb{A}}_{ij}^{-1} \psi_i(x) \psi_j(y)$$

$$= \underbrace{\sum_{i=1}^{N} \sum_{j=1}^{N} S_{ij}^{-1} \psi_i^v(x) \psi_j^v(y)}_{G_{hp}^v(x,y)} + \underbrace{\sum_{i=1}^{N-M} \sum_{j=1}^{N-M} D_{ij}^{-1} \psi_{M+i}^b(x) \psi_{M+j}^b(y)}_{G_{hp}^b(x,y)}$$

Vertex and higher-order basis functions



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Sufficient conditions

If

(a) $\psi_i^v \geq 0$

(b) $\mathcal{B}(\psi_i^v, \psi_j^v) \leq 0$ for $i \neq j$

(c) $G_{hp}(x,y) \geq 0$ in K_k^2

then $G_{hp}(x,y) \geq 0$ in Ω^2.

Stiffness matrices

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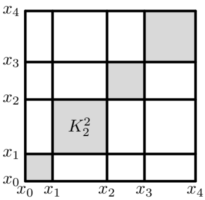

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Analysis of (a)–(c) for 1D diffusion-reaction

Theorem

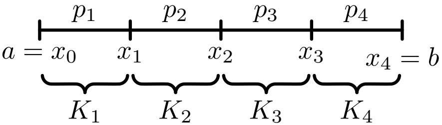
If

- $H_{\mathrm{rel}}^K \leq 1/3$
- $\kappa^2 h_K^2 \leq \min\left\{\alpha^{p_K}, \beta^{p_K}, \gamma^{p_K} \frac{H_{\mathrm{rel}}^K}{1-H_{\mathrm{rel}}^K} + \delta^{p_K}\right\} \qquad \forall K \in \mathcal{T}_{hp}$

then DMP.

$$-u'' + \kappa^2 u = f \quad \text{in } \Omega = (a,b) \qquad u(a) = u(b) = 0$$

- $\mathcal{T}_{hp}$ mesh
- K element
- $h_K = \operatorname{diam} K$
- $H_{\mathrm{rel}}^K = h_K / \operatorname{diam} \Omega$
- p_K poly. degree.



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then DMP.

p	α^p	β^p	γ^p	δ^p
1	∞	6	0	∞
2	20/3	∞	0	∞
3	38.61	25.89	5.608	0
4	18.91	∞	2.936	3.614
5	49.44	59.82	7.799	0
6	37.56	∞	7.247	0.887
7	72.82	107.81	9.791	0
8	62.62	∞	9.709	0
9	104.09	169.85	11.510	0
10	94.10	∞	10.644	0

Thank you for your attention

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