Recent Results about the Discrete Maximum Principle for Higher-Order Finite Elements

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Outline

- Diffusion-reaction problem
- *hp*-FEM
- Discrete Maximum Principle (DMP)
- Discrete Green's Function (DGF)
- Examples
- Idea of the proof
- Nonnegativity of a polynomial – verification

Diffusion-Reaction Problem

- Classical

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

- Weak

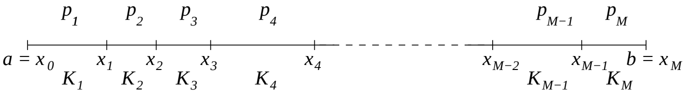
$$u \in V = H^1_0(\Omega): \qquad \underbrace{\mathcal{B}(u,v)}_{\int_\Omega (\nabla u \cdot \nabla v + \kappa^2 uv)\,\mathrm{d}x} = \underbrace{(f,v)}_{\int_\Omega fv\,\mathrm{d}x} \qquad \forall v \in V$$

- *hp*-FEM

$$u_{hp} \in V_{hp} \subset V: \quad \mathcal{B}(u_{hp}, v_{hp}) = (f, v_{hp}) \quad \forall v_{hp} \in V_{hp}$$

- $V_{hp} = \{v_{hp} \in V : v_{hp}|_{K_i} \in P^{p_i}(K_i),\ K_i \in \mathcal{T}_{hp}\}$

  hp-mesh $\mathcal{T}_{hp}$ in Ω



Discrete Maximum Principle (DMP)

Definition A discretization based on a mesh $\mathcal{T}_{hp}$ satisfies DMP if

$$f \geq 0 \text{ a.e. in } \Omega \quad \Rightarrow \quad u_{hp} \geq 0 \text{ in } \Omega$$

- Varga (1966): ... finite differences
- etc.
- etc.
- higher-order FEM
  - Höhn, Mittelmann (1981)
    no "strengthened" DMP for $p = 2, 3$ in 2D
  - Vejchodský, Šolín (2007)
    DMP for $-u'' = f$, $1 \leq p \leq 100$ in 1D

Discrete Green's Function (DGF)

Definition For all $y \in \Omega$ define

$$G_{hp,y} \in V_{hp} : \quad \mathcal{B}(v_{hp}, G_{hp,y}) = v_{hp}(y) \quad \forall v_{hp} \in V_{hp}$$

Notation

$$G_{hp}(x,y) = G_{hp,y}(x) \quad \text{for } (x,y) \in \Omega^2$$

Properties

- $u_{hp}(y) = \int_\Omega G_{hp}(x,y) f(x) \, \mathrm{d}x$
- $G_{hp}(x,y) = \sum_{i=1}^{N} \sum_{j=1}^{N} (\mathbb{A}^{-1})_{ij} \varphi_i(x) \varphi_j(y)$
  - φ_i, $i = 1, 2, \dots, N$ ... any basis in V_{hp}
  - $\mathbb{A}_{ij} = \mathcal{B}(\varphi_j, \varphi_i)$ ... the stiffness matrix

Theorem

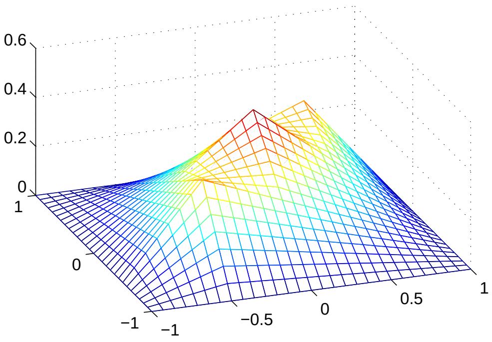
$$\text{DMP} \quad \Leftrightarrow \quad G_{hp}(x,y) \geq 0 \quad \forall (x,y) \in \Omega^2$$

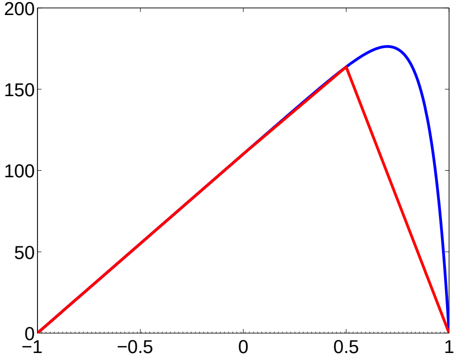
Example 1 – Poisson Equation

$$-u'' = f \text{ in } (-1,1) \qquad u(-1) = u(1) = 0$$

Linear FEM $\Rightarrow$ DMP O.K.

$$f(x) = \exp(10x)$$





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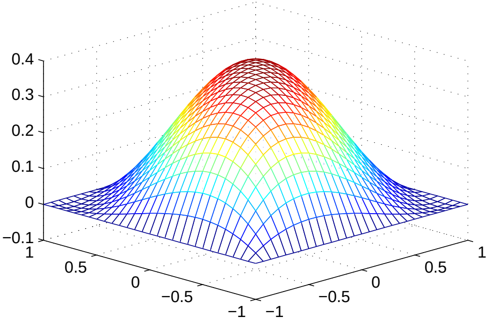
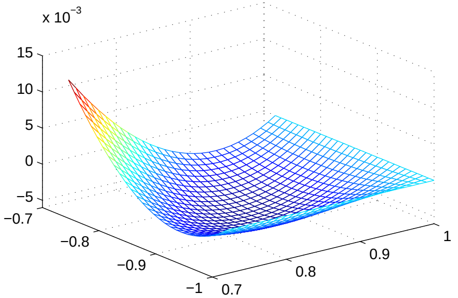
One element of degree 3 $\Rightarrow$ no DMP





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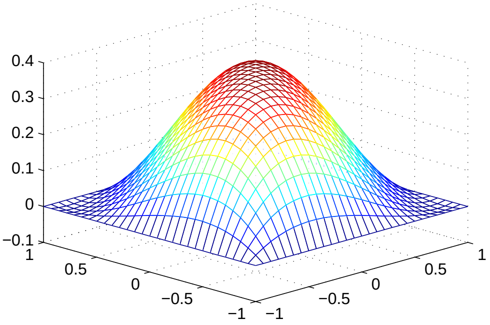
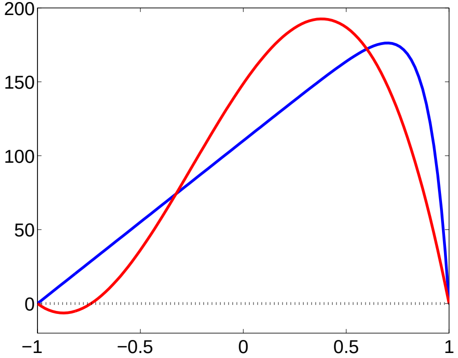


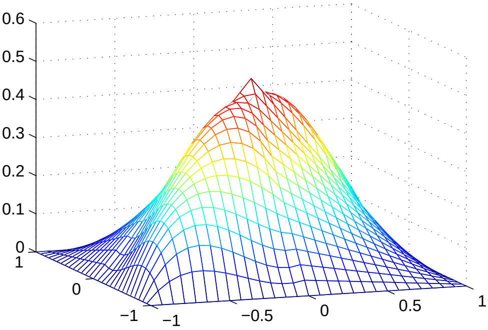


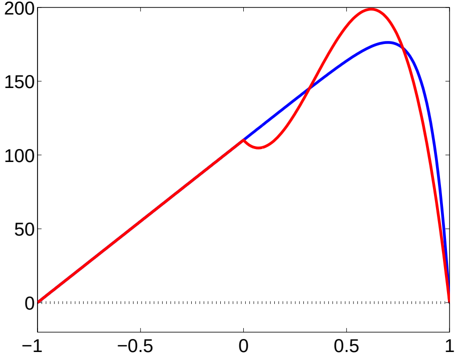
Example 1 – Poisson Equation

$$-u'' = f \text{ in } (-1,1) \qquad u(-1) = u(1) = 0$$

Two (and more) elements of degree 3 $\Rightarrow$ DMP O.K.

$$f(x) = \exp(10x)$$





Example 2 – Diffusion-Reaction

$$-u'' + \kappa^2 u = f \text{ in } (-1,1) \qquad u(-1) = u(1) = 0$$

- $p = 1$: $\kappa^2 h_K^2 \leq 6 \quad \Leftrightarrow$ DMP
- $p = 2$: $\kappa^2 h_K^2 \leq 20/3 \Rightarrow$ DMP
- $p \geq 3$: more complicated

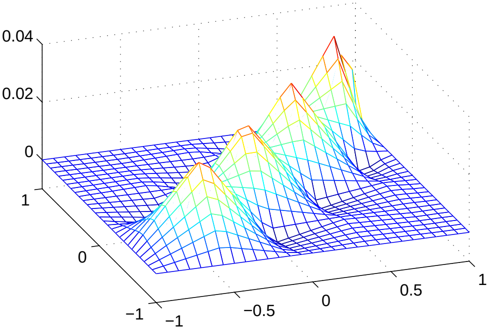
Example 2 – Diffusion-Reaction

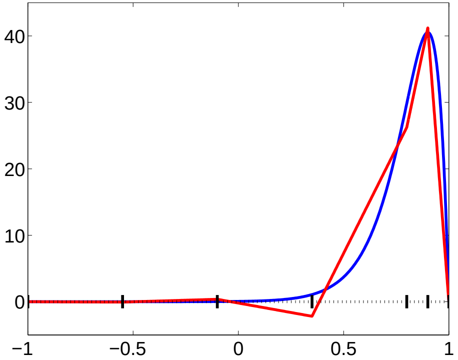
$$-u'' + \kappa^2 u = f \text{ in } (-1,1) \qquad u(-1) = u(1) = 0$$

Example:

- 6 linear elements
- $\kappa = 10$

$$f(x) = \exp(10x)$$





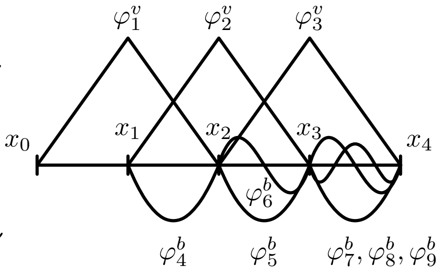
Vertex and Higher-Order Basis Functions

- Standard basis

  $\underbrace{\varphi_1^v, \varphi_2^v, \dots, \varphi_M^v}_{\text{vertex funs.}}, \underbrace{\varphi_{M+1}^b, \dots, \varphi_N^b}_{\text{higher-order funs.}}$

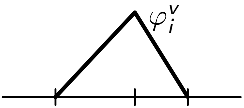
- New basis

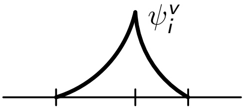
  $\underbrace{\psi_1^v, \psi_2^v, \dots, \psi_M^v}_{\text{vertex funs.}}, \underbrace{\psi_{M+1}^b, \dots, \psi_N^b}_{\text{higher-order funs.}}$



$$\psi_i^b = \varphi_i^b$$

$$\psi_i^v = \varphi_i^v - \sum_{j=1}^{M} c_{ij} \varphi_{M+j}^b \quad \text{such that} \quad \mathcal{B}(\psi_i^v, \varphi_j^b) = 0 \quad \forall j$$





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Stiffness matrices

$$\mathbb{A} = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$$

$$\widetilde{\mathbb{A}} = \begin{pmatrix} S & 0 \\ 0 & D \end{pmatrix}$$

$$S = A - BD^{-1}B^T$$

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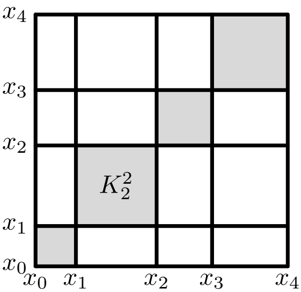
$$G_{hp}(x,y) = \sum_{i=1}^{N} \sum_{j=1}^{N} \widetilde{\mathbb{A}}_{ij}^{-1} \psi_i(x) \psi_j(y)$$

$$= \underbrace{\sum_{i=1}^{N} \sum_{j=1}^{N} S_{ij}^{-1} \psi_i^v(x) \psi_j^v(y)}_{G_{hp}^v(x,y)} + \underbrace{\sum_{i=1}^{N-M} \sum_{j=1}^{N-M} D_{ij}^{-1} \psi_{M+i}^b(x) \psi_{M+j}^b(y)}_{G_{hp}^b(x,y)}$$

Vertex and Higher-Order Basis Functions



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Sufficient Conditions

Stiffness matrices

$$\mathbb{A} = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$$

$$\widetilde{\mathbb{A}} = \begin{pmatrix} S & 0 \\ 0 & D \end{pmatrix}$$

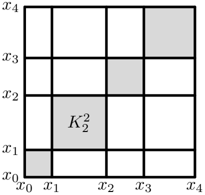
If

(a) $\psi_i^v \geq 0$

(b) $\mathcal{B}(\psi_i^v, \psi_j^v) \leq 0$ for $i \neq j$

(c) $G_{hp}(x,y) \geq 0$ in K_k^2

then $G_{hp}(x,y) \geq 0$ in Ω^2.



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Analysis of (a)–(c) for 1D Diffusion-Reaction

Theorem

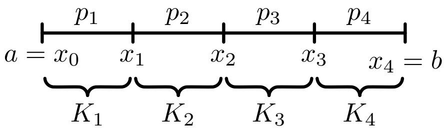
If technical assumptions and if

- $h_K \leq |\Omega|/3$
- $\kappa^2 h_K^2 \leq \min\left\{\alpha^{p_K}, \beta^{p_K}, \gamma^{p_K}\frac{h_K}{|\Omega| - h_K} + \delta^{p_K}\right\} \qquad \forall K \in \mathcal{T}_{hp}$

then DMP.

$$-u'' + \kappa^2 u = f \quad \text{in } \Omega = (a, b) \qquad u(a) = u(b) = 0$$

- $\mathcal{T}_{hp}$ mesh
- K element
- $h_K = \operatorname{diam} K$
- p_K poly. degree.



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then DMP.

p	α^p	β^p	γ^p	δ^p
1	∞	6	0	∞
2	20/3	∞	0	∞
3	38.61	25.89	5.608	0
4	18.91	∞	2.936	3.614
5	49.44	59.82	7.799	0
6	37.56	∞	7.247	0.887
7	72.82	107.81	9.791	0
8	62.62	∞	9.709	0
9	104.09	169.85	11.510	0
10	94.10	∞	10.644	0

Technical Assumptions (for $p \geq 3$)

- $\gamma^p \geq 3/2$
- $\omega^p(\theta, \gamma^p\theta + \delta^p, \xi, \eta) \geq 0 \quad$ for all $\theta \in (0, 1/2], \quad (\xi, \eta) \in [-1, 1]^2$

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$$\omega^p(\theta, \zeta, \xi, \eta) = s(0, \theta, \zeta)\Psi_1^p(\zeta, \xi)\Psi_1^p(\zeta, \eta) + \ell_0(\xi)\ell_0(\eta)\mathrm{Ker}^{b,p}(\zeta, \xi, \eta)$$

$$s(0, \theta, \zeta) = \left(r^p(\zeta) + \theta + \zeta/(3\theta)\right)^{-1}$$

$$\psi_1^{\mathrm{ref}}(\zeta, \xi) = \ell_1(\xi)\Psi_1^p(\zeta, \xi)$$

$$\ell_0(\xi) = (1 - \xi)/2, \qquad \ell_1(\xi) = (1 + \xi)/2$$

$$\mathrm{Ker}^{b,p}(\zeta, \xi, \eta) = \sum_{i=1}^{p-1} (1 + \zeta\mu_i^p)^{-1} \mathcal{K}_{i+1}^p(\xi)\mathcal{K}_{j+1}^p(\eta)$$

$$\mu_i^p = 1/(4\lambda_{i+1}^p)$$

$$\ell_i^p(\xi) = \ell_0(\xi)\ell_1(\xi)\mathcal{K}_i^p(\xi)$$

$$\left((\ell_i^p)', v'\right)_{[-1,1]} = \lambda_i^p(\ell_i^p, v)_{[-1,1]} \quad \forall v \in \mathbb{P}_0^p([-1, 1])$$

etc.

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17th Hilbert problem:

Any nonnegative polynomial can be written as a sum of squares of rational functions.

[Proved by Emil Artin in 1927].

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Interval arithmetics:

I, J intervals

$*$ an arithmetic operation

$R = I * J$ is an interval such that $\{r = a * b : a \in I,\ b \in J\} \subset R$

To verify that $f(x) \geq 0$ on I

1. If $f(I) \subset [0, \infty)$ then $f(x) \geq 0$ for all $x \in I$ and return to (a) or (b).
2. If not split $I = I_1 \cup I_2$ and
   (a) go to 1. with $I := I_1$
   (b) go to 1. with $I := I_2$

Matlab package INTLAB

Thank you for your attention

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Auxiliary DGF

$$0 \le \widehat{G}_{hp} \le \widetilde{G}_{hp} \le G_{hp} \quad \text{in } K^2$$