

Computable upper bounds on Friedrichs' and trace constants

Tomáš Vejchodský

Ivana Šebestová

Institute of Mathematics
Academy of Sciences
Czech Republic



Faculty of Mathematics and Physics
Charles University in Prague



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- ▶ Friedrichs', trace, and similar inequalities
- ▶ Relationship with the smallest eigenvalue
- ▶ Rayleigh–Ritz approximation
- ▶ Method of a priori–a posteriori inequalities
- ▶ Application for the trace constant
- ▶ Examples

Inequalities



Friedrichs' inequality:

$$\|v\|_{0,\Omega} \leq C_F \|\nabla v\|_{0,\Omega} \\ \forall v \in H_0^1(\Omega)$$

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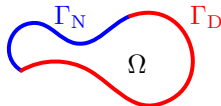
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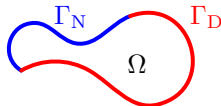
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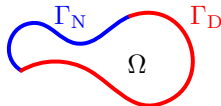
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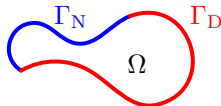
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General form:

$$\|u\|_c \leq C_{ac} \|u\|_a \quad \forall u \in V$$

V Hilbert with inner product $a(\cdot, \cdot)$

a, c continuous bilinear forms on V

$\|u\|_c^2 = c(u, u)$ seminorm on V

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$$u_i \in V$$

$$\begin{aligned} c(u_i, v) &= \lambda_i a(u_i, v) \\ &\quad \forall v \in V \end{aligned}$$

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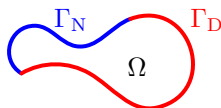
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Relation with the smallest eigenvalue

Space: $V = H_{\Gamma_D}^1(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$

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Theorem:

$$C_T^2 = \frac{1}{\lambda_1}, \quad \lambda_1 = \min_i \lambda_i$$

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Proof:

Weak formulation: $u_i \in V : (\nabla u_i, \nabla v)_\Omega = \lambda_i \langle u_i, v \rangle_{\Gamma_N} \quad \forall v \in V$

$$\lambda_1 = \inf_{v \in V} \frac{\|\nabla v\|_{0,\Omega}^2}{\|v\|_{0,\Gamma_N}^2} \Leftrightarrow \frac{1}{\lambda_1} = \sup_{v \in V} \frac{\|v\|_{0,\Gamma_N}^2}{\|\nabla v\|_{0,\Omega}^2} \quad \square$$

Rayleigh–Ritz approximation of λ_1

Weak formulation:

$$u_i \in V : \quad (\nabla u_i, \nabla v)_\Omega = \lambda_i \langle u_i, v \rangle_{\Gamma_N} \quad \forall v \in V$$

Rayleigh–Ritz method: $V^h \subset V$, $\dim V^h < \infty$

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Theorem: $\lambda_1 \leq \lambda_1^h$

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$$\lambda_1 = \inf_{v \in V} \frac{\|\nabla v\|_{0,\Omega}^2}{\|v\|_{0,\Gamma_N}^2} \leq \inf_{v^h \in V^h} \frac{\|\nabla v\|_{0,\Omega}^2}{\|v\|_{0,\Gamma_N}^2} = \lambda_1^h$$

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Corollary: $C_T^h \leq C_T$

Theorem (Kuttler and Sigillito, 1978):

- ▶ H a separable Hilbert space
- ▶ $A : H \mapsto H$ symmetric with pure point spectrum
- ▶ $Au_i = \lambda_i u_i$, $\{u_i\}$ ON basis of H
- ▶ $\lambda_* \in \mathbb{R}$, $u_* \in H$ arbitrary, $\|u_*\|_H = 1$
- ▶ $w \in H$ such that $Aw = Au_* - \lambda_* u_*$

Then

$$\min_i \left| \frac{\lambda_i - \lambda_*}{\lambda_i} \right| \leq \|w\|_H$$

- ▶ $Au_i = \lambda_i u_i$, $\lambda_* \in \mathbb{R}$, $u_* \in H$ arbitrary, $\|u_*\|_H = 1$
- ▶ $w \in H$: $Aw = Au_* - \lambda_* u_*$
- ▶ $1 = \|u_*\|_H^2 = \sum_i |\langle u_*, u_i \rangle_H|^2$

$$\begin{aligned}
 \min_i \left| \frac{\lambda_i - \lambda_*}{\lambda_i} \right|^2 \|u_*\|_H^2 &\leq \sum_i \left| \frac{\lambda_i - \lambda_*}{\lambda_i} \langle u_*, u_i \rangle_H \right|^2 \\
 &= \sum_i \frac{1}{\lambda_i^2} |\langle u_*, \lambda_i u_i \rangle_H - \langle \lambda_* u_*, u_i \rangle_H|^2 \\
 &= \sum_i \frac{1}{\lambda_i^2} |\langle u_*, Au_i \rangle_H - \langle Au_*, u_i \rangle_H + \langle Aw, u_i \rangle_H|^2 \\
 &= \sum_i \frac{1}{\lambda_i^2} |\langle Aw, u_i \rangle_H|^2 = \sum_i \frac{1}{\lambda_i^2} |\langle w, Au_i \rangle_H|^2 \\
 &= \sum_i \frac{1}{\lambda_i^2} |\langle w, \lambda_i u_i \rangle_H|^2 = \sum_i |\langle w, u_i \rangle_H|^2 = \|w\|_H^2
 \end{aligned}$$

Application to C_T



Steklov eigenproblem: $u_i \in V : (\nabla u_i, \nabla v)_\Omega = \lambda_i \langle u_i, v \rangle_{\Gamma_N} \quad \forall v \in V$

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Residual: $w \in V : (\nabla w, \nabla v)_\Omega = (\nabla u_1^h, \nabla v)_\Omega - \lambda_1^h \langle u_1^h, v \rangle_{\Gamma_N} \quad \forall v \in V$

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Theorem $\Rightarrow$ $\min_i \left| \frac{\lambda_i - \lambda_1^h}{\lambda_i} \right| \leq \|w\|_{0,\Gamma_N}$

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Lemma: $\forall \mathbf{q} \in \mathbf{H}(\operatorname{div}, \Omega)$
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Proof:

$$\begin{aligned} \|\nabla w\|^2 &= (\nabla u_1^h, \nabla w) - \langle \lambda_1^h u_1^h, w \rangle - (\mathbf{q}, \nabla w) - (\text{div } \mathbf{q}, w) + \langle \mathbf{q} \cdot \mathbf{n}, w \rangle \\ &= (\nabla u_1^h - \mathbf{q}, \nabla w) - (\text{div } \mathbf{q}, w) + \langle \mathbf{q} \cdot \mathbf{n} - \lambda_1^h u_1^h, w \rangle \\ &\leq \|\nabla u_1^h - \mathbf{q}\| \|\nabla w\| + \underbrace{\|\text{div } \mathbf{q}\|}_{\leq C_F \|\nabla w\|} \|w\| + \underbrace{\|\mathbf{q} \cdot \mathbf{n} - \lambda_1^h u_1^h\|_{0,\Gamma_N} \|w\|_{0,\Gamma_N}}_{\leq C_T \|\nabla w\|} \\ &\leq C_F \|\nabla w\| + C_T \|\nabla w\| \end{aligned}$$

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Lemma: $\forall \mathbf{q} \in \mathbf{H}(\text{div}, \Omega)$
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Proof:

$$\begin{aligned} \|\nabla w\|^2 &= (\nabla u_1^h, \nabla w) - \langle \lambda_1^h u_1^h, w \rangle - (\mathbf{q}, \nabla w) - (\text{div } \mathbf{q}, w) + \langle \mathbf{q} \cdot \mathbf{n}, w \rangle \\ &= (\nabla u_1^h - \mathbf{q}, \nabla w) - (\text{div } \mathbf{q}, w) + \langle \mathbf{q} \cdot \mathbf{n} - \lambda_1^h u_1^h, w \rangle \\ &\leq \|\nabla u_1^h - \mathbf{q}\| \|\nabla w\| + \underbrace{\|\text{div } \mathbf{q}\|}_{\leq C_F \|\nabla w\|} \|w\| + \underbrace{\|\mathbf{q} \cdot \mathbf{n} - \lambda_1^h u_1^h\|_{0,\Gamma_N}}_{\leq C_T \|\nabla w\|} \|w\|_{0,\Gamma_N} \end{aligned}$$

Application to C_T



Steklov eigenproblem: $u_i \in V : (\nabla u_i, \nabla v)_\Omega = \lambda_i \langle u_i, v \rangle_{\Gamma_N} \quad \forall v \in V$

Rayleigh–Ritz approximations: $\lambda_1^h \in \mathbb{R}, u_1^h \in V, \|u_1^h\|_{0,\Gamma_N} = 1$

Residual: $w \in V : (\nabla w, \nabla v)_\Omega = (\nabla u_1^h, \nabla v)_\Omega - \lambda_1^h \langle u_1^h, v \rangle_{\Gamma_N} \quad \forall v \in V$

Theorem $\Rightarrow \min_i \left| \frac{\lambda_i - \lambda_1^h}{\lambda_i} \right| \leq \|w\|_{0,\Gamma_N} \leq C_T \|\nabla w\|_{0,\Omega}$

Lemma: $\forall \mathbf{q} \in \mathbf{H}(\text{div}, \Omega)$
 $\|\nabla w\|_{0,\Omega} \leq \|\nabla u_1^h - \mathbf{q}\|_{0,\Omega} + C_F \|\text{div } \mathbf{q}\|_{0,\Omega} + C_T \|\mathbf{q} \cdot \mathbf{n} - \lambda_1^h u_1^h\|_{0,\Gamma_N}$

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Application to C_T



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Theorem $\Rightarrow$ $\min_i \left| \frac{\lambda_i - \lambda_1^h}{\lambda_i} \right| \leq \|w\|_{0,\Gamma_N} \leq C_T \|\nabla w\|_{0,\Omega} \leq C_T \alpha + C_T^2 \beta$

Lemma:

$$\|\nabla w\|_{0,\Omega} \leq \underbrace{\|\nabla u_1^h - \mathbf{q}\|_{0,\Omega} + C_F \|\operatorname{div} \mathbf{q}\|_{0,\Omega}}_{\alpha} + C_T \underbrace{\|\mathbf{q} \cdot \mathbf{n} - \lambda_1^h u_1^h\|_{0,\Gamma_N}}_{\beta} \quad \forall \mathbf{q} \in \mathbf{H}(\operatorname{div}, \Omega)$$

Application to C_T

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Assume: λ_1^h is closest to λ_1 .

Use: $C_T = 1/\sqrt{\lambda_1}$

$$\frac{\lambda_1^h - \lambda_1}{\lambda_1} \leq \frac{1}{\sqrt{\lambda_1}} \alpha + \frac{1}{\lambda_1} \beta$$

Application to C_T



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Use: $C_T = 1/\sqrt{\lambda_1}$

$$\frac{\lambda_1^h - \lambda_1}{\lambda_1} \leq \frac{1}{\sqrt{\lambda_1}} \alpha + \frac{1}{\lambda_1} \beta$$

$$\Leftrightarrow X_2^2 \leq \lambda_1, \quad \text{where } X_2 = \left(\sqrt{\alpha^2 + 4(\lambda_1^h - \beta)} - \alpha \right) / 2$$

Application to C_T

Steklov eigenproblem: $u_i \in V : (\nabla u_i, \nabla v)_\Omega = \lambda_i \langle u_i, v \rangle_{\Gamma_N} \quad \forall v \in V$

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Theorem $\Rightarrow$ $\min_i \left| \frac{\lambda_i - \lambda_1^h}{\lambda_i} \right| \leq \|w\|_{0,\Gamma_N} \leq C_T \|\nabla w\|_{0,\Omega} \leq C_T \alpha + C_T^2 \beta$

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$$\|\nabla w\|_{0,\Omega} \leq \underbrace{\|\nabla u_1^h - \mathbf{q}\|_{0,\Omega} + C_F \|\operatorname{div} \mathbf{q}\|_{0,\Omega}}_{\alpha} + C_T \underbrace{\|\mathbf{q} \cdot \mathbf{n} - \lambda_1^h u_1^h\|_{0,\Gamma_N}}_{\beta} \quad \forall \mathbf{q} \in \mathbf{H}(\operatorname{div}, \Omega)$$

Assume: λ_1^h is closest to λ_1 .

Use: $C_T = 1/\sqrt{\lambda_1}$

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$$\Leftrightarrow X_2^2 \leq \lambda_1, \quad \text{where } X_2 = \left(\sqrt{\alpha^2 + 4(\lambda_1^h - \beta)} - \alpha \right) / 2$$

$$\Leftrightarrow C_T \leq 1/X_2$$

Computing $\mathbf{q} \in \mathbf{H}(\text{div}, \Omega)$

Approximately minimize: $\eta(\mathbf{q}) \approx \alpha + C_T \beta$ $C_T \approx C_{T,h} = (\lambda_1^h)^{-1/2}$
 $C_F \approx C_{F,h}$

$$\begin{aligned} \eta^2(\mathbf{q}) &= \left(\underbrace{\|\nabla u_1^h - \mathbf{q}\|_{0,\Omega}}_A + \underbrace{C_{F,h} \|\text{div } \mathbf{q}\|_{0,\Omega}}_B + \underbrace{C_{T,h} \|\mathbf{q} \cdot \mathbf{n} - \lambda_1^h u_1^h\|_{0,\Gamma_N}}_C \right)^2 \\ &\leq \underbrace{(1 + \varrho^{-1})(1 + \sigma^{-1})A^2 + (1 + \varrho^{-1})(1 + \sigma)B^2 + (1 + \varrho)C^2}_{\tilde{\eta}^2(\varrho, \sigma, \mathbf{q})} \quad \forall \varrho > 0, \sigma > 0 \end{aligned}$$

Minimize $\tilde{\eta}(\varrho, \sigma, \mathbf{q})^2$ over $W_h \subset \mathbf{H}(\text{div}, \Omega)$: Find $\mathbf{q}_h \in W_h$:

$$\begin{aligned} &\frac{C_{F,h}^2 \lambda_1^h}{\varrho} (\text{div } \mathbf{q}_h, \text{div } \psi_h)_\Omega + \frac{\lambda_1^h}{\varrho \sigma} (\mathbf{q}_h, \psi_h)_\Omega + \frac{1}{1 + \sigma} \langle \mathbf{q}_h \cdot \mathbf{n}, \psi_h \cdot \mathbf{n} \rangle_{\Gamma_N} \\ &= \frac{\lambda_1^h}{\varrho \sigma} (\nabla u_1^h, \psi_h)_\Omega + \frac{1}{1 + \sigma} \langle \lambda_1^h u_1^h, \psi_h \cdot \mathbf{n} \rangle_{\Gamma_N} \quad \forall \psi \in W_h \end{aligned}$$

Solve by standard Raviart-Thomas finite elements.

Example 1

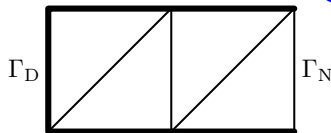
$$\Omega = (0, 2) \times (0, 1)$$

$$\Gamma_D = \{(x, y) \in \partial\Omega : x < 2\}$$

$$\Gamma_N = \{(x, y) \in \partial\Omega : x = 2\}$$

$$\|v\|_{0,\Gamma_N} \leq C_T \|\nabla v\|_{0,\Omega} \quad \forall v \in H_{\Gamma_D}^1(\Omega)$$

$$C_T = (\pi \coth 2\pi)^{-1/2}$$



Example 1

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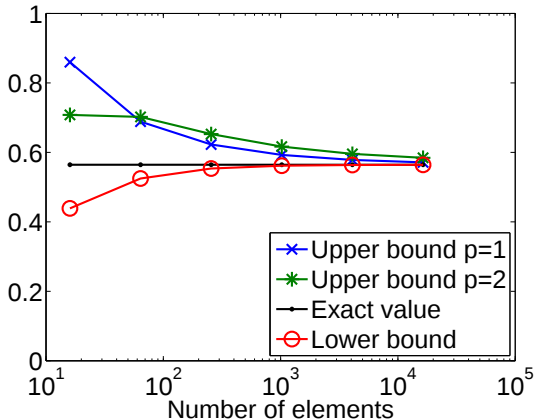
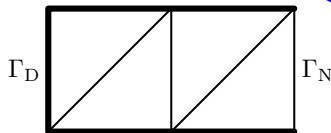
$$C_T = (\pi \coth 2\pi)^{-1/2}$$

Trace constant – Example 1

$$C_T^{\text{up}} = 0.571154$$

$$C_T \approx 0.564188$$

$$C_T^{\text{low}} = 0.564017$$



Example 2

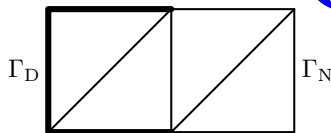
$$\Omega = (0, 2) \times (0, 1)$$

$$\Gamma_D = \{(x, y) \in \partial\Omega : x < 1\}$$

$$\Gamma_N = \{(x, y) \in \partial\Omega : x > 1\}$$

$$\|v\|_{0, \Gamma_N} \leq C_T \|\nabla v\|_{0, \Omega} \quad \forall v \in H_{\Gamma_D}^1(\Omega)$$

$$C_T = ?$$



Example 2



$$\Omega = (0, 2) \times (0, 1)$$

$$\Gamma_D = \{(x, y) \in \partial\Omega : x < 1\}$$

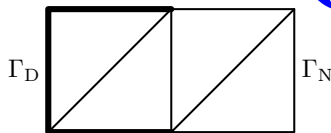
$$\Gamma_N = \{(x, y) \in \partial\Omega : x > 1\}$$

$$\|v\|_{0,\Gamma_N} \leq C_T \|\nabla v\|_{0,\Omega} \quad \forall v \in H_{\Gamma_D}^1(\Omega)$$

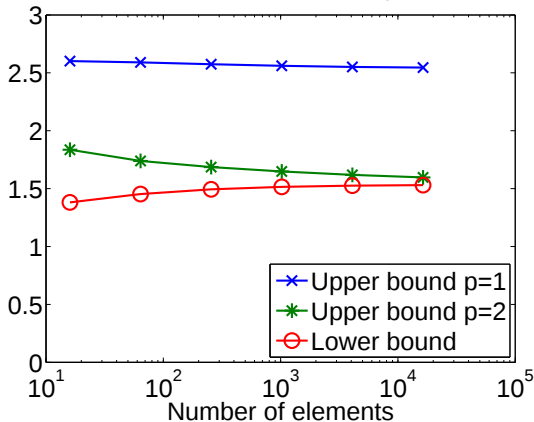
$$C_T = ?$$

$$C_T^{\text{up}} = 1.59698$$

$$C_T^{\text{low}} = 1.53059$$



Trace constant – Example 2



- ▶ Practical method
- ▶ Upper bound
- ▶ General: trace, Friedrichs', Poincaré, Korn's, ... constants
- ▶ Computationally demanding
- ▶ Exact representation of the domain Ω

Thank you for your attention

Tomáš Vejchodský

Ivana Šebestová

Institute of Mathematics
Academy of Sciences
Czech Republic



Faculty of Mathematics and Physics
Charles University in Prague



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