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On some aspects of the hp-FEM for time-harmonic Maxwell's equations

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Time harmonic Maxwell's equations

$$\mathbf{curl}\left(\mu_{\mathsf{r}}^{-1}\,\mathrm{curl}\,\mathbf{E}\right) - \kappa^2\epsilon_{\mathsf{r}}\mathbf{E} = \mathbf{F} \quad \text{in } \Omega,$$

where

- $\mathbf{curl} = (\partial/\partial x_2, -\partial/\partial x_1)^\top$
- $\mathrm{curl}\,\mathbf{E} = \partial E_2/\partial x_1 - \partial E_1/\partial x_2$
- $\Omega \subset \mathbb{R}^2$
- $\mu_{\mathsf{r}} = \mu_{\mathsf{r}}(x) \in \mathbb{R}$ relative permeability
- $\epsilon_{\mathsf{r}} = \epsilon_{\mathsf{r}}(x) \in \mathbb{C}^{2\times 2}$ relative permittivity
- $\mathbf{E} = \mathbf{E}(x) \in \mathbb{C}^2$ phaser of the electric field intensity
- $\mathbf{F} = \mathbf{F}(x) \in \mathbb{C}^2$
- $\kappa \in \mathbb{R}$ the wave number

Time harmonic Maxwell's equations + boundary conditions

$$\mathbf{curl}\left(\mu_{\mathsf{r}}^{-1}\,\mathsf{curl}\,\mathbf{E}\right) - \kappa^2\epsilon_{\mathsf{r}}\mathbf{E} = \mathbf{F} \quad \text{in } \Omega$$

Perfect conducting boundary conditions:

$$\mathbf{E}\cdot\tau = 0, \quad \text{on } \Gamma_P.$$

Impedance boundary conditions:

$$\mu_{\mathsf{r}}^{-1}\,\mathsf{curl}\,\mathbf{E} - \mathrm{i}\kappa\lambda\mathbf{E}\cdot\tau = \mathbf{g}\cdot\tau \quad \text{on } \Gamma_I.$$

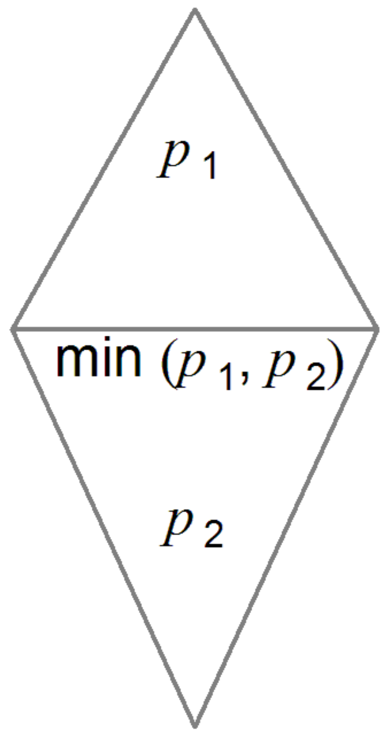
Here,

- $\tau = (-\nu_2, \nu_1)^\top$ positively oriented unit tangent vector
- $\lambda = \lambda(x) > 0$ impedance
- $\mathbf{g} = \mathbf{g}(x) \in \mathbb{C}^2$

Weak and hp-FEM formulations

$$V = \{\mathbf{E} \in \mathbf{H}(\mathrm{curl}, \Omega) : \mathbf{E} \cdot \tau = 0 \text{ on } \Gamma_P\}$$

$$\mathbf{E} \in V : \quad \boxed{a(\mathbf{E}, \mathbf{\Phi}) = \mathcal{F}(\mathbf{\Phi})} \quad \forall \mathbf{\Phi} \in V$$



$$V_h = \Big\{\mathbf{E}_h \in V : \mathbf{E}_h|_{K_j} \in \mathbf{P}^{p_j}(K_j) \text{ and } \mathbf{E}_h \cdot \tau_k \text{ is continuous on each edge } e_k\}$$

$$\mathbf{E}_h \in V_h : \quad \boxed{a(\mathbf{E}_h, \mathbf{\Phi}_h) = \mathcal{F}(\mathbf{\Phi}_h)} \quad \forall \mathbf{\Phi}_h \in V_h$$

$$a(\mathbf{E}, \mathbf{\Phi}) = \left(\mu_{\mathrm{r}}^{-1} \mathrm{curl}\, \mathbf{E}, \mathrm{curl}\, \mathbf{\Phi}\right) - \kappa^2 (\epsilon_{\mathrm{r}} \mathbf{E}, \mathbf{\Phi}) - \mathrm{i}\kappa \langle \lambda \mathbf{E} \cdot \tau, \mathbf{\Phi} \cdot \tau \rangle$$

$$\mathcal{F}(\mathbf{\Phi}) = (F, \mathbf{\Phi}) + \langle \mathbf{g}, \mathbf{\Phi} \cdot \tau \rangle$$

$$\boxed{\mathbf{E}_h = \sum_j^N \underbrace{c_j}_{\in \mathbb{C}} \psi_j} \qquad \psi_j \ \ldots \ \text{hierarchic basis}$$

Shape functions

Whitney functions:

$$\hat{\psi}_0^{e1} = \frac{1}{\|e_1\|}\left(\frac{\lambda_3 \mathbf{n}_2}{\mathbf{n}_2 \cdot \mathbf{t}_1} + \frac{\lambda_2 \mathbf{n}_3}{\mathbf{n}_3 \cdot \mathbf{t}_1}\right)$$

$$\hat{\psi}_0^{e2} = \frac{1}{\|e_2\|}\left(\frac{\lambda_1 \mathbf{n}_3}{\mathbf{n}_3 \cdot \mathbf{t}_2} + \frac{\lambda_3 \mathbf{n}_1}{\mathbf{n}_1 \cdot \mathbf{t}_2}\right)$$

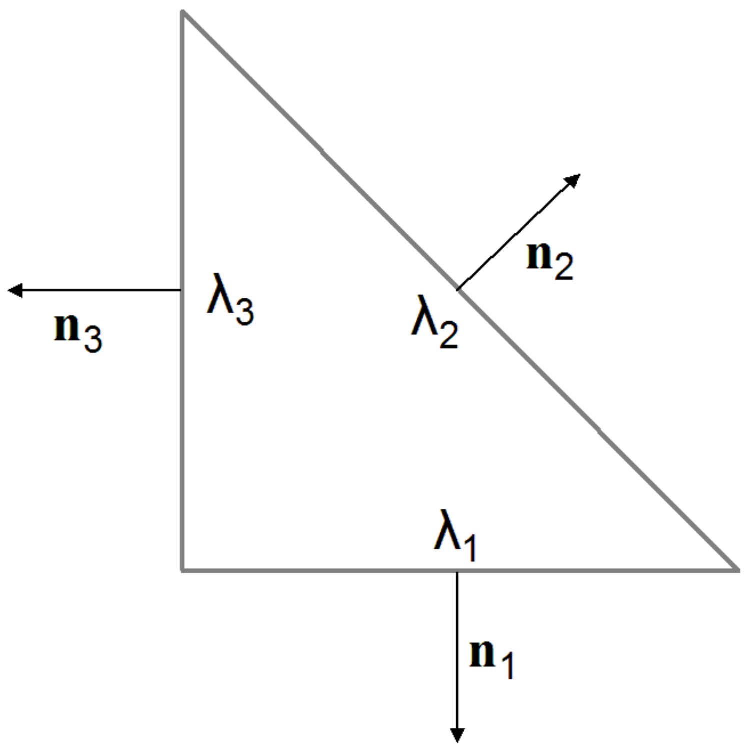
$$\hat{\psi}_0^{e3} = \frac{1}{\|e_3\|}\left(\frac{\lambda_2 \mathbf{n}_1}{\mathbf{n}_1 \cdot \mathbf{t}_3} + \frac{\lambda_1 \mathbf{n}_2}{\mathbf{n}_2 \cdot \mathbf{t}_3}\right)$$

First order functions:

$$\hat{\psi}_1^{e1} = \frac{1}{\|e_1\|}\left(\frac{\lambda_3 \mathbf{n}_2}{\mathbf{n}_2 \cdot \mathbf{t}_1} - \frac{\lambda_2 \mathbf{n}_3}{\mathbf{n}_3 \cdot \mathbf{t}_1}\right)$$

$$\hat{\psi}_1^{e2} = \frac{1}{\|e_2\|}\left(\frac{\lambda_1 \mathbf{n}_3}{\mathbf{n}_3 \cdot \mathbf{t}_2} - \frac{\lambda_3 \mathbf{n}_1}{\mathbf{n}_1 \cdot \mathbf{t}_2}\right)$$

$$\hat{\psi}_1^{e3} = \frac{1}{\|e_3\|}\left(\frac{\lambda_2 \mathbf{n}_1}{\mathbf{n}_1 \cdot \mathbf{t}_3} - \frac{\lambda_1 \mathbf{n}_2}{\mathbf{n}_2 \cdot \mathbf{t}_3}\right)$$



$$\mathbf{t}_i = \begin{bmatrix} -\mathbf{n}_{i,2} \\ \mathbf{n}_{i,1} \end{bmatrix}$$

Edge functions:

$$\widehat{\psi}_k^{e_1} = \frac{2k-1}{k} L_{k-1}(\lambda_3 - \lambda_2)\widehat{\psi}_1^{e_1} - \frac{k-1}{k} L_{k-2}(\lambda_3 - \lambda_2)\widehat{\psi}_0^{e_1},$$
$$\widehat{\psi}_k^{e_2} = \frac{2k-1}{k} L_{k-1}(\lambda_1 - \lambda_3)\widehat{\psi}_1^{e_2} - \frac{k-1}{k} L_{k-2}(\lambda_1 - \lambda_3)\widehat{\psi}_0^{e_2},$$
$$\widehat{\psi}_k^{e_3} = \frac{2k-1}{k} L_{k-1}(\lambda_2 - \lambda_1)\widehat{\psi}_1^{e_3} - \frac{k-1}{k} L_{k-2}(\lambda_2 - \lambda_1)\widehat{\psi}_0^{e_3}, \quad k = 2, 3, \ldots$$

Edge based bubble functions:

$$\widehat{\psi}_k^{b,e_1} = \lambda_3\lambda_2 L_{k-2}(\lambda_3 - \lambda_2)\mathbf{n}_1,$$
$$\widehat{\psi}_k^{b,e_2} = \lambda_1\lambda_3 L_{k-2}(\lambda_1 - \lambda_3)\mathbf{n}_2,$$
$$\widehat{\psi}_k^{b,e_3} = \lambda_2\lambda_1 L_{k-2}(\lambda_2 - \lambda_1)\mathbf{n}_3, \quad k = 2, 3, \ldots$$

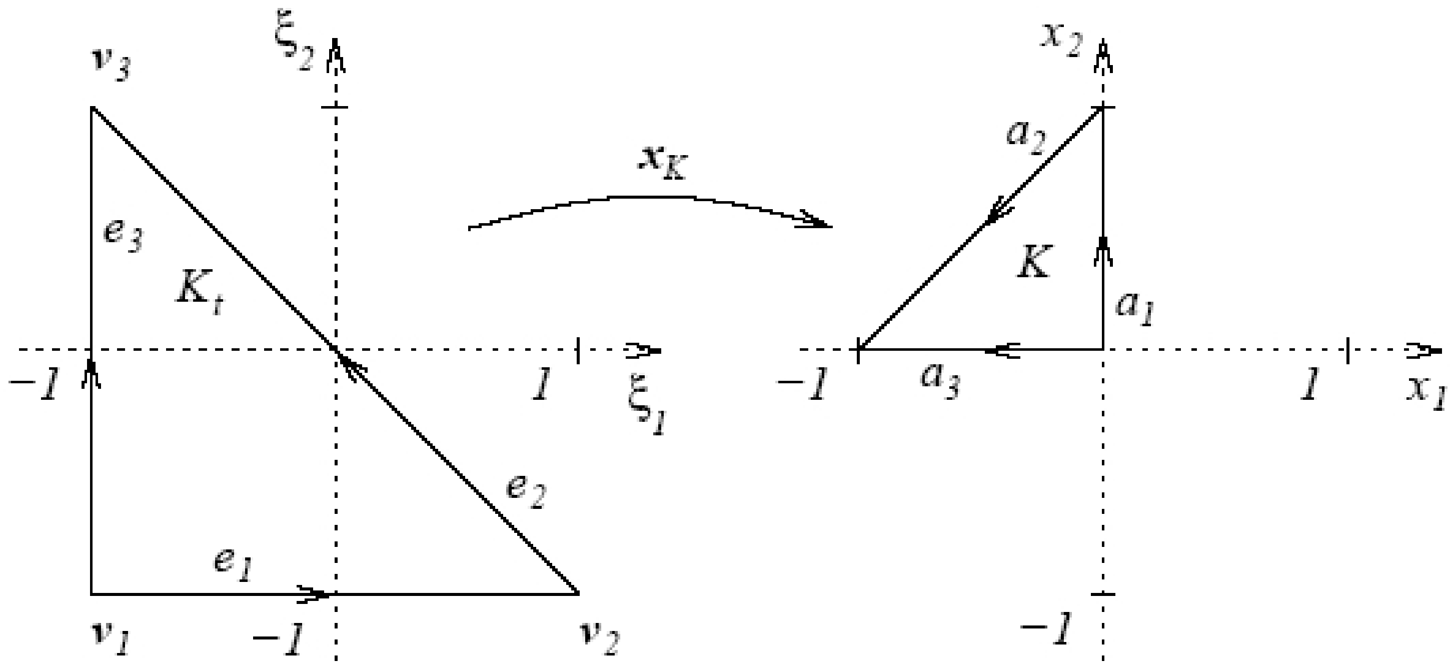
Genuine bubble functions:

$$\widehat{\psi}_{n_1,n_2}^{b,1} = \lambda_1\lambda_2\lambda_3 L_{n_1-1}(\lambda_3 - \lambda_2) L_{n_2-1}(\lambda_2 - \lambda_1) \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$\widehat{\psi}_{n_1,n_2}^{b,2} = \lambda_1\lambda_2\lambda_3 L_{n_1-1}(\lambda_3 - \lambda_2) L_{n_2-1}(\lambda_2 - \lambda_1) \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad 1 \le n_1, n_2$$

Simple transformation does not work

$$x_K^{-1}(x) = \begin{bmatrix} 2x_2 - 1 \\ -2x_1 - 1 \end{bmatrix} \qquad \hat{\psi}_0^{e_1}(\xi) = \frac{1}{4}\begin{bmatrix} 1 - \xi_2 \\ 1 + \xi_1 \end{bmatrix}$$

$$\psi_{K,0}^{a_1}(x) = \hat{\psi}_0^{e_1}\left(x_K^{-1}(x)\right) = \frac{1}{2}\begin{bmatrix} 1 + x_1 \\ x_2 \end{bmatrix} \qquad \psi_{K,0}^{a_1} \cdot \mathbf{t_1} = \frac{1}{2}x_2 \text{ non-constant}$$



Correct transformation

$$\xi = x_K^{-1}(x)$$

$$\hat{\mathbf{t}}_i = \frac{\|a_i\|}{\|e_i\|}\left(\frac{\mathsf{D}x_K}{\mathsf{D}\xi}\right)^{-1}\mathbf{t}_i$$

$$\mathsf{d}\xi = \frac{\|e_i\|}{\|a_i\|}\,\mathsf{d}x$$

$$\int_{e_i}\widehat{\psi}(\xi)\cdot\hat{\mathbf{t}}_i\,\mathsf{d}\xi = \int_{a_i}\psi_K(x)\cdot\mathbf{t}_i\,\mathsf{d}x$$

$$\int_{a_i}\widehat{\psi}(x_K^{-1}(x))\cdot\left[\frac{\|a_i\|}{\|e_i\|}\left(\frac{\mathsf{D}x_K}{\mathsf{D}\xi}\right)^{-1}\mathbf{t}_i\right]\,\frac{\|e_i\|}{\|a_i\|}\,\mathsf{d}x = \int_{a_i}\psi_K(x)\cdot\mathbf{t}_i\,\mathsf{d}x$$

$$\int_{a_i}\left[\underline{\left(\frac{\mathsf{D}x_K}{\mathsf{D}\xi}\right)^{-\top}\widehat{\psi}(x_K^{-1}(x))}\right]\cdot\mathbf{t}_i\,\mathsf{d}x = \int_{a_i}\underline{\psi_K(x)}\cdot\mathbf{t}_i\,\mathsf{d}x$$

$$\boxed{\psi_K(x) = \left(\frac{\mathsf{D}x_K}{\mathsf{D}\xi}\right)^{-\top}\widehat{\psi}(x_K^{-1}(x))}$$

HERMES_2D – hp-FEM solver

$\boxed{H^1}$ Systems of elliptic (non)linear PDE

$$\frac{\partial}{\partial x}\left(P_1\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial x}\left(P_2\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(P_3\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(P_4\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial x}(P_5 u)+\frac{\partial}{\partial y}(P_6 u)+P_7 u = f$$

$\boxed{\mathbf{H}(\text{curl})}$ Time harmonic Maxwell's equations

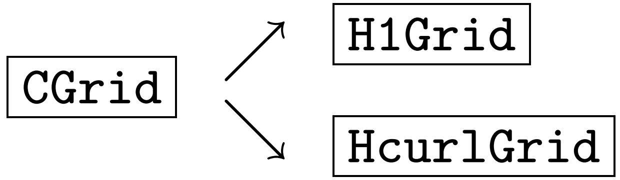
$\boxed{\mathbf{H}(\text{div})}$ . . .

Modularity of HERMES_2D

Independent modules

- Quadrature
- I/O
- sMatrix
  - own solvers
  - external libraries (`Trilinos`, `PETSc`, `UMFPACK`)

Modular hp-FEM code



Elements
Vertices
Nodes
Read Grid file
Preprocessing
Refinement

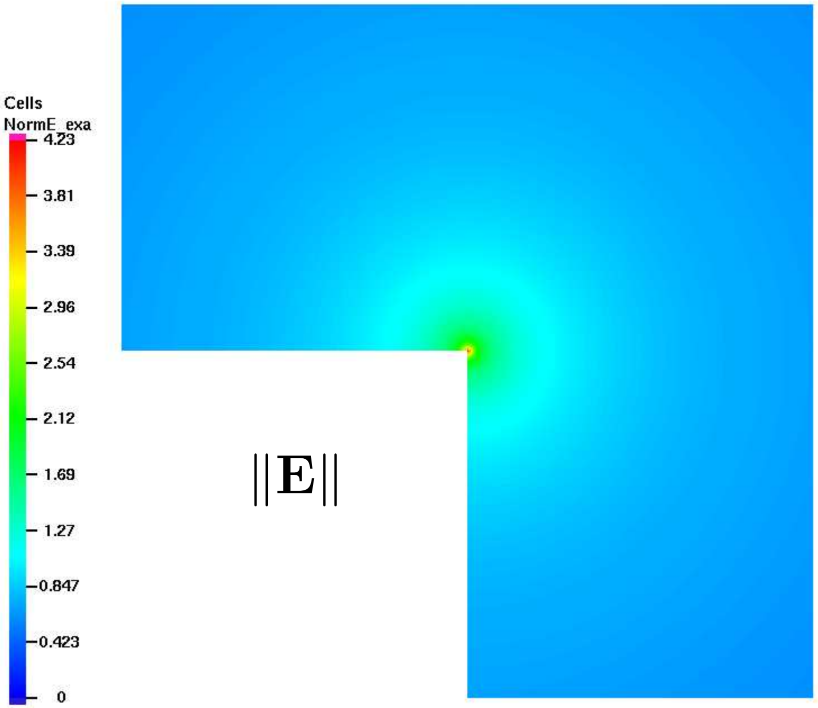
Boundary cond.
DOFs Allocation

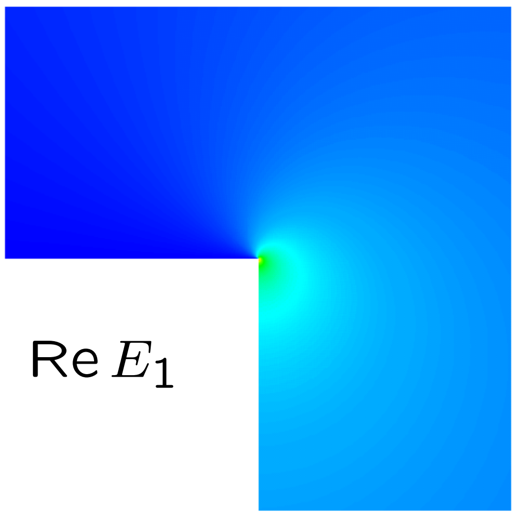
New bubble functions

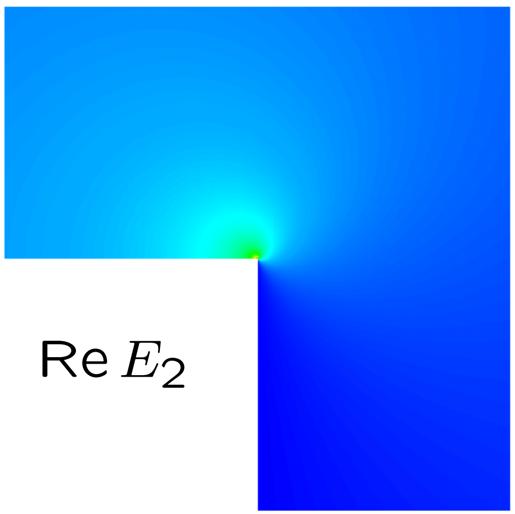
Orthonormal in $(\text{curl}\, u, \text{curl}\, v) + (u, v)$ on a reference triangle.

Condition number				
p	# DOF	old bubbles	new bubbles	improvement
0	80	$3.9 \cdot 10^{2}$	$3.9 \cdot 10^{2}$	$1.0 \cdot 10^{0}$
1	160	$1.0 \cdot 10^{3}$	$1.0 \cdot 10^{3}$	$1.0 \cdot 10^{0}$
2	384	$7.8 \cdot 10^{3}$	$3.3 \cdot 10^{3}$	$2.3 \cdot 10^{0}$
3	704	$3.3 \cdot 10^{5}$	$7.5 \cdot 10^{3}$	$4.4 \cdot 10^{1}$
4	1120	$4.9 \cdot 10^{6}$	$1.9 \cdot 10^{4}$	$2.6 \cdot 10^{2}$
5	1632	$9.1 \cdot 10^{7}$	$4.7 \cdot 10^{4}$	$1.9 \cdot 10^{3}$
6	2240	$1.3 \cdot 10^{9}$	$9.8 \cdot 10^{4}$	$1.3 \cdot 10^{4}$
7	2944	$3.2 \cdot 10^{10}$	$1.8 \cdot 10^{5}$	$1.8 \cdot 10^{5}$
8	3744	$7.0 \cdot 10^{11}$	$3.6 \cdot 10^{5}$	$1.9 \cdot 10^{6}$
9	4640	$1.4 \cdot 10^{13}$	$5.9 \cdot 10^{5}$	$2.4 \cdot 10^{7}$
10	5632	$3.3 \cdot 10^{14}$	$9.3 \cdot 10^{5}$	$3.5 \cdot 10^{8}$

Example 1







$$u = r^{\frac{2}{3}} \sin\left(\tfrac{2}{3}\theta + \tfrac{\pi}{3}\right)$$

$$\mathbf{E} = \nabla u$$

$$\mathbf{E} = \tfrac{2}{3} r^{-\frac{1}{3}} \begin{bmatrix} \cos\left(\frac{\pi}{6} + \frac{\theta}{3}\right) \\ \sin\left(\frac{\pi}{6} + \frac{\theta}{3}\right) \end{bmatrix}$$

$$\mathbf{F} = -\mathbf{E}$$

$$\mu_{\mathrm{r}} = 1$$

$$\epsilon_{\mathrm{r}} = I$$

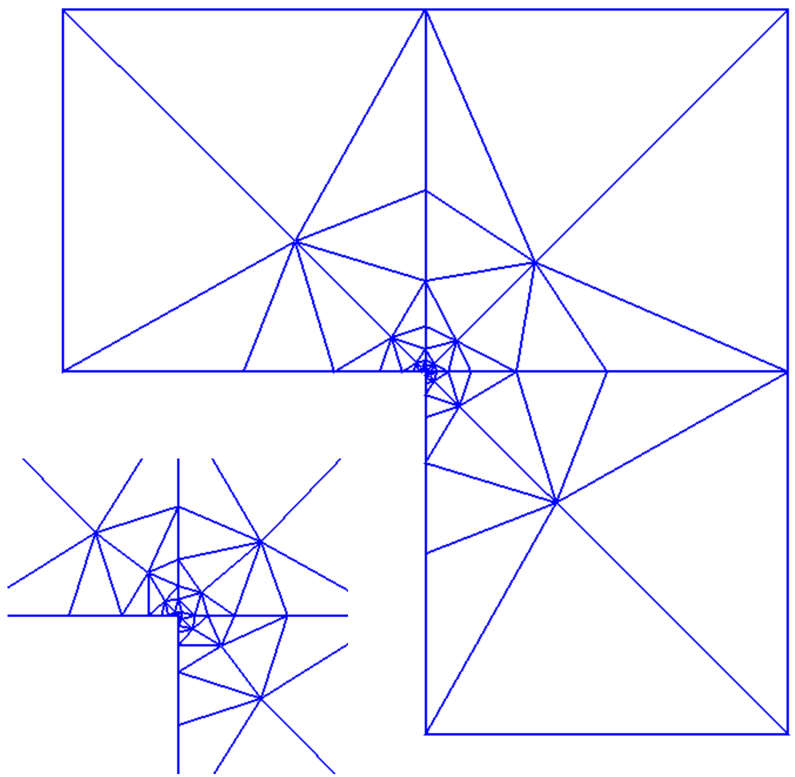
$$\kappa = 1$$

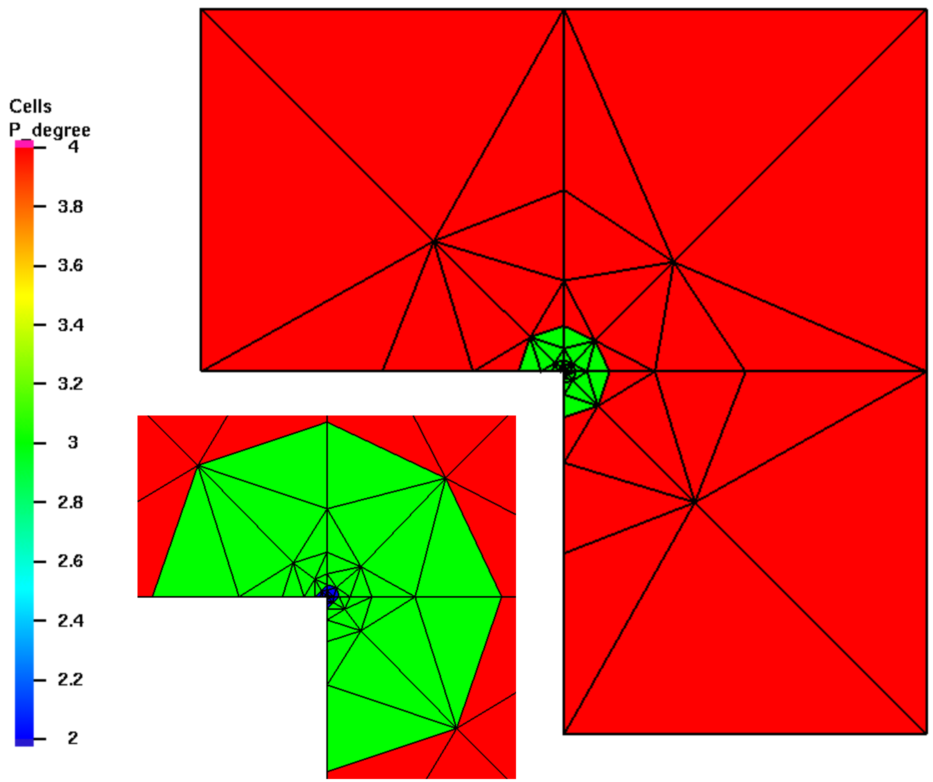
$$\lambda = 1$$

$$\mathbf{g} = \ldots$$

Example 1

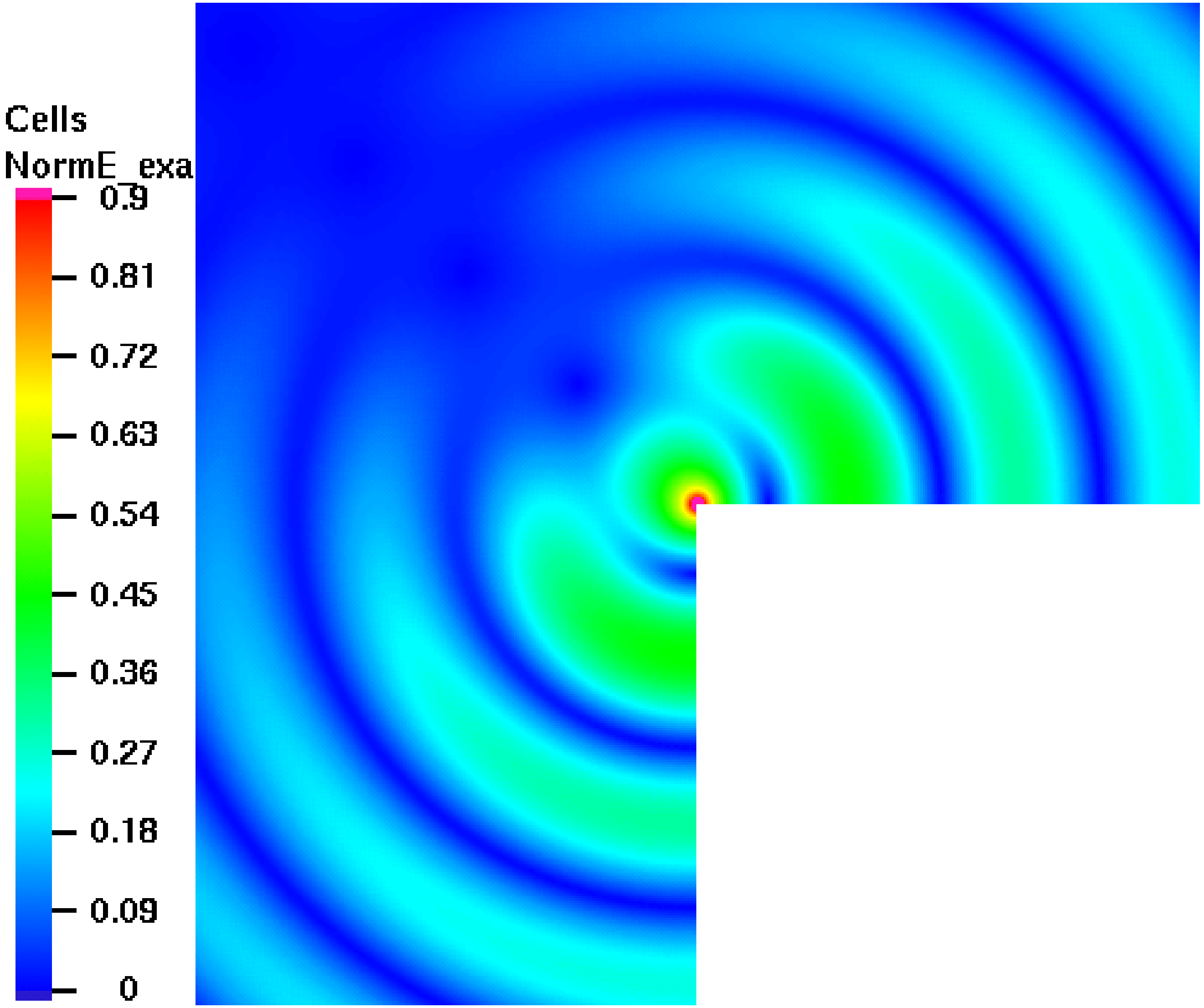
	DOFs	CPU time	$\|\mathrm{Err}\|_{\mathbf{H}(\mathrm{curl})}/\|\mathbf{E}\|_{\mathbf{H}(\mathrm{curl})}$
$p = 0$	2 758 400	11 min 26 s	0.156 %
hp	2 732	0.55 s	0.138 %
Improvement	1 010×	1 247×	





refinement 100

Example 2 (P. Monk, 2003)



$u = J_{\frac{2}{3}}(r)\cos\left(\frac{2}{3}\theta\right)$

$\mathbf{E} = \mathbf{curl}\, u$

$\mathbf{F} = 0$

$\mu_{\mathrm{r}} = 1$

$\epsilon_{\mathrm{r}} = I$

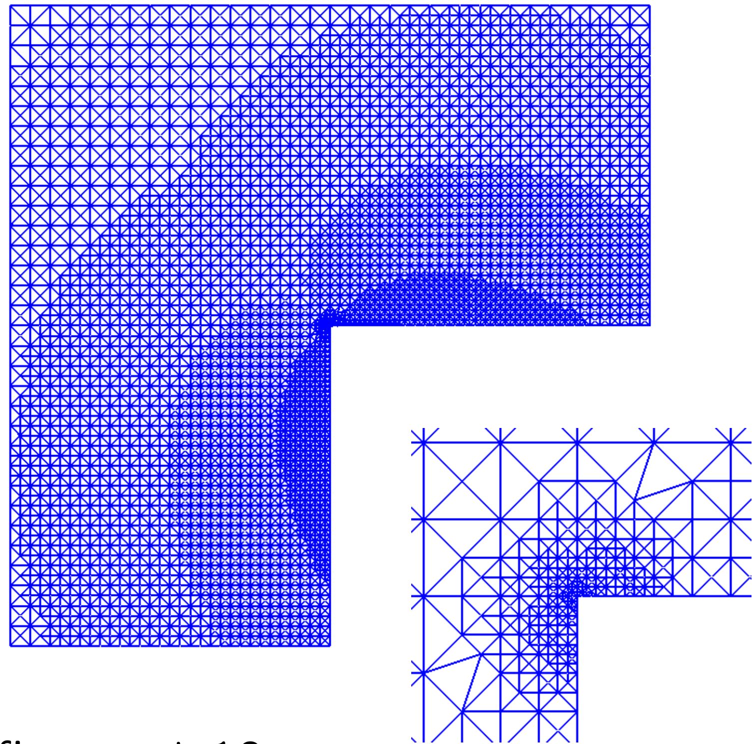
$\kappa = 1$

$\lambda = 1$

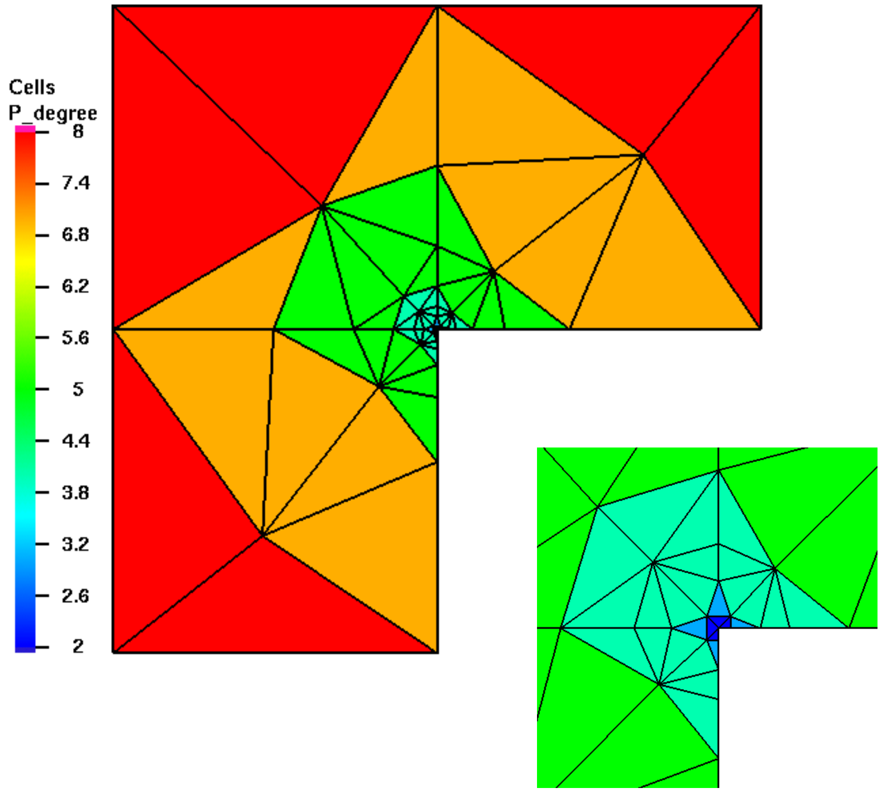
$\mathbf{g} = \dots$

Example 2

	DOFs	CPU time	$\|\text{Err}\|_{\mathbf{H}(\text{curl})}/\|\mathbf{E}\|_{\mathbf{H}(\text{curl})}$
$p = 0$	2 586 540	21 min 12 s	0.645 %
hp	4 324	2.49 s	0.621 %
Improvement	598×	511×	



refinement 10



Outlook

- H^1 and $\mathbf{H}$(curl) conforming elements in 3D
- $\mathbf{H}$(div) conforming elements in 2D and 3D
- parallelization
- a posteriori error estimates
- automatic hp-adaptivity
- orthonormalization of the bubble functions
  (investigation of the non-affine hierarchic elements)
- . . . .

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Thank you for your attention.

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