Discrete Green's function and Maximum Principles

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Model problem

Classical formulation

$$
\begin{aligned}
-\operatorname{div}(\mathcal{A}\nabla u) + cu &= f && \text{in } \Omega\\
u &= 0 && \text{on } \Gamma_{\mathrm{D}}\\
\alpha u + (\mathcal{A}\nabla u)\cdot\nu &= g_{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}}
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Weak formulation:

$$V = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_{\mathrm{D}}\}$$

$$u \in V : a(u,v) = F(v) \quad \forall v \in V$$

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\begin{aligned}
a(u,v) &= \int_\Omega (\mathcal{A}\nabla u)\cdot\nabla v \,\mathrm{d}x + \int_\Omega cuv \,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} \alpha uv \,\mathrm{d}s \\
F(v) &= \int_\Omega fv \,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} g_{\mathrm{N}} v \,\mathrm{d}s
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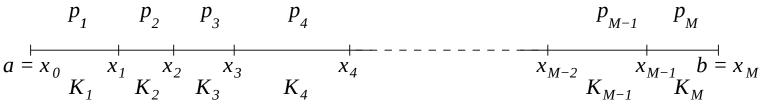
$$F(v) = \int_\Omega fv \,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} g_{\mathrm{N}} v \,\mathrm{d}s$$

For $y \in \overline{\Omega}$ find $\quad G_y \in V : a(w, G_y) = \delta_y(w) \quad \forall w \in V$

$$u(y) = \delta_y(u) = a(u, G_y) = F(G_y)$$

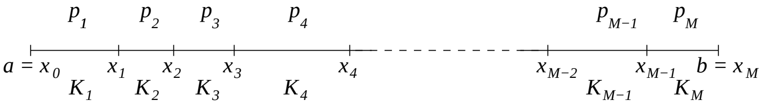
Discretization by *hp*-FEM



$$V_{hp} = \{ v_{hp} \in V : v_{hp}|_{K_i} \in P^{p_i}(K_i) \}$$

$$u_{hp} \in V_{hp} : a(u_{hp}, v_{hp}) = F(v_{hp}) \quad \forall v_{hp} \in V_{hp}$$

Discretization by *hp*-FEM



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Discrete Green's function: for $y \in \overline{\Omega}$

$$G_{hp,y} \in V_{hp} : a(w_{hp}, G_{hp,y}) = \delta_y(w_{hp}) \quad \forall w_{hp} \in V_{hp}$$

$$u_{hp}(y) = \delta_y(u_{hp}) = a(u_{hp}, G_{hp,y}) = F(G_{hp,y})$$

$$u_{hp}(y) = \int_\Omega f(x) G_{hp}(x,y) \,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} g_\mathrm{N}(s) G_{hp}(s,y) \,\mathrm{d}s$$

$$G_{hp}(x,y) = G_{hp,y}(x)$$

Properties of DGF

Lemma

Let $\{\varphi_1, \varphi_2, \dots, \varphi_N\}$ *be a basis in* V_{hp}*. If* $A_{ij} = a(\varphi_j, \varphi_i)$ *then*

$$G_{hp}(x,y) = \sum_{j=1}^{N} \sum_{k=1}^{N} A^{-1}_{jk} \varphi_k(x) \varphi_j(y), \quad \text{where} \sum_{j=1}^{N} A_{ij} A^{-1}_{jk} = \delta_{ik}.$$

Proof.

$$a(v_{hp}, G_{hp,y}) = v_{hp}(y)$$

$$G_{hp}(x,y) = \sum_{i=1}^{N} c_i(y) \varphi_i(x)$$

$$v_{hp} = \varphi_j$$

$$\sum_{i=1}^{N} c_i(y) \underbrace{a(\varphi_j, \varphi_i)}_{A_{ij}} = \varphi_j(y)$$

$$c_k(y) = \sum_{j=1}^{N} \varphi_j(y) A^{-1}_{jk}$$

□

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Corollary

If $a(\cdot,\cdot)$ *is symmetric then* $G_{hp}(x,y) = G_{hp}(y,x)$*.*

Properties of DGF

Lemma

Let $\{\varphi_1, \varphi_2, \dots, \varphi_N\}$ *be a basis in* V_{hp}. *If* $A_{ij} = a(\varphi_j, \varphi_i)$ *then*

$$G_{hp}(x,y) = \sum_{j=1}^{N} \sum_{k=1}^{N} A_{jk}^{-1} \varphi_k(x) \varphi_j(y), \quad \textit{where} \sum_{j=1}^{N} A_{ij} A_{jk}^{-1} = \delta_{ik}.$$

Corollary

If $a(\cdot,\cdot)$ *is symmetric then* $G_{hp}(x,y) = G_{hp}(y,x)$.

Corollary

Let $\{l_1, l_2, \cdots, l_N\}$ *be a basis of* V_{hp} *such that* $a(l_i, l_j) = \delta_{ij}$. *Then*

$$G_{hp}(x,z) = \sum_{i=1}^{N} l_i(x) l_i(z).$$

Discrete Comparison Principle

Definition

$f \geq 0$ and $g_{\mathrm{N}} \geq 0 \quad \Rightarrow \quad u_{hp} \geq 0$

Lemma

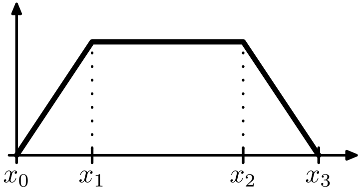
Discrete comparison principle $\quad \Leftrightarrow \quad G_{hp}(x,y) \geq 0$ *in* $\overline{\Omega}^2$.

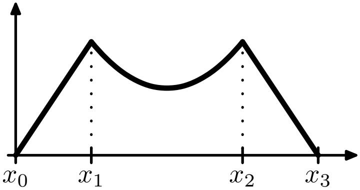
Proof.

$$u_{hp}(y) = \int_{\Omega} f(x) G_{hp}(x,y)\,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} g_{\mathrm{N}}(s) G_{hp}(s,y)\,\mathrm{d}s$$

□

Troubles with higher-order elements





DCP for mixed BC in 1D

$$-u'' = f \quad \text{in } \Omega = (a,b) \qquad a(u,v) = \int_a^b u'v' \, \mathrm{d}x$$

$$u(a) = 0 \qquad F(v) = \int_a^b fv \, \mathrm{d}x + g_{\mathrm{N}} v(b)$$

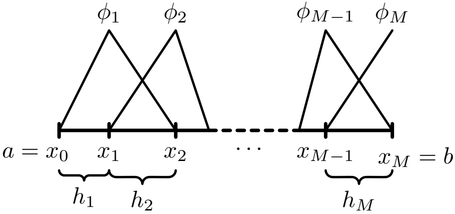
$$-u'(b) = g_{\mathrm{N}} \quad g_{\mathrm{N}} \in \mathbb{R}$$

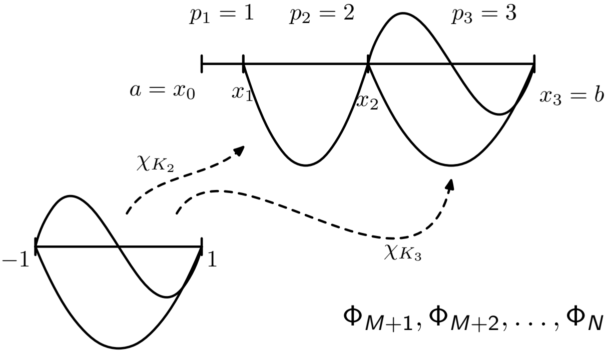
$$u_{hp} \in V_{hp} : a(u_{hp}, v_{hp}) = F(v_{hp}) \quad \forall v_{hp} \in V_{hp}$$

$$G_{hp,y} \in V_{hp} : a(w_{hp}, G_{hp,y}) = \delta_y(w_{hp}) \quad \forall w_{hp} \in V_{hp}$$

$$u_{hp}(y) = F(G_{hp,y}) = \int_a^b f(x) G_{hp}(x,y) \, \mathrm{d}x + g_{\mathrm{N}} G_{hp}(b,y)$$

hp-FEM basis in 1D



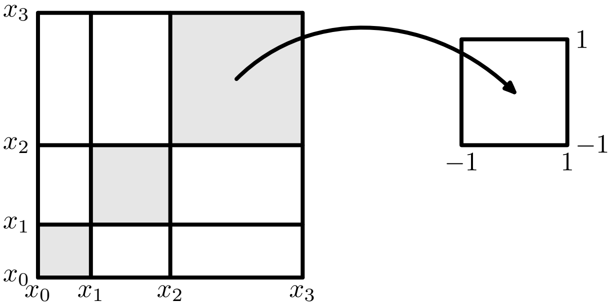


$\Phi_{M+1}, \Phi_{M+2}, \ldots, \Phi_N$

Explicit expression of DGF

$$A = \begin{pmatrix} A^L & 0 \\ 0 & D \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} (A^L)^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix}$$

$$G_{hp}(x,y) = \sum_{j=1}^{N} \sum_{k=1}^{N} A^{-1}_{jk} \Phi_k(x) \Phi_j(y) = \underbrace{G^L_{hp}(x,y)}_{\geq 0} + \underbrace{G^B_{hp}(x,y)}_{\ngeq 0}$$



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$$K_i = [x_{i-1}, x_i] \qquad G_{hp}(x,y)|_{K_i \times K_i} \mapsto \hat{G}_{hp}(\xi, \eta)$$

$$\hat{G}_{hp}(\xi,\eta) = \underbrace{(x_{i-1} - a)}_{\geq 0} + \underbrace{(x_i - x_{i-1}) l_1(\xi) l_1(\eta)}_{\geq 0}$$

$$\underbrace{\left[1 + \frac{1}{2} l_0(\xi) l_0(\eta) \sum_{k=2}^{p} \kappa_k(\xi) \kappa_k(\eta) \right]}_{\geq 9/10}$$

$$\hat{G}_{hp}(\xi,\eta) = (x_{i-1} - a) + (x_i - x_{i-1}) l_1(\xi) l_1(\eta) \left[1 + \frac{1}{2} l_0(\xi) l_0(\eta) \sum_{k=2}^{p} \kappa_k(\xi)\kappa_k(\eta)\right]$$

$$\xi \in \hat{K} = [-1, 1]$$

$$l_0(\xi) = (1-\xi)/2$$

$$l_1(\xi) = (1+\xi)/2$$

$$l_j(\xi) = \sqrt{\frac{2j-1}{2}} \int_{-1}^{\xi} P_{j-1}(x)\,\mathrm{d}x$$

$$\int_{-1}^{1} l_i'(\xi) l_j'(\xi)\,\mathrm{d}\xi = \delta_{ij}$$

$$i, j = 2, 3, \dots$$

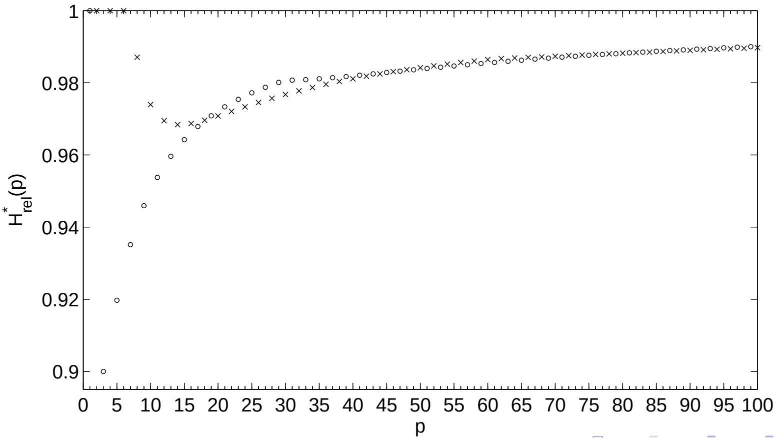
$$l_j(\xi) = l_0(\xi) l_1(\xi) \kappa_j(\xi)$$

$$\kappa_j(\xi) = \sqrt{\frac{2j-1}{2}} \frac{4}{j(1-j)} P_{j-1}'(\xi)$$

$$\int_{-1}^{1} \frac{(1-\xi^2)}{4} \kappa_i(\xi)\kappa_j(\xi)\,\mathrm{d}\xi = 0, \;\; i \neq j$$

$$= \frac{4}{j(j-1)}, \;\; i = j$$

$$\hat{G}_{hp}(\xi,\eta) = (x_{i-1} - a) + (x_i - x_{i-1}) l_1(\xi) l_1(\eta) \left[1 + \frac{1}{2} l_0(\xi) l_0(\eta) \sum_{k=2}^{p} \kappa_k(\xi)\kappa_k(\eta)\right]$$



Conclusions

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- DMP for Dirichlet BC in 1D is valid if $H_{\mathrm{rel}} \leq 9/10$.
- Generalization

$$-(au')' = f, \quad a \text{ is piecewise constant.}$$

Thank you for your attention

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