Angle Conditions for Discrete Maximum Principles in Higher-Order FEM

Tomáš Vejchodský (vejchod@math.cas.cz)

Institute of Mathematics, Academy of Sciences
Žitná 25, 115 67 Prague 1
Czech Republic

ENUMATH 2009, June 29 – July 3, Uppsala, Sweden

Poisson problem

- Classical formulation: $$-\Delta u = f \quad \text{in } \Omega$$ $$u = 0 \quad \text{on } \partial\Omega$$

- Weak formulation: $u \in V: \quad a(u,v) = F(v) \quad \forall v \in V$
  - $V = H_0^1(\Omega)$
  - $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x, \quad F(v) = \int_\Omega f v \,\mathrm{d}x$

- *hp*-FEM: $u_{hp} \in V_{hp}: \quad a(u_{hp}, v_{hp}) = F(v_{hp}) \quad \forall v_{hp} \in V_{hp}$
  - $V_{hp} = \{v_{hp} \in V : v_{hp}|_K \in P^{p_K}(K), K \in \mathcal{T}_{hp}\}$
  - p_K polynomial degree on K

(Continuous) maximum principle

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

MaxP : $f \leq 0 \quad \Rightarrow \quad \max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u = 0$

$\Updownarrow$

ComP : $f \geq 0 \quad \Rightarrow \quad u \geq 0$

$\Updownarrow$

$G(x, y) \geq 0$ in Ω^2

$$u(y) = \int_\Omega f(x) G(x, y) \, \mathrm{d}x$$

$$\begin{aligned} -\Delta G_y &= \delta_y \quad \text{in } \Omega \\ G_y &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

$$G(x, y) = G_y(x)$$

Discrete Maximum Principle (DMP)

DMP : $f \geq 0$ in $\Omega \quad \Rightarrow \quad u_{hp} \geq 0$ in Ω

Theorem (main)

$$DMP \quad \Leftrightarrow \quad G_{hp}(x, y) \geq 0 \quad \forall (x, y) \in \Omega^2$$

Discrete Maximum Principle (DMP)

DMP : $f \geq 0$ in $\Omega \quad \Rightarrow \quad u_{hp} \geq 0$ in Ω

Theorem (main)

$$DMP \quad \Leftrightarrow \quad G_{hp}(x,y) \geq 0 \quad \forall (x,y) \in \Omega^2$$

- $G_{hp,y} \in V_{hp} \; : \; a(v_{hp}, G_{hp,y}) = v_{hp}(y) \quad \forall v_{hp} \in V_{hp}, \; y \in \Omega$

  $G_{hp}(x,y) = G_{hp,y}(x)$

Discrete Maximum Principle (DMP)

DMP : $f \geq 0$ in $\Omega \quad \Rightarrow \quad u_{hp} \geq 0$ in Ω

Theorem (main)

$$DMP \quad \Leftrightarrow \quad G_{hp}(x,y) \geq 0 \quad \forall (x,y) \in \Omega^2$$

- $G_{hp,y} \in V_{hp} \ : \ a(v_{hp}, G_{hp,y}) = v_{hp}(y) \quad \forall v_{hp} \in V_{hp}, \ y \in \Omega$

  $G_{hp}(x,y) = G_{hp,y}(x)$
- $u_{hp}(y) = a(u_{hp}, G_{hp,y}) = \int_\Omega f(x) G_{hp}(x,y) \, \mathrm{d}x$

Discrete Maximum Principle (DMP)

DMP : $f \geq 0$ in Ω $\quad\Rightarrow\quad$ $u_{hp} \geq 0$ in Ω

Theorem (main)

$$DMP \quad \Leftrightarrow \quad G_{hp}(x,y) \geq 0 \quad \forall (x,y) \in \Omega^2$$

- $G_{hp,y} \in V_{hp} \ : \ a(v_{hp}, G_{hp,y}) = v_{hp}(y) \quad \forall v_{hp} \in V_{hp}, \ y \in \Omega$

  $G_{hp}(x,y) = G_{hp,y}(x)$
- $u_{hp}(y) = \int_\Omega f(x) G_{hp}(x,y) \, \mathrm{d}x$

Discrete Maximum Principle (DMP)

DMP : $f \geq 0$ in $\Omega \quad \Rightarrow \quad u_{hp} \geq 0$ in Ω

Theorem (main)

$$DMP \quad \Leftrightarrow \quad G_{hp}(x,y) \geq 0 \quad \forall (x,y) \in \Omega^2$$

- $G_{hp,y} \in V_{hp} \;:\; a(v_{hp}, G_{hp,y}) = v_{hp}(y) \quad \forall v_{hp} \in V_{hp}, \; y \in \Omega$
  $G_{hp}(x,y) = G_{hp,y}(x)$
- $u_{hp}(y) = \int_\Omega f(x) G_{hp}(x,y) \, \mathrm{d}x$
- $G_{hp}(x,y) = \sum_{i=1}^{N} \sum_{j=1}^{N} (A^{-1})_{ij} \varphi_i(x) \varphi_j(y), \quad A_{ij} = a(\varphi_j, \varphi_i)$

Discrete Maximum Principle (DMP)

DMP : $f \geq 0$ in $\Omega \quad \Rightarrow \quad u_{hp} \geq 0$ in Ω

Theorem (main)

$$DMP \quad \Leftrightarrow \quad G_{hp}(x,y) \geq 0 \quad \forall (x,y) \in \Omega^2$$

Remark (1D):

$\Omega = (a_\Omega, b_\Omega) \subset \mathbb{R}^1$

$$-u'' = f \quad \text{in } (a_\Omega, b_\Omega)$$
$$u(a_\Omega) = u(b_\Omega) = 0$$

- $p = 1, 2, 4, 6 \quad \Rightarrow$ DMP for any mesh
- $p = 3, 5, 7, 8, \ldots, 100 \Rightarrow$ DMP if $h \leq 0.9|\Omega|$

(Vejchodský, Šolín, Math. Comp. 2007)

Linear FEM

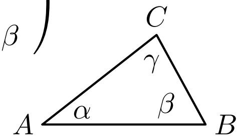
- DMP $\Leftrightarrow$ $G_{hp} \geq 0$ $\Leftrightarrow$ $A^{-1} \geq 0$ $\Leftrightarrow$ A monotone
- A s.p.d., off-diag$(A) \leq 0$ $\Leftrightarrow$ A M-matrix $\Rightarrow$ A monotone
- Element matrices: off-diag$(A^K) \leq 0$ $\Rightarrow$ off-diag$(A) \leq 0$
  - $A = \sum_{K \in \mathcal{T}_{hp}} A^K, \quad A_{ij}^K = a_K(\varphi_j, \varphi_i) = \int_K \nabla\varphi_i \cdot \nabla\varphi_j$

Linear FEM

- DMP $\Leftrightarrow$ $G_{hp} \geq 0$ $\Leftrightarrow$ $A^{-1} \geq 0$ $\Leftrightarrow$ A monotone
- A s.p.d., off-diag$(A) \leq 0$ $\Leftrightarrow$ A M-matrix $\Rightarrow$ A monotone
- Element matrices: off-diag$(A^K) \leq 0$ $\Rightarrow$ off-diag$(A) \leq 0$
  - $A = \sum_{K \in \mathcal{T}_{hp}} A^K, \quad A^K_{ij} = a_K(\varphi_j, \varphi_i) = \int_K \nabla\varphi_i \cdot \nabla\varphi_j$

Example (triangles):

$$A^K = \frac{1}{2}\begin{pmatrix} \cot\beta + \cot\gamma & -\cot\gamma & -\cot\beta \\ -\cot\gamma & \cot\alpha + \cot\gamma & -\cot\alpha \\ -\cot\beta & -\cot\alpha & \cot\alpha + \cot\beta \end{pmatrix}$$

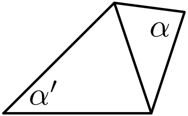


Linear FEM

- DMP $\Leftrightarrow$ $G_{hp} \geq 0$ $\Leftrightarrow$ $A^{-1} \geq 0$ $\Leftrightarrow$ A monotone
- A s.p.d., off-diag$(A) \leq 0$ $\Leftrightarrow$ A M-matrix $\Rightarrow$ A monotone
- Element matrices: off-diag$(A^K) \leq 0$ $\Rightarrow$ off-diag$(A) \leq 0$
  - $A = \sum_{K \in \mathcal{T}_{hp}} A^K, \quad A^K_{ij} = a_K(\varphi_j, \varphi_i) = \int_K \nabla \varphi_i \cdot \nabla \varphi_j$

Example (triangles):

- $\alpha_{\max} \leq \pi/2$ $\Rightarrow$ off-diag$(A^K) \leq 0$
- $\alpha + \alpha' \leq \pi$ $\Leftrightarrow$ off-diag$(A) \leq 0$



Linear FEM

- DMP $\Leftrightarrow$ $G_{hp} \geq 0$ $\Leftrightarrow$ $A^{-1} \geq 0$ $\Leftrightarrow$ A monotone
- A s.p.d., off-diag$(A) \leq 0$ $\Leftrightarrow$ A M-matrix $\Rightarrow$ A monotone
- Element matrices: off-diag$(A^K) \leq 0$ $\Rightarrow$ off-diag$(A) \leq 0$
  - $A = \sum_{K \in \mathcal{T}_{hp}} A^K, \quad A^K_{ij} = a_K(\varphi_j, \varphi_i) = \int_K \nabla\varphi_i \cdot \nabla\varphi_j$

Gap: A monotone but not M-matrix

$\Rightarrow$ numerical tests

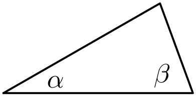
Higher order FEM

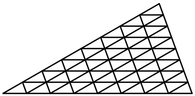
- DMP $\Leftrightarrow$ $G_{hp} \geq 0$ $\not\Leftarrow$ $A^{-1} \geq 0$
- $G_{hp}(x_i^{\mathrm{V}}, x_j^{\mathrm{V}}) \geq 0$ $\Leftrightarrow$ $S^{-1} \geq 0$
  - x_i^{V}, $i = 1, \dots, N_{\mathrm{vert}}$ vertices (nodes) in $\mathcal{T}_{hp}$
  - $S = A_{\mathrm{VV}} - A_{\mathrm{VN}} A_{\mathrm{NN}}^{-1} A_{\mathrm{NV}}$
  - $A = \begin{pmatrix} A_{\mathrm{VV}} & A_{\mathrm{VN}} \\ A_{\mathrm{NV}} & A_{\mathrm{NN}} \end{pmatrix}$
  - $A_{\mathrm{VV}} \in \mathbb{R}^{N_{\mathrm{vert}} \times N_{\mathrm{vert}}}$, $A_{\mathrm{NN}} \in \mathbb{R}^{N_{\mathrm{nonv}} \times N_{\mathrm{nonv}}}$, $N_{\mathrm{dof}} = N_{\mathrm{vert}} + N_{\mathrm{nonv}}$
  - *hp*-FEM basis: $\underbrace{\varphi_1^{\mathrm{V}}, \dots, \varphi_{N_{\mathrm{vert}}}^{\mathrm{V}}}_{\text{vertex fun.}}, \underbrace{\varphi_{N_{\mathrm{vert}}+1}^{\mathrm{N}}, \dots, \varphi_{N_{\mathrm{dof}}}^{\mathrm{N}}}_{\text{edge, bubble fun.}}$

Test 1

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

$\alpha = 30^\circ$
$\beta = 70^\circ$



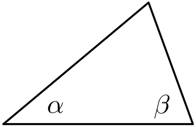


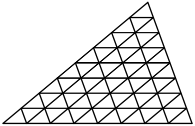
$\alpha = 1^\circ, 2^\circ, \ldots, 179^\circ \qquad \beta = 1^\circ, 2^\circ, \ldots, 179^\circ \qquad \alpha + \beta < 180^\circ$

Test 1

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

$\alpha = 40^\circ$
$\beta = 70^\circ$



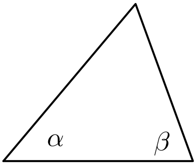


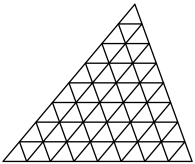
$\alpha = 1^\circ, 2^\circ, \ldots, 179^\circ \qquad \beta = 1^\circ, 2^\circ, \ldots, 179^\circ \qquad \alpha + \beta < 180^\circ$

Test 1

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

$\alpha = 50°$
$\beta = 70°$



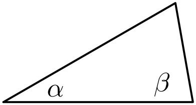


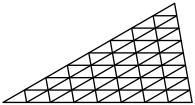
$\alpha = 1°, 2°, \ldots, 179°$ $\quad \beta = 1°, 2°, \ldots, 179°$ $\quad \alpha + \beta < 180°$

Test 1

$$
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

$\alpha = 30^\circ$
$\beta = 80^\circ$



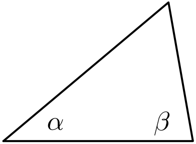


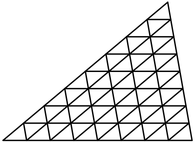
$\alpha = 1^\circ, 2^\circ, \ldots, 179^\circ$ $\quad$ $\beta = 1^\circ, 2^\circ, \ldots, 179^\circ$ $\quad$ $\alpha + \beta < 180^\circ$

Test 1

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

$\alpha = 40^\circ$
$\beta = 80^\circ$



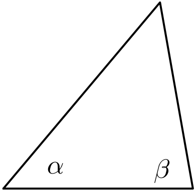


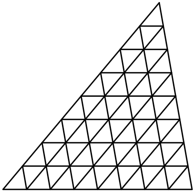
$\alpha = 1^\circ, 2^\circ, \ldots, 179^\circ \qquad \beta = 1^\circ, 2^\circ, \ldots, 179^\circ \qquad \alpha + \beta < 180^\circ$

Test 1

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

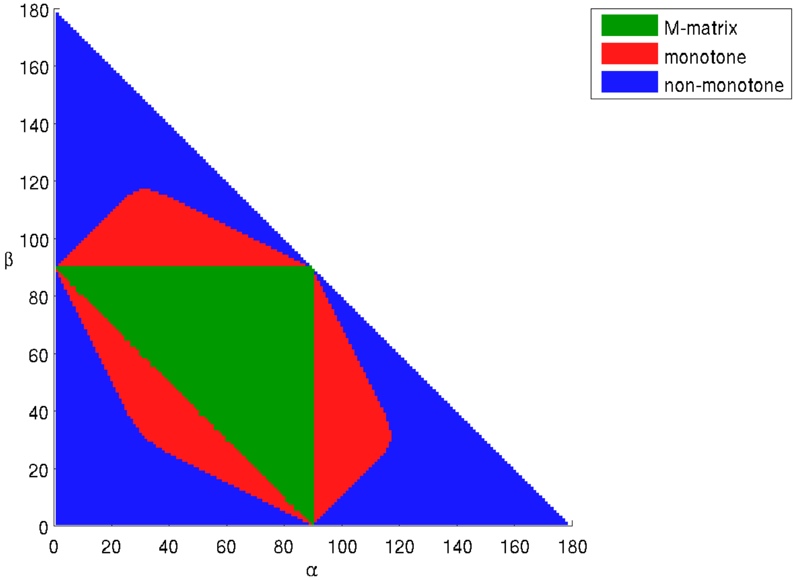
$\alpha = 50^\circ$
$\beta = 80^\circ$





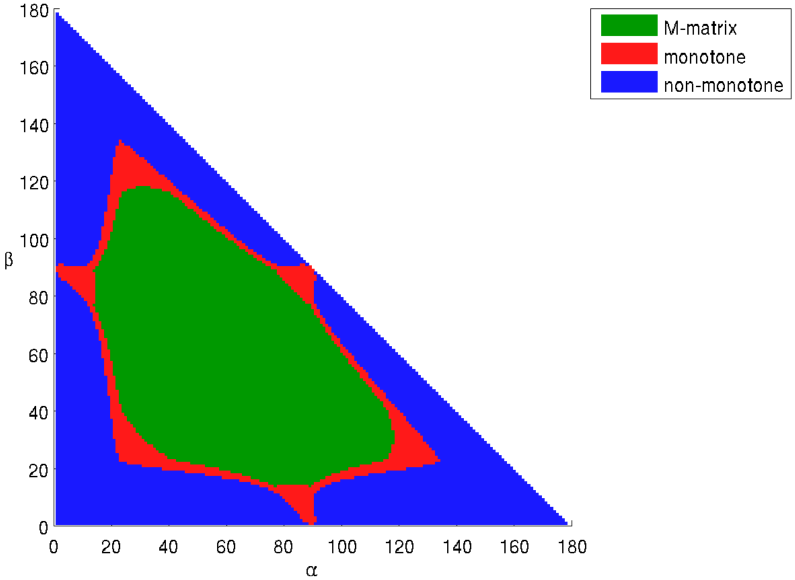
$\alpha = 1^\circ, 2^\circ, \ldots, 179^\circ \qquad \beta = 1^\circ, 2^\circ, \ldots, 179^\circ \qquad \alpha + \beta < 180^\circ$

Test 1 – $p = 1$



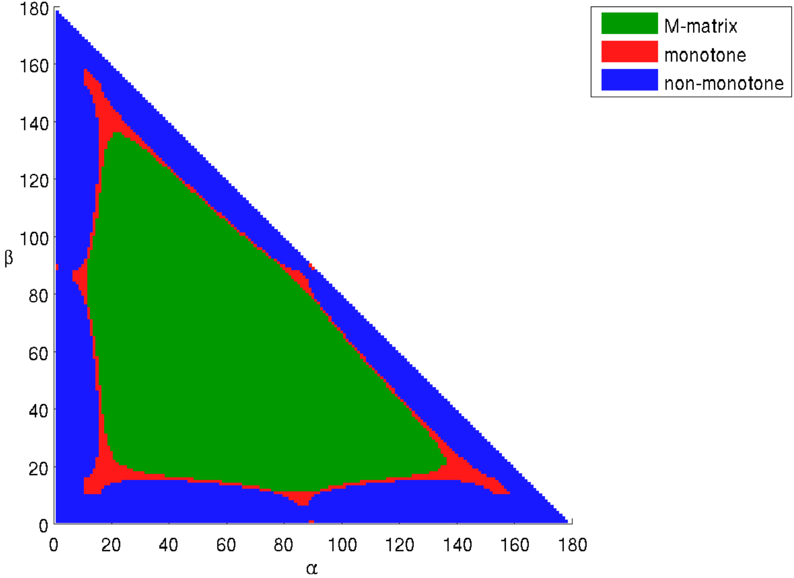
M-matrix
monotone
non-monotone
β
α
0
20
40
60
80
100
120
140
160
180

Test 1 – $p = 3$



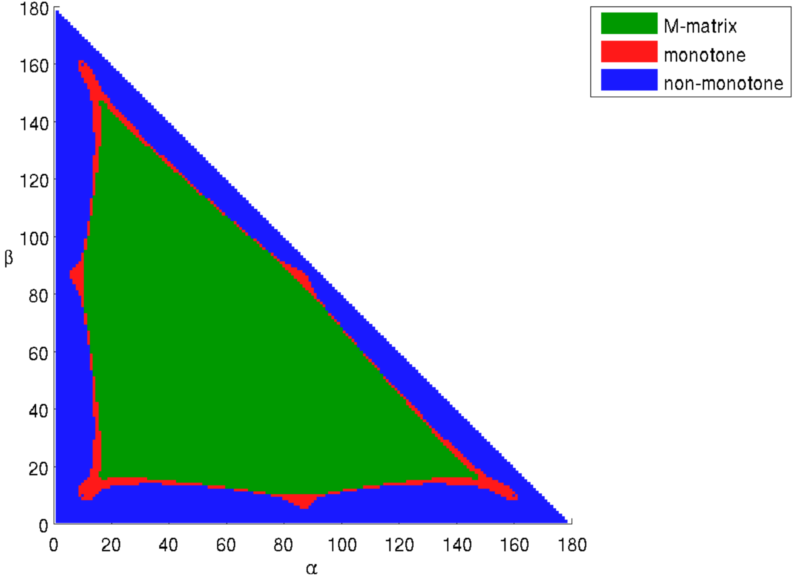
M-matrix
monotone
non-monotone
β
α
0
20
40
60
80
100
120
140
160
180

Test 1 – $p = 5$

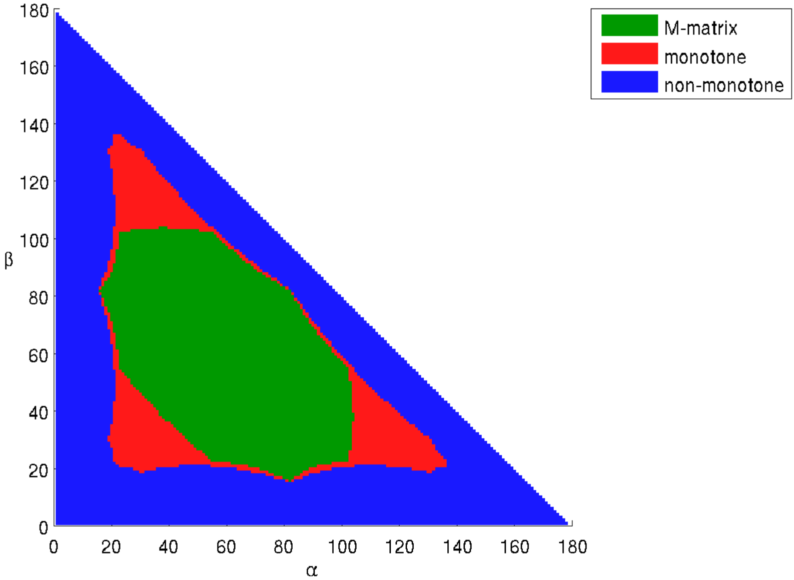


M-matrix
monotone
non-monotone
180
160
140
120
100
80
60
40
20
0
β
0
20
40
60
80
100
120
140
160
180
α

Test 1 – $p = 7$

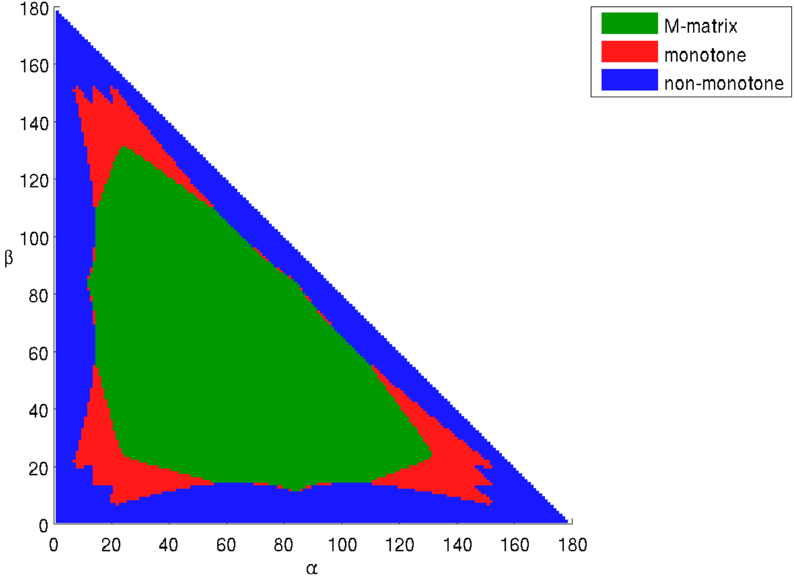

M-matrix
monotone
non-monotone
β
α
0
20
40
60
80
100
120
140
160
180

Test 1 – $p = 2$



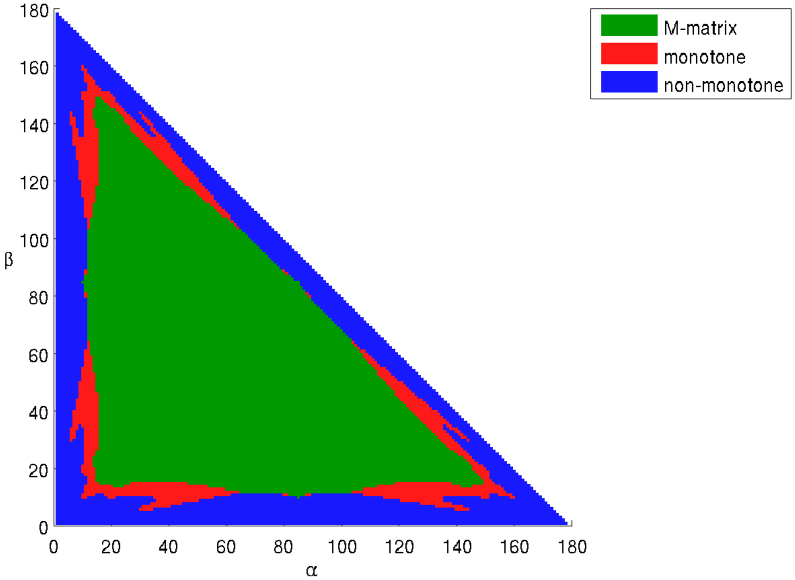
M-matrix
monotone
non-monotone
β
α
0
20
40
60
80
100
120
140
160
180

Test 1 – $p = 4$



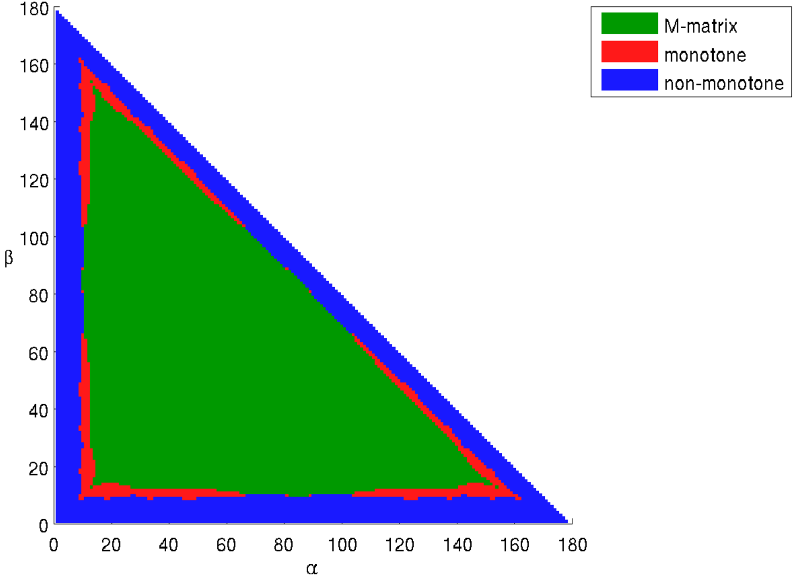
M-matrix
monotone
non-monotone
β
α
0
20
40
60
80
100
120
140
160
180

Test 1 – $p = 6$



M-matrix
monotone
non-monotone
β
α
0
20
40
60
80
100
120
140
160
180

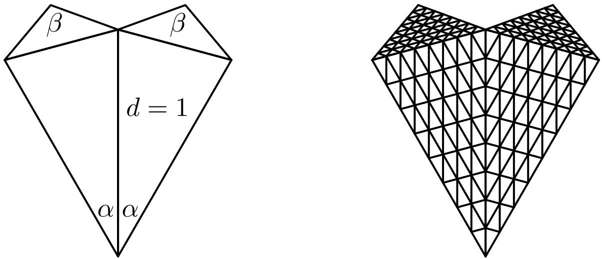
Test 1 – $p = 8$



M-matrix
monotone
non-monotone
β
α
0
20
40
60
80
100
120
140
160
180

Test 2

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

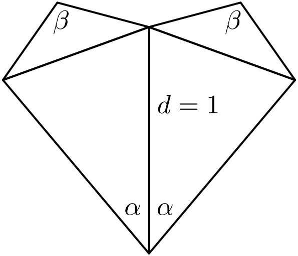


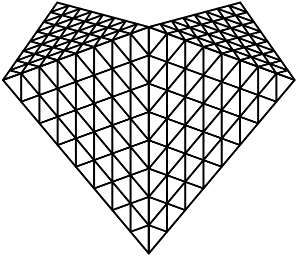
$\alpha = 30^\circ$
$\beta = 110^\circ$

$\alpha = 1^\circ, 2^\circ, \ldots, 179^\circ \qquad \beta = 1^\circ, 2^\circ, \ldots, 179^\circ$

Test 2

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$



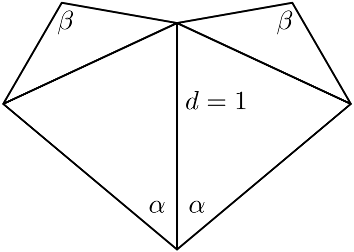


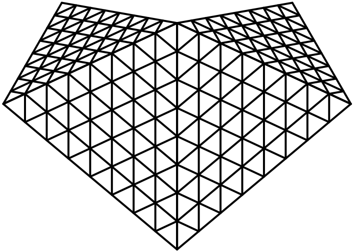
$\alpha = 40^\circ$
$\beta = 110^\circ$

$\alpha = 1^\circ, 2^\circ, \dots, 179^\circ \qquad \beta = 1^\circ, 2^\circ, \dots, 179^\circ$

Test 2

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

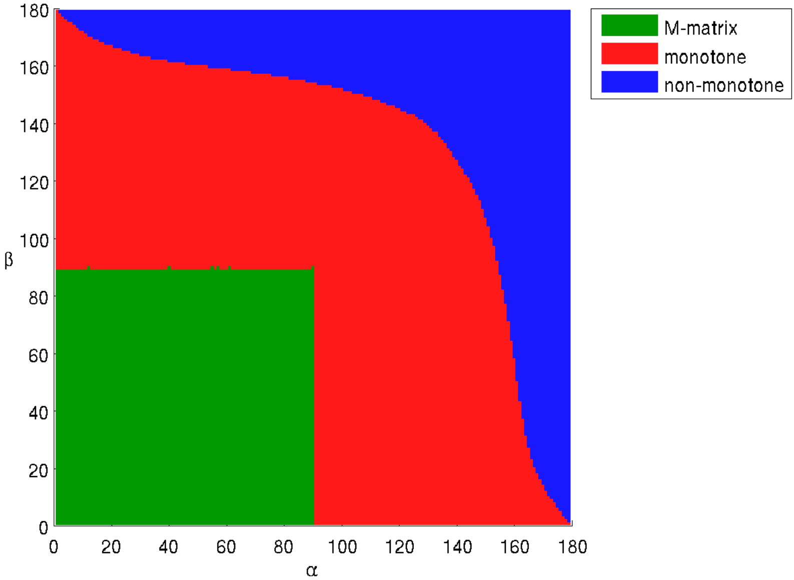




$\alpha = 50^\circ$
$\beta = 110^\circ$

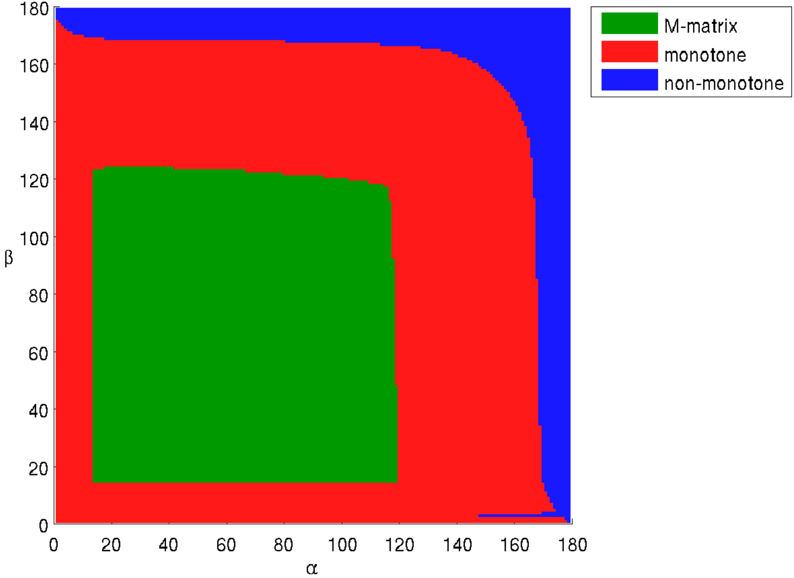
$\alpha = 1^\circ, 2^\circ, \ldots, 179^\circ \qquad \beta = 1^\circ, 2^\circ, \ldots, 179^\circ$

Test 2 – $p = 1$



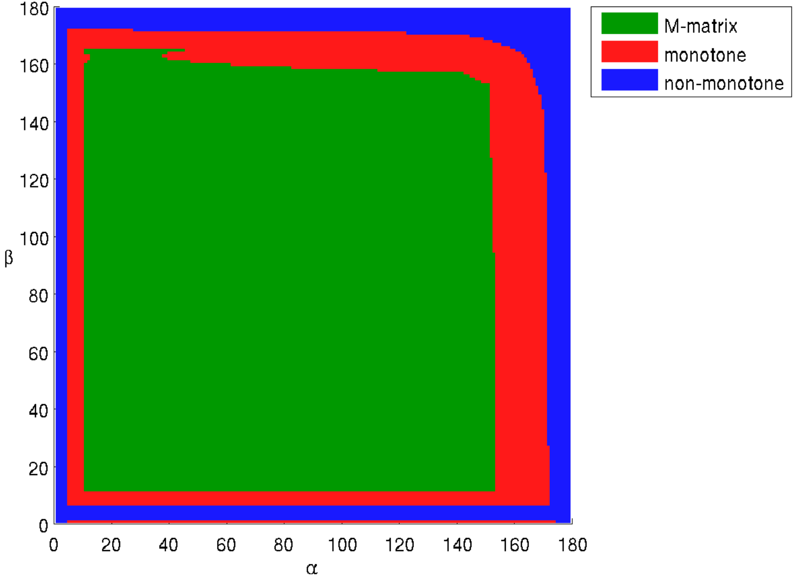
180
160
140
120
100
80
60
40
20
0
β
0
20
40
60
80
100
120
140
160
180
α
M-matrix
monotone
non-monotone

Test 2 – $p = 3$



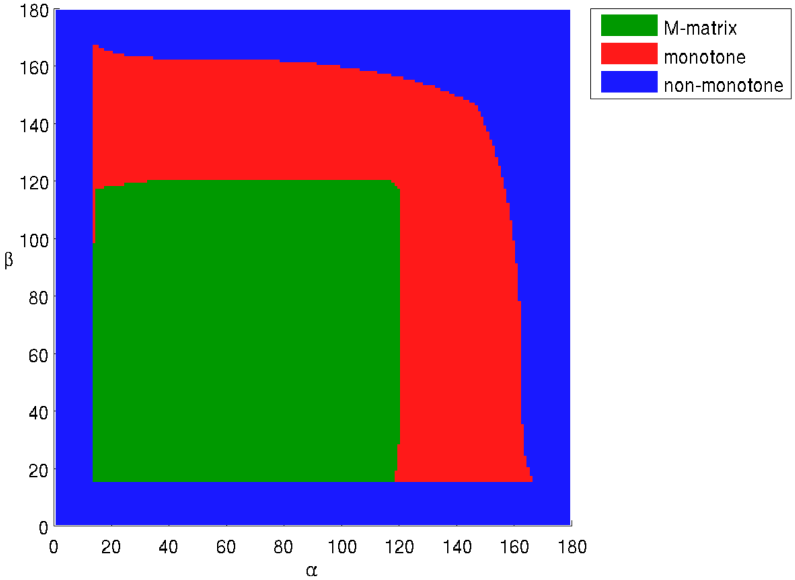
M-matrix
monotone
non-monotone
β
α
0
20
40
60
80
100
120
140
160
180

Test 2 – $p = 5$



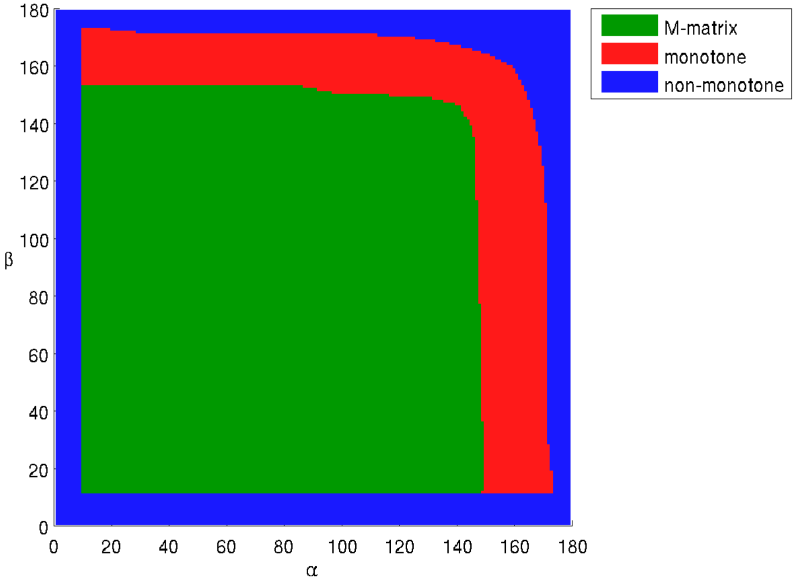
180
160
140
120
100
80
60
40
20
0
β
0
20
40
60
80
100
120
140
160
180
α
M-matrix
monotone
non-monotone

Test 2 – $p = 2$



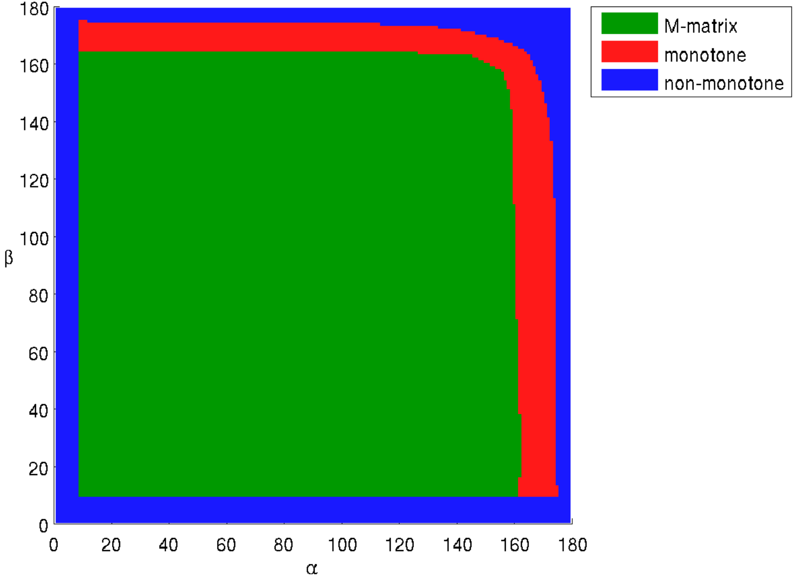
M-matrix
monotone
non-monotone
β
α
0
20
40
60
80
100
120
140
160
180

Test 2 – $p = 4$



M-matrix
monotone
non-monotone
β
α
0
20
40
60
80
100
120
140
160
180

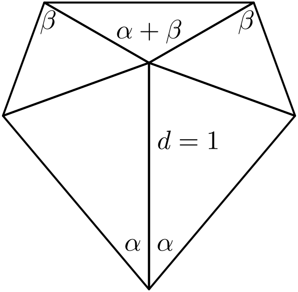
Test 2 – $p = 6$



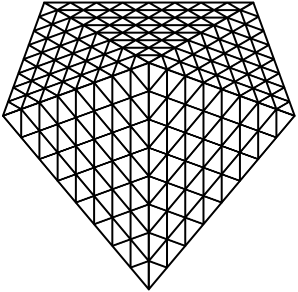
180
160
140
120
100
80
60
40
20
0
β
0
20
40
60
80
100
120
140
160
180
α
M-matrix
monotone
non-monotone

Test 3

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

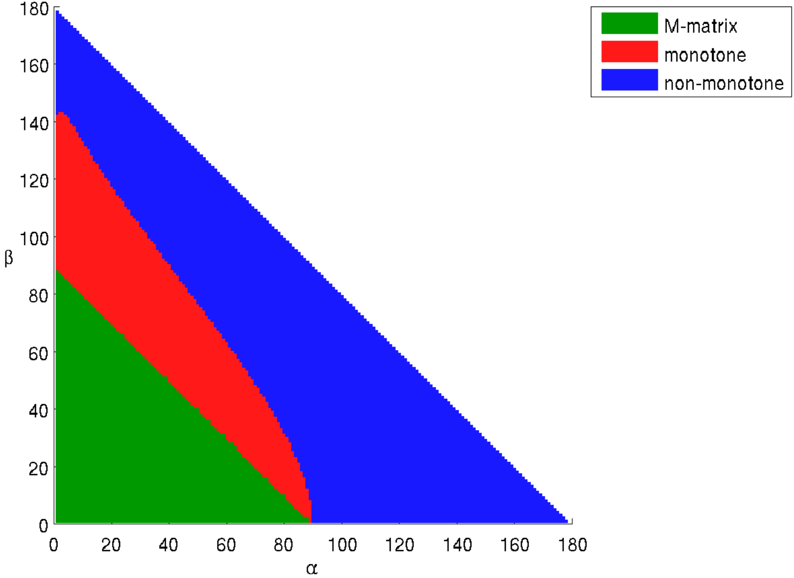


$\alpha = 40^\circ$
$\beta = 110^\circ$



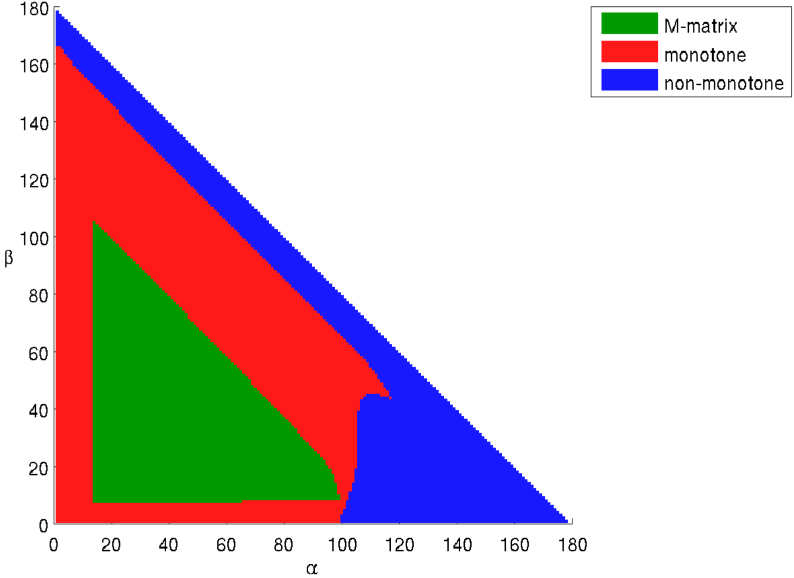
$\alpha = 1^\circ, 2^\circ, \ldots, 179^\circ$ $\qquad \beta = 1^\circ, 2^\circ, \ldots, 179^\circ$ $\qquad \alpha + \beta < 180^\circ$

Test 3 – $p = 1$



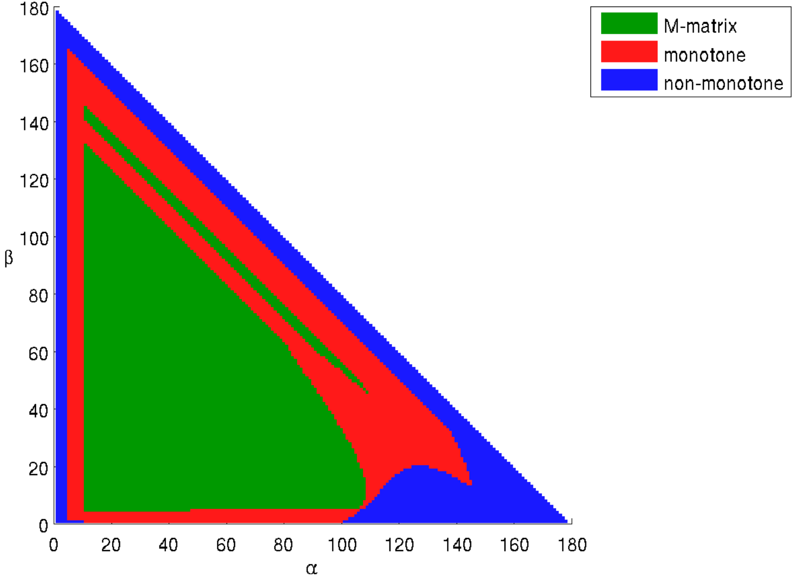
M-matrix
monotone
non-monotone
β
α
0
20
40
60
80
100
120
140
160
180

Test 3 – $p = 3$



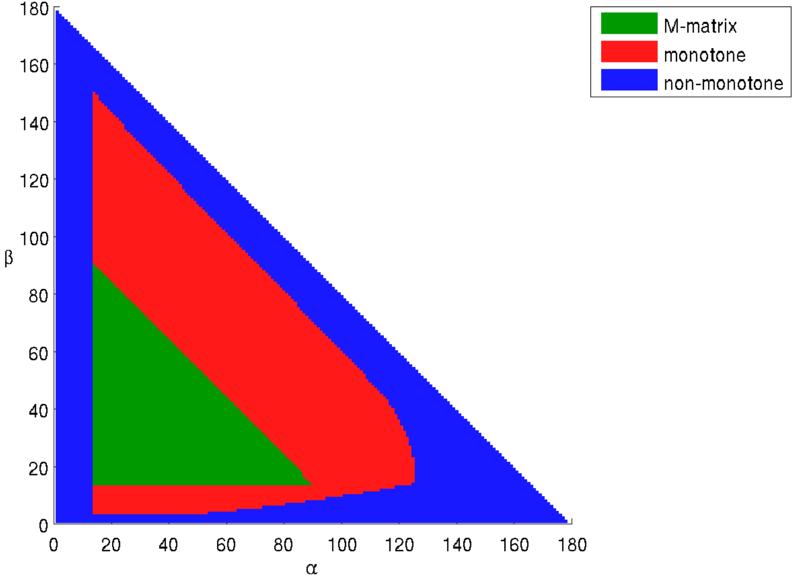
M-matrix
monotone
non-monotone
β
α
0
20
40
60
80
100
120
140
160
180

Test 3 – $p = 5$



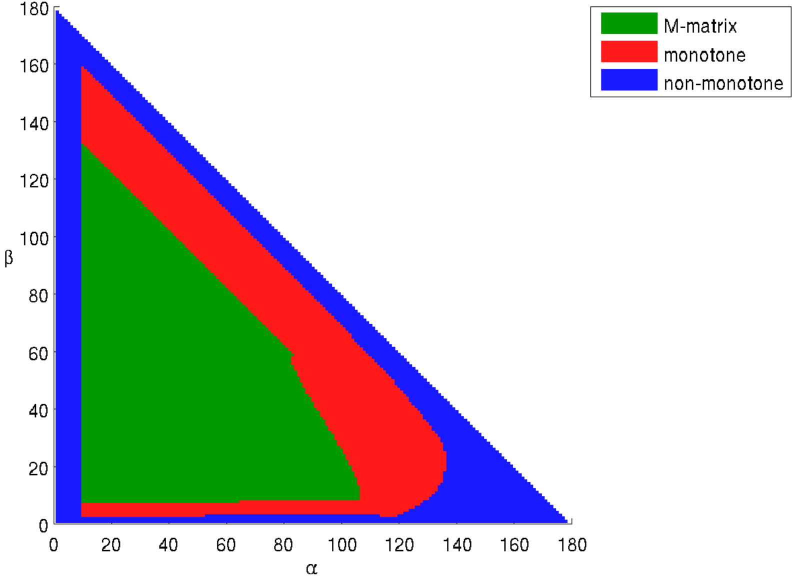
180
160
140
120
100
80
60
40
20
0
β
0
20
40
60
80
100
120
140
160
180
α
M-matrix
monotone
non-monotone

Test 3 – $p = 2$



M-matrix
monotone
non-monotone
β
α
0
20
40
60
80
100
120
140
160
180

Test 3 – $p = 4$



M-matrix
monotone
non-monotone
180
160
140
120
100
80
60
40
20
0
β
0
20
40
60
80
100
120
140
160
180
α

Conclusions

- Quite a gap: A monotone but not M-matrix
- Nodal values $G_{hp}(x_i^{\mathsf{V}}, x_j^{\mathsf{V}})$:
  nonnegative for wider range of angles if p grows
- Dichotomy of odd and even polynomial degrees
- Wide space for new theory

Thank you for your attention

Tomáš Vejchodský (vejchod@math.cas.cz)

Institute of Mathematics, Academy of Sciences
Žitná 25, 115 67 Prague 1
Czech Republic

ENUMATH 2009, June 29 – July 3, Uppsala, Sweden