Admissibility and unification with parameters

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Overview

The plan for this talk:

- General remarks on admissibility and unification
- Known results on admissibility in modal, intuitionistic, and Łukasiewicz logics
- Admissibility with parameters in modal and intuitionistic logics
- Admissibility with parameters in Łukasiewicz logic

Admissibility and unification in propositional logics

Propositional logics

Propositional logic L:

Language: formulas Form_L built freely from atoms (variables) $\{x_n : n \in \omega\}$ using a fixed set of connectives of finite arity

Consequence relation $\vdash_L$: finitary structural Tarski-style consequence operator
I.e.: a relation $\Gamma \vdash_L \varphi$ between finite sets of formulas and formulas such that

- $\varphi \vdash_L \varphi$
- $\Gamma \vdash_L \varphi$ implies $\Gamma, \Gamma' \vdash_L \varphi$
- $\Gamma \vdash_L \varphi$ and $\Gamma, \varphi \vdash_L \psi$ imply $\Gamma \vdash_L \psi$
- $\Gamma \vdash_L \varphi$ implies $\sigma(\Gamma) \vdash_L \sigma(\varphi)$ for every substitution σ

Algebraization

L is finitely algebraizable wrt a class K of algebras if there is a finite set $F(u,v)$ of formulas and a finite set $E(x)$ of equations such that

- $\Gamma \vdash_L \varphi \Leftrightarrow E(\Gamma) \vDash_K E(\varphi)$
- $\Theta \vDash_K t \approx s \Leftrightarrow F(\Theta) \vdash_L F(t,s)$
- $x \dashv\vdash_L F(E(x))$
- $u \approx v \mathrel{=\!\!|\!\!\vDash}_K E(F(u,v))$

We may assume K is a quasivariety

In the cases in this talk, we will always have:
$E(x) = \{x \approx 1\}$, $F(u,v) = \{u \leftrightarrow v\}$, K is a variety

Equational unification

Let Θ be an equational theory (or a variety of algebras):

- Θ-unifier of a set Γ of equations:
  a substitution σ s.t. $\vDash_\Theta \sigma(t) \approx \sigma(s)$ for all $t \approx s \in \Gamma$
- Γ is Θ-unifiable if it has a Θ-unifier
- $\sigma \equiv_\Theta \tau$ iff $\vDash_\Theta \sigma(u) \approx \tau(u)$ for every variable u
- $\sigma \preceq_\Theta \tau$ (τ is more general than σ) if $\exists \varrho\, \sigma \equiv_\Theta \varrho \circ \tau$
- Complete set of unifiers of Γ: a set X of unifiers of Γ such that every unifier of Γ is less general than some $\tau \in X$
- Θ has finitary unification type if every finite Γ has a finite complete set of unifiers

Unification in propositional logics

If L is a logic finitely algebraizable wrt a variety K, we can express K-unification in terms of L:

An L-unifier of a formula φ is σ such that $\vdash_L \sigma(\varphi)$

Then we have:

- L-unifier of φ = K-unifier of $E(\varphi)$
- K-unifier of $t \approx s$ = L-unifier of $F(t, s)$
- $\sigma \equiv_L \tau$ iff $\vdash_L F(\sigma(x), \tau(x))$ for every x
  (in our case: $\vdash_L \sigma(x) \leftrightarrow \tau(x)$)
- . . .

Admissible rules

Single-conclusion rule: $\Gamma \mathbin{/} \varphi$ (Γ finite set of formulas)

Multiple-conclusion rule: $\Gamma \mathbin{/} \Delta$ (Γ, Δ finite sets of formulas)

- $\Gamma \mathbin{/} \Delta$ is L-derivable (or valid) if $\Gamma \vdash_L \delta$ for some $\delta \in \Delta$
- $\Gamma \mathbin{/} \Delta$ is L-admissible (written as $\Gamma \mathrel{|\!\!\sim}_L \Delta$) if every L-unifier of Γ also unifies some $\delta \in \Delta$

$$E(\Gamma \mathbin{/} \Delta) := \bigwedge_{\gamma \in \Gamma} E(\gamma) \to \bigvee_{\delta \in \Delta} E(\delta):$$

- $\Gamma \mathbin{/} \Delta$ is derivable iff $E(\Gamma \mathbin{/} \Delta)$ holds in all K-algebras
- $\Gamma \mathbin{/} \Delta$ is admissible iff $E(\Gamma \mathbin{/} \Delta)$ holds in free K-algebras

Note: Γ is unifiable iff $\Gamma \mathrel{|\!\!\not\sim}_L \varnothing$

Multiple-conclusion consequence relations

Single-conc. admissible rules form a consequence relation

Multiple-conc. admissible rules form a (finitary structural) multiple-conclusion consequence relation:

- $\varphi \mathrel{|\!\sim} \varphi$
- $\Gamma \mathrel{|\!\sim} \Delta$ implies $\Gamma, \Gamma' \mathrel{|\!\sim} \Delta, \Delta'$
- $\Gamma \mathrel{|\!\sim} \varphi, \Delta$ and $\Gamma, \varphi \mathrel{|\!\sim} \Delta$ imply $\Gamma \mathrel{|\!\sim} \Delta$
- $\Gamma \mathrel{|\!\sim} \Delta$ implies $\sigma(\Gamma) \mathrel{|\!\sim} \sigma(\Delta)$ for every substitution σ

A set B of rules is a basis of L-admissible rules if $\mathrel{|\!\sim}_L$ is the smallest m.-c. c. r. containing $\vdash_L$ and B

Admissibly saturated approximation

Γ is admissibly saturated if $\Gamma \mathrel{|\!\!\sim}_L \Delta$ implies $\Gamma \vdash_L \Delta$ for any Δ

Assume for simplicity that L has a well-behaved conjunction.

Admissibly saturated approximation of Γ:
a finite set of formulas Π_Γ such that

- each $\pi \in \Pi_\Gamma$ is admissibly saturated
- $\Gamma \mathrel{|\!\!\sim}_L \Pi_\Gamma$
- $\pi \vdash_L \varphi$ for each $\pi \in \Pi_\Gamma$ and $\varphi \in \Gamma$

Application of admissible saturation

Assuming every Γ has an a.s. approximation Π_Γ:

- Reduction of $\mathrel{|\!\sim}_L$ to $\vdash_L$:

$$\Gamma \mathrel{|\!\sim}_L \Delta \quad \text{iff} \quad \forall \pi \in \Pi_\Gamma \, \exists \psi \in \Delta \, \pi \vdash_L \psi$$

- If $\Gamma \mapsto \Pi_\Gamma$ is computable and $\vdash_L$ is decidable, then $\mathrel{|\!\sim}_L$ is decidable
- If $\Gamma \,/\, \Pi_\Gamma$ is derivable in $\vdash_L +$ a set of rules $B \subseteq \mathrel{|\!\sim}_L$, then B is a basis of admissible rules
- If each $\pi \in \Pi_\Gamma$ has an mgu σ_π, then $\{\sigma_\pi : \pi \in \Pi_\Gamma\}$ is a complete set of unifiers for Γ
  $\Rightarrow$ finitary unification

Projective formulas

π is projective if it has a unifier σ such that $\pi \vdash_L x \leftrightarrow \sigma(x)$
(in general: $\pi \vdash_L F(x, \sigma(x))$) for every variable x

- Every projective formula is admissibly saturated
- σ is an mgu of π: if τ is a unifier of π, then $\tau \equiv_L \tau \circ \sigma$
- Projective formula $\approx$ presentation of a projective algebra

Projective approximation := admissibly saturated approximation consisting of projective formulas

If projective approximations exist:
convenient tool for analysis of unification and admissibility

Exact formulas

φ is exact if there exists σ such that

$$\vdash_L \sigma(\psi) \quad \text{iff} \quad \varphi \vdash_L \psi$$

for all formulas ψ

- projective $\Rightarrow$ exact $\Rightarrow$ admissibly saturated
- in general: can't be reversed
- if projective approximations exist:
  projective = exact = admissibly saturated
- exact formulas do not need to have mgu
  $\Rightarrow$ can coexist with bad unification type

Parameters

In real life, propositional atoms model both “variables” and “constants”

We don’t want to allow substitution for constants

Example (description logic):

(1) $\forall \mathrm{child}.(\neg \mathrm{HasSon} \sqcap \exists \mathrm{spouse}.\top)$

(2) $\forall \mathrm{child}.\forall \mathrm{child}.\neg \mathrm{Male} \sqcap \forall \mathrm{child}.\mathrm{Married}$

(3) $\forall \mathrm{child}.\forall \mathrm{child}.\neg \mathrm{Female} \sqcap \forall \mathrm{child}.\mathrm{Married}$

Good: Unify (1) with (2) by $\mathrm{HasSon} \mapsto \exists \mathrm{child}.\mathrm{Male}$, $\mathrm{Married} \mapsto \exists \mathrm{spouse}.\top$

Bad: Unify (2) with (3) by $\mathrm{Male} \mapsto \mathrm{Female}$

Admissibility with parameters

In equational unification theory, it is customary to consider a setup with two kinds of atoms:

- variables $\{x_n : n \in \omega\}$
- parameters $\{p_n : n \in \omega\}$
  (aka constants, metavariables, coefficients)

Substitutions only modify variables, we require $\sigma(p_n) = p_n$

Adapt accordingly the definitions of other notions:

- Unifiers, admissible rules, bases, a.s. formulas and approximations, projective formulas, ...

Exception: “Propositional logic” is always assumed to be closed under substitution for parameters

Inheritance

L' inherits admissible rules of L if $\Gamma \mathrel{|\!\!\sim}_L \Delta \Rightarrow \Gamma \mathrel{|\!\!\sim}_{L'} \Delta$

Parameter-free examples:

- $\mathbf{S4Grz}$ inherits admissible rules of $\mathbf{S4}$
- $\mathbf{KC}$ inherits single-conclusion admissible rules of $\mathbf{IPC}$

Admissible rules with parameters cannot be inherited in a nontrivial way: L and L' have the same theorems

$$\vdash_L \varphi \;\Rightarrow\; \mathrel{|\!\!\sim}_L \varphi \;\Rightarrow\; \vdash_{L'} \varphi$$

$$\nvdash_L \varphi \;\Rightarrow\; \varphi(\vec{p}) \mathrel{|\!\!\sim}_L q \;\Rightarrow\; \nvdash_{L'} \varphi$$

Transitive modal logics

Transitive modal logics

Normal modal logics with a single modality $\Box$, include the transitivity axiom $\Box x \to \Box\Box x$ (i.e., $L \supseteq \mathbf{K4}$)

Common examples: various combinations of

logic	axiom (on top of $\mathbf{K4}$)	finite rooted transitive frames
$\mathbf{S4}$	$\Box x \to x$	reflexive
$\mathbf{D4}$	$\Diamond\top$	final clusters reflexive
$\mathbf{GL}$	$\Box(\Box x \to x) \to \Box x$	irreflexive
$\mathbf{K4Grz}$	$\Box(\Box(x \to \Box x) \to x) \to \Box x$	no proper clusters
$\mathbf{K4.1}$	$\boxdot\Diamond x \to \diamonddot\Box x$	no proper final clusters
$\mathbf{K4.2}$	$\Diamond\boxdot x \to \Box\diamonddot x$	unique final cluster
$\mathbf{K4.3}$	$\Box(\boxdot x \to y) \lor \Box(\boxdot y \to x)$	linear (chain of clusters)
$\mathbf{K4B}$	$x \to \Box\Diamond x$	lone cluster
$\mathbf{S5}$	$= \mathbf{S4} \oplus \mathbf{B}$	lone reflexive cluster

Some classes of transitive logics

Cofinal-subframe (csf) logics:

- complete wrt a class of frames closed under the removal of a subset of non-final points
- all combinations of logics from the table are csf

Extensible logics:

- If a frame F has a unique root r whose reflexivity is compatible with L, and $F \smallsetminus \{r\} \vDash L$, then $F \vDash L$
- $\mathbf{K4}$, $\mathbf{S4}$, $\mathbf{GL}$, $\mathbf{K4Grz}$, $\mathbf{S4Grz}$, $\mathbf{D4}$, $\mathbf{K4.1}$, ... (not $\mathbf{K4.2}$, ...)

Linear extensible logics:

- $\mathbf{K4.3}$, $\mathbf{S4.3}$, $\mathbf{GL.3}$, ...

Admissibility in transitive modal logics

A lot is known about admissibility without parameters:

- Admissibility is decidable in a large class of logics (Rybakov)
- Extensible logics have projective approximations (Ghilardi)
  - finitary unification type
  - complete sets of unifiers computable
- Bases of admissible rules for extensible logics (J.)
- Computational complexity of admissibility (J.)
  - Lower bounds for a quite general class of logics
  - Matching upper bounds for csf extensible logics
- . . . and more . . .

Projectivity in modal logics

Fix $L \supseteq \mathbf{K4}$ with the finite model property (fmp)

Extension property: if F is a finite L-model with a unique root r and $x \vDash \varphi$ for every $x \in F \smallsetminus \{r\}$, then we can change valuation of variables in r to make $r \vDash \varphi$

Theorem [Ghilardi]: The following are equivalent:

- φ is projective
- φ has the extension property
- θ_φ is a unifier of φ

where θ_φ is an explicitly defined composition of substitutions of the form $\sigma(x) = \boxdot\varphi \wedge x$ or $\sigma(x) = \boxdot\varphi \to x$

Bases of admissible rules

If L is an extensible logic, it has a basis of admissible rules consisting of

$$\frac{\Box y \to \Box x_1 \vee \cdots \vee \Box x_n}{\boxdot y \to x_1, \ldots, \boxdot y \to x_n} \qquad (n \in \omega)$$

if L admits an irreflexive point, and

$$\frac{\boxdot(y \leftrightarrow \Box y) \to \Box x_1 \vee \cdots \vee \Box x_n}{\boxdot y \to x_1, \ldots, \boxdot y \to x_n} \qquad (n \in \omega)$$

if L admits a reflexive point

For L linear extensible, take only $n = 0, 1$

Complexity of admissible rules

Lower bound:
Assume $L \supseteq \mathbf{K4}$ and every depth-3 tree is a skeleton of an L-frame with prescribed final clusters.
Then L-admissibility is coNEXP-hard.

Upper bounds: Admissibility in

- csf extensible logics is coNEXP-complete
- csf linearly extensible logics is coNP-complete

Intuitionistic logic

Admissible rules of $\mathbf{IPC}$ and some intermediate logics ($\mathbf{KC}$, $\mathbf{LC}$, ...) can be analyzed similarly to the modal case:

- Admissibility is decidable (Rybakov)
- Projective approximations exist (Ghilardi)
  - finitary unification type
  - complete sets of unifiers computable
- Bases of admissible rules (Iemhoff)
- Computational complexity of admissibility (J.)
- ...

Translation for intermediate logics

In fact, admissibility in intermediate logics can be directly reduced to modal logics by means of the Blok–Esakia isomorphism, using the following result of Rybakov:

Theorem:
If $L \supseteq \mathbf{IPC}$ and σL is its largest modal companion, then

$$\Gamma \mathrel{|\!\!\sim}_L \Delta \Leftrightarrow \mathsf{T}(\Gamma) \mathrel{|\!\!\sim}_{\sigma L} \mathsf{T}(\Delta),$$

where T is the Gödel translation

Example: $\sigma\mathbf{IPC} = \mathbf{S4Grz}$, $\sigma\mathbf{KC} = \mathbf{S4.2Grz}$, $\sigma\mathbf{LC} = \mathbf{S4.3Grz}$, $\sigma\mathbf{CPC} = \mathbf{Triv}$

Łukasiewicz logic

Admissibility in Łukasiewicz logic

Parameter-free admissible rules of **Ł** are fairly well understood:

- Admissibility is equivalent to validity in the 1-generated free MV-algebra
- Semantic (geometric) description of admissible rules and admissibly saturated formulas
- All formulas have admissibly saturated approximations
- Admissibility in **Ł** is decidable (PSPACE-complete)
- Explicit basis of admissible rules
- Admissibly saturated formulas are exact [Cabrer]
- OTOH: **Ł** has nullary unification type [Marra&Spada]
  $\Rightarrow$ projective approximations in general do not exist

Anchoredness

If $X \subseteq \mathbb{R}^n$, let $A(X)$ be its affine hull and $C(X)$ its convex hull

X is anchored if $A(X) \cap \mathbb{Z}^n \neq \varnothing$

Using Hermite normal form, we obtain:

- $X \subseteq \mathbb{Q}^n$ is anchored iff

$$\forall u \in \mathbb{Z}^n \, \forall a \in \mathbb{Q} \, [\forall x \in X \, (u^{\mathsf{T}} x = a) \Rightarrow a \in \mathbb{Z}]$$

  (Whenever X is contained in a hyperplane defined by an affine function with integral linear coefficients, its constant coefficients must be integral, too.)

- Given $x_0, \dots, x_k \in \mathbb{Q}^n$, it is decidable in polynomial time whether $\{x_0, \dots, x_k\}$ is anchored

Characterization of admissibility in Ł

Theorem [J.]: Write $t(\Gamma) = \{x \in [0,1]^n : \forall\varphi \in \Gamma\, \varphi(x) = 1\}$ as a union of rational polytopes $\bigcup_{j<r} C_j$.

Then $\Gamma \not\mathrel{|\!\!\sim}_{\text{Ł}} \Delta$ iff $\exists a \in \{0,1\}^n\ \forall\psi \in \Delta\ \exists j_0, \dots, j_k < r$ such that

- $a \in C_{j_0}$
- each C_{j_i} is anchored
- $C_{j_i} \cap C_{j_{i+1}} \neq \varnothing$
- $\psi(x) < 1$ for some $x \in C_{j_k}$

Corollary: Admissibility in Ł is decidable

Computational complexity

- $\Gamma \not\mathrel{|\!\!\sim}_{\text{Ł}} \Delta$ is reducible to reachability in an exponentially large graph with poly-time edge relation:
  - vertices: anchored polytopes in $t(\Gamma)$
  - edges: C, C' connected iff $C \cap C' \neq \varnothing$

  $\Rightarrow \mathrel{|\!\!\sim}_{\text{Ł}} \in \mathrm{PSPACE}$
- In fact: $\mathrel{|\!\!\sim}_{\text{Ł}}$ is PSPACE-complete
- In contrast, $\mathrm{Th}(\text{Ł})$ and $\vdash_{\text{Ł}}$ are coNP-complete [Mundici]

Admissibly saturated formulas

The characterization of $\mathrel{|\!\sim}_{\text{Ł}}$ easily implies:

- $\varphi \in F_n$ is admissibly saturated in **Ł** iff $t(\varphi)$
  - is connected,
  - intersects $\{0,1\}^n$, and
  - is piecewise anchored
    (i.e., a finite union of anchored polytopes)
- In **Ł**, every formula φ has an admissibly saturated approximation

Exact and projective formulas

- Cabrer gave a description of exact formulas in **Ł**, which implies the equivalence of:
  - φ is admissibly saturated
  - φ is exact
  - $t(\varphi)$ is connected and $\vdash_{\text{Ł}} \varphi \leftrightarrow \bigvee_i \pi_i$ with projective π_i
- Marra & Spada proved that **Ł** has nullary unification type $\Rightarrow$ it can't have projective approximations
  - Example: $x \vee \neg x \vee y \vee \neg y$ is admissibly saturated, but not projective

Multiple-conclusion basis

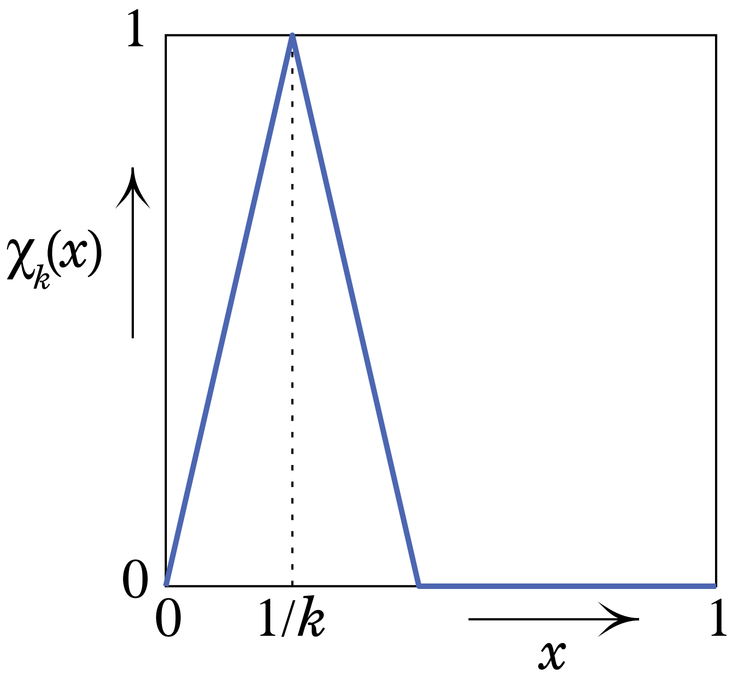
The construction of a.s. approximations can be simulated by simple rules:

Theorem [J.]: $\{NA_p : p \text{ is a prime}\} + CC_3 + WDP$ is an independent basis of multiple-conclusion Ł-admissible rules

$$NA_k = \frac{x \lor \chi_k(y)}{x}$$

$$CC_n = \frac{\neg(y \lor \neg y)^n}{}$$

$$WDP = \frac{x \lor \neg x}{x, \neg x}$$



Admissibility with parameters in modal logics

Known results

Not that much is known about admissibility in transitive modal logics in the presence of parameters:

- Rybakov's results on decidability of admissibility also apply to admissibility with parameters
- Recently, he expanded the results to effectively construct complete sets of unifiers $\Rightarrow$ finitary unification type

Terminology: From now on, admissibility and unification always allow parameters

New results

Parameters complicate matters, but typical properties carry over:

- Ghilardi-style characterization of projective formulas
- Existence of projective approximations for cluster-extensible (clx) logics [defined on the next slide]
- Semantic description of admissibility in clx logics
- Explicit bases of admissible rules for clx logics
- Computational complexity:
  - Lower bounds on unification in wide classes of transitive logics
  - Matching upper bounds for admissibility in clx logics
- Translation of these results to intuitionistic logic

Cluster-extensible logics

Let L be a transitive modal logic with fmp, $n \in \omega$, and C a finite cluster.

A finite rooted frame F is of type $\langle n, C \rangle$ if its root cluster $\mathrm{rcl}(F)$ is isomorphic to C and has n immediate successor clusters.

L is $\langle n, C \rangle$-extensible if:
For every type-$\langle n, C \rangle$ frame F, if $F \smallsetminus \mathrm{rcl}(F)$ is an L-frame, then so is F.

L is cluster-extensible (clx), if it is $\langle n, C \rangle$-extensible whenever there exists a type-$\langle n, C \rangle$ L-frame.

Properties of clx logics

Examples: All combinations of $\mathbf{K4}$, $\mathbf{S4}$, $\mathbf{GL}$, $\mathbf{D4}$, $\mathbf{K4Grz}$, $\mathbf{K4.1}$, $\mathbf{K4.3}$, $\mathbf{K4B}$, $\mathbf{S5}$, $\pm$ bounded branching

Nonexamples: $\mathbf{K4.2}$, $\mathbf{S4.2}$, $\dots$

For every clx logic L:

- L is finitely axiomatizable
- L has the exponential-size model property
- L is $\forall\exists$-definable on finite frames
- L is decidable in PSPACE
  (if width ≥ 2, PSPACE-complete)

Projective formulas: the extension property

Fix $L \supseteq \mathbf{K4}$ with the fmp, and P and V finite sets of parameters and variables, resp.

- If F is a rooted model with valuation of $P \cup V$, its variant is any model F' which differs from F only by changing the value of some variables $x \in V$ in $\mathrm{rcl}(F)$
- A set M of finite rooted L-models evaluating $P \cup V$ has the model extension property, if:
  every L-model F whose all rooted generated proper submodels belong to M has a variant $F' \in M$
- A formula φ in atoms $P \cup V$ has the model extension property if $\mathrm{Mod}_L(\varphi) := \{F : \forall x \in F\ (x \vDash \varphi)\}$ does

Projective formulas: Löwenheim substitutions

Let φ be a formula in atoms $P \cup V$

- For every $D = \{\beta_x : x \in V\}$, where each β_x is a Boolean function of the parameters P, define the substitution

$$\theta_D(x) = (\boxdot\varphi \wedge x) \vee (\neg\boxdot\varphi \wedge \beta_x)$$

- Let θ_φ be the composition of substitutions θ_D for all the $2^{2^{|P|}|V|}$ possible D's, in arbitrary order

Projective formulas: a characterization

Theorem:
Let $L \supseteq \mathbf{K4}$ have the fmp, and φ be a formula in finitely many parameters P and variables V. Tfae:

- φ is projective
- φ has the model extension property
- θ_φ^N is a unifier of φ

where $N = (|B| + 1)(2^{|P|} + 1)$, $B = \{\psi : \Box\psi \subseteq \varphi\}$

Remark: If $P = \varnothing$, we have $N \le 2|\varphi|$.
Ghilardi's original proof gives N nonelementary
(tower of exponentials of height $\mathrm{md}(\varphi)$)

Projective approximations

Theorem:

If L is a clx logic, every formula φ has a projective approximation Π_φ.

Moreover, every $\pi \in \Pi_\varphi$ is a Boolean combination of subformulas of φ.

Corollary:

- $\{\theta_\pi : \pi \in \Pi_\varphi\}$ is a complete set of unifiers of φ
- Admissibility in L is decidable
- If $n = |\varphi|$, then $|\Pi_\varphi| \le 2^{2^n}$, and $|\pi| = O(n2^n)\ \forall \pi \in \Pi_\varphi$
- $|\theta_\pi|$ is doubly exponential in $|B| + |V|$, and triply exponential in $|P|$. This is likely improvable.

Size of projective approximations

The bounds $|\Pi_\varphi| = 2^{2^{O(n)}}$ and $|\pi| = 2^{O(n)}$ for $\pi \in \Pi_\varphi$ are asymptotically optimal, even if $P = \varnothing$:

- If L is $\langle 2, \bullet \rangle$-extensible (e.g., $\mathbf{K4}$, $\mathbf{GL}$), consider

$$\varphi_n = \bigwedge_{i<n} (\Box x_i \vee \Box \neg x_i) \to \Box y \vee \Box \neg y$$

$$\Pi_{\varphi_n} = \Big\{ \bigwedge_{i<n} (\boxdot x_i \vee \boxdot \neg x_i) \to (y \leftrightarrow \beta(\vec{x})) \;\Big|\; \beta \colon \mathbf{2}^n \to \mathbf{2} \Big\}$$

- Similar examples work for $\langle 2, \circ \rangle$-extensible logics ($\mathbf{S4}$)

Irreflexive extension rules

Let $n < \omega$, and P a finite set of parameters.

$\mathrm{Ext}^P_{n,\bullet}$ is the set of rules

$$\frac{P^e \wedge \Box y \to \Box x_1 \vee \cdots \vee \Box x_n}{\boxdot y \to x_1, \dots, \boxdot y \to x_n}$$

for each assignment $e\colon P \to \mathbf{2}$

Notation:

$\varphi^1 = \varphi$, $\varphi^0 = \neg\varphi$, $P^e = \bigwedge_{p \in P} p^{e(p)}$, $\mathbf{2}^P = \{e \mid e\colon P \to \mathbf{2}\}$

Reflexive extension rules

Let C be a finite reflexive cluster

$\mathrm{Ext}^P_{n,C}$ is the set of the following rules:
Pick $E\colon C \to \mathbf{2}^P$ and $e_0 \in E(C)$, and consider

$$\frac{P^{e_0} \wedge \boxdot\Big(y \to \bigvee_{e\in E(C)} \Box(P^e \to y)\Big) \wedge \bigwedge_{e\in E(C)} \boxdot\big(\Box(P^e \to \Box y) \to y\big) \to \Box x_1 \vee \cdots \vee \Box x_n}{\boxdot y \to x_1, \dots, \boxdot y \to x_n}$$

Tight predecessors

P a finite set of parameters, C a finite cluster, $n < \omega$

- A P-L-frame is a (Kripke or general) L-frame W together with a fixed valuation of parameters $p \in P$
- If $X = \{w_1, \dots, w_n\} \subseteq W$ and $E \colon C \to \mathbf{2}^P$, a tight E-predecessor (E-tp) of X is $\{u_c : c \in C\} \subseteq W$ such that

$$u_c \vDash P^{E(c)}, \qquad u_c{\uparrow} = X{\Uparrow} \cup \{u_d : d \in c{\uparrow}\}$$

  (Note: $c{\uparrow} = C$ if C is reflexive, $c{\uparrow} = \varnothing$ if irreflexive)
- W is $\langle n, C\rangle$-extensible if every $\{w_1, \dots, w_n\} \subseteq W$ has an E-tp for every $E \colon C \to \mathbf{2}^P$
- If L is a clx logic, W is L-extensible if it is $\langle n, C\rangle$-extensible whenever L is

Correspondence and completeness

Theorem: If P is a finite set of parameters and W is a descriptive or Kripke P-$\mathbf{K4}$-frame, tfae:

- $W \vDash \mathrm{Ext}^P_{n,C}$
- W is $\langle n, C\rangle$-extensible

Corollary: For a logic $L \supseteq \mathbf{K4}$, tfae:

- L is $\langle n, C\rangle$-extensible
- $\mathrm{Ext}^P_{n,C}$ is L-admissible for every P

Theorem: If L has fmp and is $\langle n, C\rangle$-extensible for all $\langle n, C\rangle \in X$, then $L + \{\mathrm{Ext}^P_{n,C} : \langle n, C\rangle \in X\}$ is complete wrt locally finite (= all rooted subframes finite) P-L-frames, $\langle n, C\rangle$-extensible for each $\langle n, C\rangle \in X$

Semantics and bases of admissible rules

Theorem:
Let L be a clx logic, and $\Gamma \mathbin{/} \Delta$ a rule in a finite set of parameters P. Then tfae:

- $\Gamma \mathrel{|\!\sim}_L \Delta$
- $\Gamma \mathbin{/} \Delta$ holds in every [locally finite] L-extensible P-L-frame
- $\Gamma \mathbin{/} \Delta$ is derivable in $\vdash_L$ extended by the rules $\mathrm{Ext}^P_{n,C}$ such that L is $\langle n, C\rangle$-extensible

Corollary: If L is a clx logic, it has a basis of admissible rules consisting of $\mathrm{Ext}^P_{n,C}$ for all finite P and all $\langle n, C\rangle$ such that L is $\langle n, C\rangle$-extensible

Complexity: wide logics

Theorem:
If $L \supseteq \mathbf{K4}$ has width ≥ 2, then unification (and thus inadmissibility) in L is NEXP-hard.

Theorem:
If L is a clx logic of width ≥ 2 and bounded cluster size, then inadmissibility (and thus unification) in L is NEXP-complete.

Examples: $\mathbf{GL}$, $\mathbf{K4Grz}$, $\mathbf{S4Grz}$, $\dots$ ($\pm$ bounded branching)

Complexity: fat logics

Theorem:
If $L \supseteq \mathbf{K4}$ has unbounded cluster size, then unification in L is coNEXP-hard.

Theorem:
If L is a clx logic of width ≤ 1 and unbounded cluster size, then inadmissibility in L is coNEXP-complete.

Examples: $\mathbf{S5}$, $\mathbf{K4.3}$, $\mathbf{S4.3}$, $\dots$

Complexity: wide and fat logics

L is "chubby" if for all $n > 0$ there is a finite rooted L-frame containing an n-element cluster C and an element incomparable with C

Recall: $\Sigma_2^{\mathrm{EXP}} = \mathrm{NEXP}^{\mathrm{NP}}$

Theorem:
If $L \supseteq \mathbf{K4}$ is chubby, then unification in L is Σ_2^{EXP}-hard.

Theorem:
If L is a clx logic of width ≥ 2 and unbounded cluster size, then inadmissibility in L is Σ_2^{EXP}-complete.

Examples: $\mathbf{K4}$, $\mathbf{S4}$, $\mathbf{S4.1}$, $\ldots$ ($\pm$ bounded branching)

Complexity: slim logics

Theorem:
If $L \supseteq \mathbf{K4}$, then unification in L is PSPACE-hard, unless L is a tabular logic of width 1.

Theorem:
If L is a clx logic of width 1, bounded cluster size, and depth > 1, then admissibility in L is PSPACE-complete.

Examples: $\mathbf{GL.3}$, $\mathbf{K4Grz.3}$, $\mathbf{S4Grz.3}$, $\dots$

Theorem:
If L is a tabular logic of width 1 and depth d, then unification and inadmissibility in L are Π^{P}_{2d}-complete.

Examples: $\mathbf{S5} + \mathbf{Alt}_n$, $\mathbf{K4} + \Box\bot$, $\dots$

Complexity: summary

We get the following classification for clx logics:

<table>
<tr><th colspan="2">logic</th><th rowspan="2">$\vdash_L$</th><th colspan="2">$\mid\!\sim_L$</th><th rowspan="2">example</th></tr>
<tr><th>cluster size</th><th>bran-ching</th><th>par.-free</th><th>with param's</th></tr>
<tr><td rowspan="2">$< \infty$</td><td>0</td><td colspan="2" rowspan="3">coNP-complete</td><td>Σ^{P}_2-c.</td><td>$\mathbf{S5} + \mathbf{Alt}_n$</td></tr>
<tr><td>1</td><td>PSPACE-c.</td><td>$\mathbf{GL.3}$</td></tr>
<tr><td>∞</td><td>≤ 1</td><td>NEXP-c.</td><td>$\mathbf{S5}$, $\mathbf{S4.3}$</td></tr>
<tr><td>$< \infty$</td><td rowspan="2">≥ 2</td><td rowspan="2">PSPACE-c.</td><td colspan="2">coNEXP-complete</td><td>$\mathbf{GL}$, $\mathbf{Grz}$</td></tr>
<tr><td>∞</td><td></td><td>Π^{EXP}_2-c.</td><td>$\mathbf{K4}$, $\mathbf{S4}$</td></tr>
</table>

With parameters, non-unifiability and admissibility have the same complexity

Logics with a top

The concept of clx logics and the whole machinery can be adapted to $\mathbf{S4.2}$ and similar logics with a single top cluster

logic: inner cl. size	logic: top cl. size	$\vdash_L$	$\mathrel{\mid\!\sim}_L$ par.-free	$\mathrel{\mid\!\sim}_L$ w/ param's	example
$< \infty$	$< \infty$	PSPACE-c.	coNEXP-complete		$\mathbf{GL.2}$, $\mathbf{Grz.2}$
	∞			Θ_2^{EXP}-c.	$\mathbf{S4.1.4} + \mathbf{S4.2}$
∞				Π_2^{EXP}-c.	$\mathbf{K4.2}$, $\mathbf{S4.2}$

Θ_2^{EXP} is the exponential version of the class Θ_2^{P}:

$$\Theta_2^{\mathrm{EXP}} := \mathrm{EXP}^{\mathrm{NP}[\mathrm{poly}]} = \mathrm{EXP}^{\|\mathrm{NP}} = \mathrm{P}^{\mathrm{NEXP}} = \mathrm{PSPACE}^{\mathrm{NEXP}}$$

Intuitionistic logic

Rybakov's translation theorem can be generalized to admissibility with parameters:

Theorem:
If $L \supseteq \mathbf{IPC}$ and σL is its largest modal companion, then

$$\Gamma \mathrel{|\!\!\sim}_L \Delta \Leftrightarrow \mathsf{T}(\Gamma) \mathrel{|\!\!\sim}_{\sigma L} \mathsf{T}(\Delta)$$

[However, $\bigwedge_{p \in P} \Box(p \to \Box p) \to \mathsf{T}(\varphi)$ is often more convenient.]

Note: Clx logics translate to $\mathbf{IPC}$ and the bounded branching logics $\mathbf{T}_n$ (incl. $\mathbf{T}_1 = \mathbf{LC}$, $\mathbf{T}_0 = \mathbf{CPC}$)
Extensions of $\mathbf{S4.2}$ give $\mathbf{KC}$ and $\mathbf{KC} + \mathbf{T}_n$

Corollaries

The translation yields:

- Char. of projective formulas in $L \supseteq \mathbf{IPC}$ with fmp
- Existence of projective approximations and semantic description of $\mathrel{|\!\sim}_L$ for $\mathbf{IPC}$, $\mathbf{KC}$, $\mathbf{T}_n$, $\mathbf{KC}+\mathbf{T}_n$
- Complexity (lower bounds need an extra argument): admissibility and non-unifiability is
  - coNEXP-complete for $\mathbf{IPC}$, $\mathbf{KC}$, $\mathbf{T}_n$, $\mathbf{KC}+\mathbf{T}_n$ ($n \geq 2$)
  - PSPACE-complete for $\mathbf{LC}$
  - Σ^{P}_{2d}-complete for $\mathbf{G}_{d+1}$
  - coNEXP-hard for any other intermediate logic

Intuitionistic extension rules

Bases of admissible rules require a separate construction:

A basis for $\mathbf{IPC}$ and $\mathbf{T}_n$ is given by the rules

$$\frac{\bigwedge P \wedge \Bigl(\bigvee_{i=1}^{n} x_i \vee \bigvee Q \to y\Bigr) \to \bigvee_{i=1}^{n} x_i \vee \bigvee Q}{\bigwedge P \wedge y \to x_1, \dots, \bigwedge P \wedge y \to x_n}$$

where P, Q are disjoint finite sets of parameters

Admissibility with parameters in Łukasiewicz logic

Overview of the situation

- Work in progress . . .
- Geometry more complicated—we can no longer restrict attention to McNaughton functions in one variable
- All formulas have admissibly saturated approximations of effectively bounded complexity
  $\Rightarrow$ problem reduces to description of a.s. formulas
- Some necessary conditions for a.s. formulas
  $\Rightarrow$ sufficient conditions for admissibility
- The conditions are complete for the case of 1 parameter

Reduction

In the parameter-free case, admissibility is detected by substitutions in one variable

In the presence of parameters, we need no variables at all:

Theorem: If $\Gamma \not\vdash_{\text{Ł}} \Delta$, where $\Gamma \cup \Delta$ are formulas in variables $x_1, \dots, x_n$ and parameters $p_1, \dots, p_m$, $m \geq 1$, then there is a substitution σ such that

- σ is a unifier of Γ
- σ is not a unifier of any $\delta \in \Delta$
- the only atoms occurring in any $\sigma(x_i)$ are $p_1, \dots, p_m$

Notation

We consider formulas $\varphi(p_1, \dots, p_m, x_1, \dots, x_n)$, $m \geq 1$

$t(\varphi) = \{v \in I^{m+n} : \varphi(v) = 1\}$, $I = [0, 1]$

π is the projection $I^{m+n} \to I^m$

Substitutions are represented by McNaughton functions $\sigma \colon I^m \to I^{m+n}$ such that $\pi \circ \sigma = \mathrm{id}$

σ is a unifier of φ iff $\mathrm{rng}(\sigma) \subseteq t(\varphi)$

We fix rational polyhedral complexes $P = \{P_i : i < r\}$ and $Q = \{Q_j : j < s\}$ such that

- $t(\varphi) = \|P\| := \bigcup_{i<r} P_i$
- $I^m = \|Q\|$
- $\forall i \, \exists j \, \pi(P_i) = Q_j$

More notation

If $X \subseteq \mathbb{R}^k$, $\mathrm{A}(X)$ denotes its affine hull

$\operatorname{Int} P_i$ is the "geometric interior" of the polytope P_i:

$$\begin{aligned} \operatorname{Int} P_i &= P_i \smallsetminus \bigcup \{P_j : P_j \subsetneq P_i\} \\ &= \text{relative topological interior of } P_i \text{ in } \mathrm{A}(P_i) \end{aligned}$$

Every point of $\|P\|$ belongs to a unique $\operatorname{Int} P_i$

Admissibly saturated approximations

Theorem: The following are equivalent:

- φ is admissibly saturated
- $\forall \varepsilon > 0$ there is a unifier σ of φ such that

$$t(\varphi) \subseteq B(\operatorname{rng}(\sigma), \varepsilon) := \{x : \operatorname{dist}(x, \operatorname{rng}(\sigma)) < \varepsilon\}$$

- there is a unifier σ of φ whose range meets $\operatorname{Int} P_i$ for every maximal $P_i \in P$

Corollary:
Every φ has an admissibly saturated approximation, whose elements are subcomplexes of P

Admissibly saturated formulas

Need a more intrinsic description of a.s. formulas

Question: How can $\operatorname{rng}(\sigma)$ look like in terms of the P_i's when σ is a unifier of φ?

Let $P(\sigma) = \{i : \operatorname{rng}(\sigma) \cap \operatorname{Int} P_i \neq \varnothing\}$

Example: For every Q_j there is $i \in P(\sigma)$ s.t. $Q_j = \pi(P_i)$

Goodness

$(a_1, \ldots, a_m, b_1, \ldots, b_n) \in \mathbb{R}^{m+n}$ is good if $b_j \in \mathbb{Z} + \sum_i a_i \mathbb{Z} \;\; \forall j$

P_i is good if $\operatorname{Int} P_i$ contains a rational good point

Note: If all $a_i \in \mathbb{Q}$, then $\mathbb{Z} + \sum_i a_i \mathbb{Z} = \frac{1}{d}\mathbb{Z}$, where $d = \operatorname{den}(a_1, \ldots, a_m)$

$\operatorname{rng}(\sigma)$ consists of good points

Corollary: If $i \in P(\sigma)$, then P_i is good

Lemma: If $\mathrm{A}(P_i)$ contains a rational good point and $\pi(P_i)$ is not a single point, then rational good points are dense in $\mathrm{A}(P_i)$, and a fortiori in $\operatorname{Int} P_i$

Projection anchoredness

P_i is projection anchored if there exists an affine map $L\colon \mathrm{A}(\pi(P_i)) \to \mathrm{A}(P_i)$ with integer coefficients s.t. $\pi \circ L = \mathrm{id}$

P_i is fully anchored if it is projection anchored and $\mathrm{A}(\pi(P_i)) = \mathbb{R}^m$

Lemma: If P_i is projection anchored and $b \in \mathrm{A}(P_i)$ is a good point, there exists L as above s.t. $L(\pi(b)) = b$

Note: A projection anchored P_i is good, unless $\pi(P_i)$ is a single point

Anchoredness of substitutions

Let $b = \sigma(a) \in \operatorname{Int} P_i$. There is a neighbourhood $U \ni a$ mapped by σ into a neighbourhood of b small enough to meet only $\operatorname{Int} P_j$ s.t. $P_j \supseteq P_i$.

If $\pi(P_j) \supsetneq \pi(P_i)$ (and thus $\operatorname{Int} \pi(P_i) \cap \operatorname{Int} \pi(P_j) = \varnothing$) for every $P_j \supsetneq P_i$, $j \in P(\sigma)$, we must have

$$\sigma \colon U \cap \pi(\operatorname{Int} P_i) \to \operatorname{Int} P_i.$$

We can restrict σ further to a relatively open subset $V \subseteq \pi(\operatorname{Int} P_i)$ where it is affine. Since $\mathrm{A}(V) = \mathrm{A}(\pi(P_i))$, P_i is projection anchored.

This motivates the following recursive definition:

Hereditary anchoredness

P_i is hereditarily anchored if

- P_i is good and projection anchored
- every $Q_k \supsetneq \pi(P_i)$ is the projection of some hereditarily anchored $P_j \supsetneq P_i$

P_i is hereditarily covered if

- P_i is good
- $\pi(P_i) = \pi(P_j)$ for some hereditarily anchored $P_j \supseteq P_i$

Corollary: If $i \in P(\sigma)$, then P_i is hereditarily covered

Note: If $P_i \in P$ is maximal, it is hereditarily anchored iff it is fully anchored

Admissibly saturated formulas

Any admissibly saturated formula satisfies:

(1) Every maximal $P_i \in P$ is hereditarily anchored

(2) Every nonempty Q_j is the projection of some hereditarily anchored P_i

(3) For every j,

$$\bigcup\{\operatorname{Int} P_i : Q_j \subseteq \pi(P_i),\ P_i \text{ hereditarily covered}\}$$

is connected

Note: Condition (3) implies that the fiber $\|P\| \cap \pi^{-1}(a)$ is connected $\forall a \in I^m$

Questions

- Is admissibility with parameters in **Ł** decidable?
- Are the given conditions for admissibly saturated formulas in **Ł** sufficient?
- Is there a general reduction of admissibility to non-unifiability (with parameters)?

Thank you for attention!

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