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LINEAR BOUNDARY VALUE TYPE PROBLEMS FOR FUNCTIONAL-DIFFERENTIAL EQUATIONS AND THEIR ADJOINTS

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0. INTRODUCTION

The paper deals with boundary value type problems for functional-differential equations

$$\dot{x}(t) = \int_{-r}^{0} [d_\vartheta P(t, \vartheta)]\, x(t + \vartheta) + f(t) \quad \text{a.e. on} \quad [a, b] \tag{0,1}$$

or

$$\dot{x}(t) = A(t)\, x(t) + B(t)\, x(t - r) + \int_{a-r}^{b} [d_s G(t, s)]\, x(s) + f(t) \text{ a.e. on } [a, b], \tag{0,2}$$

where $-\infty < a < b < \infty$ and the functions $P(t, \vartheta)$, $G(t, s)$, $A(t)$, $B(t)$ and $f(t)$ fulfil some natural assumptions. In particular, we derive their adjoints and in some special cases prove the Fredholm alternative. (The results of A. HALANAY [5] or E. A. LIFŠIC [9] on the existence of periodic solutions to the equation (0,1) and the results of [12] on integral boundary value problems for ordinary integro-differential equations are included.) Our approach is based on the ideas of D. WEXLER [14] and ŠT. SCHWABIK [11] and differs from that of A. Halanay [6] or D. HENRY [8] (cf. also J. K. HALE [7]). The adjoint problems obtained seem to be more natural than those of D. Henry [8] and follow directly from the principles of functional analysis. (It is shown that after some artificial steps our adjoint reduces to that of D. Henry.) Initial functions are continuous on $[a - r, a]$ or of bounded variation on $[a - r, a]$. In § 4 boundary value type problems for hereditary differential equations considered in the sense of M. C. DELFOUR, S. K. MITTER [3] (with square integrable initial functions) are treated.

Let $-\infty < \alpha < \beta < +\infty$. The closed interval $\alpha \leqq t \leqq \beta$ is denoted by $[\alpha, \beta]$, its interior $\alpha < t < \beta$ by (α, β) and the corresponding half-open intervals by $[\alpha, \beta)$ and $(\alpha, \beta]$. Given a $p \times q$-matrix $M = (m_{i,j})_{i=1,\dots,p, j=1,\dots,q}$, M' denotes its transpose and

$$\|M\| = \max_{i=1,\dots,p} \sum_{j=1}^{q} |m_{i,j}| .$$

$\mathscr{R}_n$ is the space of real column n-vectors with the norm $\|x\| = \max_{i=1,\dots,p} |x_i|$. The space of real row n-vectors is $\mathscr{R}_n^*$. (Elements of $\mathscr{R}_n^*$ are denoted by x', where $x \in \mathscr{R}_n$;

$$\|x'\| = \sum_{i=1}^{n} |x_i| .)$$

$\mathscr{C}_n(\alpha, \beta)$ is the Banach space (B-space) of continuous functions $u : [\alpha, \beta] \to \mathscr{R}_n$ with the norm $\|u\|_{\mathscr{C}} = \sup_{t \in [\alpha,\beta]} \|u(t)\|$; $\mathscr{BV}_n(\alpha, \beta)$ is the B-space of functions $u : [\alpha, \beta] \to \mathscr{R}_n$ of bounded variation on $[\alpha, \beta]$ with the norm $\|u\|_{\mathscr{BV}} = \|u(\beta)\| + \operatorname{var}_\alpha^\beta u$; $\mathscr{V}_n^0(\alpha, \beta)$ is the set of functions $u' : [\alpha, \beta] \to \mathscr{R}_n^*$ of bounded variation on $[\alpha, \beta]$, right continuous on (α, β) and vanishing at β (being equipped with the norm $\|u'\|_{\mathscr{BV}}$, $\mathscr{V}_n^0(\alpha, \beta)$ becomes a B-space); $\mathscr{AC}_n(\alpha, \beta)$ is the B-space of absolutely continuous functions $u : [\alpha, \beta] \to \mathscr{R}_n$ with the norm $\|u\|_{\mathscr{AC}} = \|u\|_{\mathscr{BV}}$; $\mathscr{L}_n(\alpha, \beta)$ is the B-space of Lebesgue integrable (L-integrable) functions $u : [\alpha, \beta] \to \mathscr{R}_n$ with the norm

$$\|u\|_{\mathscr{L}} = \int_\alpha^\beta \|u(t)\| \, \mathrm{d}t ;$$

$\mathscr{L}_n^\infty(\alpha, \beta)$ is the B-space of essentially bounded functions $u' : [\alpha, \beta] \to \mathscr{R}_n^*$ with the norm $\|u'\| = \sup \operatorname{ess}_{t \in [\alpha,\beta]} \|u'(t)\|$.

Given a B-space $\mathscr{X}$, $\mathscr{X}^*$ denotes its dual and the value of a functional $y \in \mathscr{X}^*$ on $x \in \mathscr{X}$ is denoted by $\langle x, y \rangle_{\mathscr{X}}$. The zero functional on $\mathscr{X}$ is denoted by $o_{\mathscr{X}}$. Hereafter $\mathscr{L}_n^*(\alpha, \beta)$ and $\mathscr{C}_n^*(\alpha, \beta)$ are identified with $\mathscr{L}_n^\infty(\alpha, \beta)$ and $\mathscr{V}_n^0(\alpha, \beta)$, respectively, while

$$\langle x, y' \rangle_{\mathscr{L}} = \int_\alpha^\beta y'(t)\, x(t) \, \mathrm{d}t \quad \text{and} \quad \langle u, v' \rangle_{\mathscr{C}} = \int_\alpha^\beta [\mathrm{d}v'(t)]\, u(t)$$

for $x \in \mathscr{L}_n(\alpha, \beta)$, $y' \in \mathscr{L}_n^\infty(\alpha, \beta)$, $u \in \mathscr{C}_n(\alpha, \beta)$ and $v' \in \mathscr{V}_n^0(\alpha, \beta)$. (There are isometric isomorphisms between $\mathscr{L}_n^*(\alpha, \beta)$ and $\mathscr{L}_n^\infty(\alpha, \beta)$ and between $\mathscr{C}_n^*(\alpha, \beta)$ and $\mathscr{V}_n^0(\alpha, \beta)$, cf. e.g. [4].)

Let $\mathscr{X}$, $\mathscr{Y}$ be B-spaces. Given a linear bounded operator $T : \mathscr{X} \to \mathscr{Y}$ (defined on the whole $\mathscr{X}$), T^* denotes its adjoint ($T^* : \mathscr{Y}^* \to \mathscr{X}^*$, $\langle Tx, y \rangle_{\mathscr{Y}} = \langle x, T^* y \rangle_{\mathscr{X}}$ for all $x \in \mathscr{X}$ and $y \in \mathscr{Y}^*$), $\operatorname{Ker}(T)$ is the set of all $x \in \mathscr{X}$ such that $Tx = 0$ = zero element of $\mathscr{Y}$ and $\operatorname{Im}(T)$ is the range of T. Given two operators $T_1 : \mathscr{X}_1 \to \mathscr{Y}$, $T_2 : \mathscr{X}_2 \to \mathscr{Y}$, the homogeneous equations $T_1 x = 0$ and $T_2 z = 0$ are said to be equivalent if there is a one-to-one correspondence between $\operatorname{Ker}(T_1)$ and $\operatorname{Ker}(T_2)$.

2. GENERAL BOUNDARY VALUE TYPE PROBLEM AND ITS ADJOINT

2,1. Assumptions. We assume $-\infty < a < b < +\infty$, $r > 0$*). $A(t)$ and $B(t)$ are $n \times n$-matrix functions L-integrable on $[a, b]$, $G(t, s)$ is a Borel measurable in (t, s) on $[a, b] \times [a - r, b]$ $n \times n$-matrix function such that $\operatorname{var}_{a-r}^{b} G(t, \cdot) < \infty$ for any $t \in [a, b]$ and

$$\int_a^b (\|G(t, b)\| + \operatorname{var}_{a-r}^{b} G(t, \cdot))\, dt < \infty ,$$

$f(t) \in \mathscr{L}_n(a, b)$. Λ is an arbitrary B-space, $l \in \Lambda$ and the operators $M : \mathscr{C}_n(a - r, a) \to \Lambda$, $N : \mathscr{AC}_n(a, b) \to \Lambda$ are linear and bounded, while $\operatorname{Im}(N^*) \subset \mathscr{C}_n^*(a, b) = \mathscr{V}_n^0(a, b)$ (i.e., given $\lambda \in \Lambda^*$, there is a function $(N^*\lambda)(t) \in \mathscr{V}_n^0(a, b)$ such that

$$\langle Nx, \lambda \rangle_\Lambda = \langle x, N^*\lambda \rangle_{\mathscr{AC}} = \int_a^b [d(N^*\lambda)(t)]\, x(t) \text{ for all } x \in \mathscr{AC}_n(a, b)) .$$

Without any loss of generality we may also assume that, given $t \in [a, b]$, the function $G(t, \cdot)$ is right continuous on $(a - r, b)$, while $G(t, b) = 0$. Given $\lambda \in \Lambda^*$, let us denote by $(M^*\lambda)(t)$ the row n-vector function such that $(M^*\lambda)(t) - (N^*\lambda)(a) \in \mathscr{V}_n^0(a - r, a)$ and

$$\langle Mu, \lambda \rangle_\Lambda = \langle u, M^*\lambda \rangle_{\mathscr{C}} = \int_{a-r}^{a} [d\{(M^*\lambda)(t) - (N^*\lambda)(a)\}]\, u(t)$$

for all $u \in \mathscr{C}_n(a - r, a)$.

We are interested in the following boundary value type problem:

2,2. Problem (P). Determine $x \in \mathscr{AC}_n(a, b)$ and $u \in \mathscr{C}_n(a - r, a)$ such that

$$(2,1) \quad \dot{x}(t) = A(t)\, x(t) + \begin{Bmatrix} B(t)\, u(t - r), & t < a + r \\ B(t)\, x(t - r), & t \geqq a + r \end{Bmatrix} + \int_{a-r}^{a} [d_s G(t, s)]\, u(s) + \int_a^b [d_s G(t, s)]\, x(s) + f(t) \quad \text{a.e. on } [a, b] ,$$

$$(2,2) \quad x(a) = u(a) ,$$

$$(2,3) \quad Mu + Nx = l ,$$

where Assumptions 2,1 are fulfilled.

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*) If $r = 0$, the equation (2,1) reduces to an ordinary integro-differential equation with initial data in R_n. The case of $r = 0$ will be treated separately later on (cf. Sec. 5,5).

2,3. Notation. Let us put

$$\mathcal{X} = \mathcal{AC}_n(a, b) \times \mathcal{C}_n(a - r, a), \quad \mathcal{Y} = \mathcal{L}_n(a, b) \times \Lambda \times \mathcal{R}_n$$

and

$$U : \begin{pmatrix} x \\ u \end{pmatrix} \in \mathcal{X} \to \begin{pmatrix} Dx - Ax - B_1 x - B_2 u - G_1 x - G_2 u \\ Mu + Nx \\ u(a) - x(a) \end{pmatrix} \in \mathcal{Y}, \tag{2,4}$$

where

$$D : x \in \mathcal{AC}_n(a, b) \to \dot{x}(t) \in \mathcal{L}_n(a, b),$$

$$A : x \in \mathcal{AC}_n(a, b) \to A(t)\, x(t) \in \mathcal{L}_n(a, b),$$

$$B_1 : x \in \mathcal{AC}_n(a, b) \to \begin{Bmatrix} 0, & t < a + r \\ B(t)\, x(t - r), & t \geqq a + r \end{Bmatrix} \in \mathcal{L}_n(a, b),$$

$$B_2 : u \in \mathcal{C}_n(a - r, a) \to \begin{Bmatrix} B(t)\, u(t - r), & t < a + r \\ 0, & t \geqq a + r \end{Bmatrix} \in \mathcal{L}_n(a, b),$$

$$G_1 : x \in \mathcal{AC}_n(a, b) \to \int_a^b [\mathrm{d}_s G(t, s)]\, x(s) \in \mathcal{L}_n(a, b),$$

$$G_2 : u \in \mathcal{C}_n(a - r, a) \to \int_{a-r}^a [\mathrm{d}_s G(t, s)]\, u(s) \in \mathcal{L}_n(a, b).$$

All these operators are linear and bounded. The given problem (P) can be reformulated as the operator equation

$$U \begin{pmatrix} x \\ u \end{pmatrix} = \begin{bmatrix} f \\ l \\ 0 \end{bmatrix}.$$

Clearly, $\mathcal{X}^* = \mathcal{AC}_n^*(a, b) \times \mathcal{V}_n^0(a - r, a)$, $\mathcal{Y}^* = \mathcal{L}_n^\infty(a, b) \times \Lambda^* \times \mathcal{R}_n^*$ and

$$\left\langle \begin{pmatrix} x \\ u \end{pmatrix}, (g, h') \right\rangle_{\mathcal{X}} = \langle x, g \rangle_{\mathcal{AC}} + \int_{a-r}^a [\mathrm{d}h'(t)]\, u(t),$$

$$\left\langle \begin{bmatrix} f \\ l \\ k \end{bmatrix}, (y', \lambda, \gamma') \right\rangle_{\mathcal{Y}} = \int_a^b y'(t) f(t)\, \mathrm{d}s + \langle l, \lambda \rangle_\Lambda + \gamma' k$$

for $x \in \mathcal{AC}_n(a, b)$, $u \in \mathcal{C}_n^*(a - r, a)$, $g \in \mathcal{AC}_n^*(a, b)$, $h' \in \mathcal{V}_n^0(a - r, a)$, $f \in \mathcal{L}_n(a, b)$, $l \in \Lambda$, $k \in \mathcal{R}_n$, $y' \in \mathcal{L}_n^\infty(a, b)$, $\lambda \in \Lambda^*$ and $\gamma' \in \mathcal{R}_n^*$. Let $\begin{pmatrix} x \\ u \end{pmatrix} \in \mathcal{X}$ and $(y', \lambda, \gamma') \in \mathcal{Y}^*$, then

$$\left\langle U \begin{pmatrix} x \\ u \end{pmatrix}, (y', \lambda, \gamma') \right\rangle_{\mathcal{Y}} = \langle Dx - Ax - B_1 x - B_2 u - G_1 x - G_2 u, y' \rangle_{\mathcal{L}} +$$

$$+ \langle Mu + Nx, \lambda \rangle_\Lambda + \gamma'(u(a) - x(a)) =$$

$$= \langle x, D^*y' - A^*y' - B_1^*y' - G_1^*y' + N^*\lambda + K_1^*\gamma'\rangle_{\mathscr{AC}} +$$
$$+ \langle u, -B_2^*y' - G_2^*y' + M^*\lambda + K_2^*\gamma'\rangle_{\mathscr{C}},$$

where

$$K_1 : x \in \mathscr{AC}_n(a, b) \to -x(a) \in \mathscr{R}_n$$

and

$$K_2 : u \in \mathscr{C}_n(a - r, a) \to u(a) \in \mathscr{R}_n .$$

Consequently

$$U^* : (y', \lambda, \gamma') \in \mathscr{Y}^* \to \begin{bmatrix} D^*y' - A^*y' - B_1^*y' - G_1^*y' + N^*\lambda + K_1^*\gamma' \\ -B_2^*y' - G_2^*y' + M^*\lambda + K_2^*\gamma' \end{bmatrix} \in \mathscr{X}^*$$

and the adjoint to (P) is the system of equations for $(y', \lambda, \gamma') \in \mathscr{Y}^*$

(2,5) $$D^*y' - A^*y' - B_1^*y' - G_1^*y' + N^*\lambda + K_1^*\gamma' = o_{\mathscr{AC}},$$
$$- B_2^*y' - G_2^*y' + M^*\lambda + K_2^*\gamma' = o_{\mathscr{C}}.$$

2,4. An analytic form of the adjoint problem. By the definition of an adjoint operator and by the unsymmetric Fubini theorem (2) it holds for all $x \in \mathscr{AC}_n(a, b)$, $u \in \mathscr{C}_n(a - r, a)$, $y' \in \mathscr{L}_n^\infty(a, b)$, $\lambda \in \Lambda^*$ and $\gamma' \in \mathscr{R}_n^*$

$$\left\langle \begin{pmatrix} x \\ u \end{pmatrix}, U^*(y', \lambda, \gamma') \right\rangle_{\mathscr{X}} = \left\langle U \begin{pmatrix} x \\ u \end{pmatrix}, (y', \lambda, \gamma') \right\rangle_{\mathscr{Y}} =$$
$$= \int_a^b y'(t)\, \dot{x}(t)\, dt - \int_a^b y'(t)\, A(t)\, x(t)\, dt - \int_{a+r}^b y'(t)\, B(t)\, x(t - r)\, dt -$$
$$- \int_a^{a+r} y'(t)\, B(t)\, u(t - r)\, dt - \int_a^b y'(t) \left(\int_a^b [d_s G(t, s)]\, x(s) \right) dt -$$
$$- \int_a^b y'(t) \left(\int_{a-r}^a [d_s G(t, s)]\, u(s) \right) dt + \langle Mu + Nx, \lambda\rangle_\Lambda + \gamma'(u(a) - x(a)) =$$
$$= \int_a^b y'(t)\, \dot{x}(t)\, dt - \int_a^b [dg'(t)]\, x(t) - \int_{a-r}^a [dh'(t)]\, u(t),$$

where

(2,6) $$g'(t) = - \int_t^b y'(s)\, A(s)\, ds + \int_a^b y'(s)\, G(s, t)\, ds - (N^*\lambda)(t) -$$
$$- \begin{Bmatrix} \int_{t+r}^b y'(s)\, B(s)\, ds, & t \leqq b - r \\ 0 & , t > b - r \end{Bmatrix} - \begin{Bmatrix} \gamma', & t = a \\ 0, & t > a \end{Bmatrix} \quad \text{for} \quad t \in [a, b],$$

$$h'(t) = - \int_{t+r}^{a+r} y'(s)\, B(s)\, ds + \int_a^b y'(s)\, (G(s, t) - G(s, a))\, ds + \begin{Bmatrix} \gamma', & t < a \\ 0, & t = a \end{Bmatrix} -$$
$$- (M^*\lambda)(t) + (N^*\lambda)(a) \quad \text{for} \quad t \in [a - r, a].$$

Now, $(y`, \lambda, \gamma`) \in \operatorname{Ker}(U^*)$ iff

$$(2,7) \qquad 0 = \int_a^b y`(t)\,\dot{x}(t)\,\mathrm{d}t - \int_a^b [\mathrm{d}g`(t)]\,x(t) - \int_{a-r}^a [\mathrm{d}h`(t)]\,u(t)$$

for all $x \in \mathscr{AC}_n(a, b)$ and $u \in \mathscr{C}_n(a - r, a)$. In particular, if $x(t) = 0$ on $[a, b]$, (2,7) means that

$$\int_{a-r}^a [\mathrm{d}h`(t)]\,u(t) = 0 \quad \text{for all} \quad u \in \mathscr{C}_n(a - r, a)\,.$$

Since $h` \in \mathscr{V}_n^0(a - r, a)$, this is possible iff $h`(t) = 0$ on $[a - r, a]$. Thus

$$(2,8) \qquad \int_{t+r}^{a+r} y`(s)\,B(s)\,\mathrm{d}s - \int_a^b y`(s)\,(G(s, t) - G(s, a))\,\mathrm{d}s + \\ + (M^*\lambda)(t) - (N^*\lambda)(a) - \gamma` = 0 \quad \text{on} \quad [a - r, a)\,.$$

The equality (2,7) now becomes (after integrating by parts)

$$(2,9) \quad \int_a^b y`(t)\,\dot{x}(t)\,\mathrm{d}t = -g`(a)\,x(a) - \int_a^b g`(t)\,\dot{x}(t)\,\mathrm{d}t \quad \text{for all} \quad x \in \mathscr{AC}_n(a, b)\,.$$

Since we may choose $x(t) = x(a) \neq 0$ on $[a, b]$, (2,9) implies furthermore $g`(a) = 0$ or

$$(2,10) \quad \gamma` = -\int_a^b y`(s)\,A(s)\,\mathrm{d}s - \int_{a+r}^b y`(s)\,B(s)\,\mathrm{d}s + \int_a^b y`(s)\,G(s, a)\,\mathrm{d}s - (N^*\lambda)(a)\,.$$

Consequently, (2,9) reduces to

$$\int_a^b y`(t)\,\dot{x}(t)\,\mathrm{d}t = -\int_a^b g`(t)\,\dot{x}(t)\,\mathrm{d}t \quad \text{for all} \quad x \in \mathscr{AC}_n(a, b)$$

or

$$\int_a^b (y`(t) + g`(t))\,z(t)\,\mathrm{d}t = 0 \quad \text{for all} \quad z \in \mathscr{L}_n(a, b)\,.$$

Hence $y`(t) = g`(t)$ a.e. on $[a, b]$, i.e.

$$(2,11) \qquad y`(t) = \int_t^b y`(s)\,A(s)\,\mathrm{d}s + \begin{Bmatrix} \int_{t+r}^b y`(s)\,B(s)\,\mathrm{d}s, & t \leqq b - r \\ 0 & , t > b - r \end{Bmatrix} - \\ - \int_a^b y`(s)\,G(s, t)\,\mathrm{d}s + (N^*\lambda)(t) \quad \text{a.e. on} \quad [a, b]\,.$$

Let $z` \in \mathscr{L}_n^\infty(a, b)$. Then $(z`, \lambda, \gamma`) \in \operatorname{Ker}(U^*)$ iff there exists $y` \in \mathscr{L}_n^\infty(a, b)$ fulfilling (2,8) and (2,10) and such that $y(t) = z(t)$ a.e. on $[a, b]$ and (2,11) holds for all $t \in$

$\in (a, b)$. Finally, inserting (2,10) into (2,8) and taking into account that the right hand side of (2,11) is of bounded variation on $[a, b]$ and right continuous on (a, b), we complete the proof of the following

2,5. Theorem. *Let* $z' \in \mathscr{L}_n^\infty(a, b)$, $\lambda \in \Lambda^*$ *and* $\gamma' \in \mathscr{R}_n^*$. *Then* $(z', \lambda, \gamma') \in \operatorname{Ker}(U^*)$ *iff there exists* $y \in \mathscr{BV}_n(a, b)$ *right continuous on* (a, b) *(the values* $y(a)$, $y(b)$ *may be arbitrary) such that* $y(t) = z(t)$ *a.e. on* $[a, b]$ *and*

$$(2,12)\quad \int_a^b y'(s)\, A(s)\, ds + \int_{t+r}^b y'(s)\, B(s)\, ds - \int_a^b y'(s)\, G(s, t)\, ds + (M^*\lambda)(t) = 0 \quad \text{for} \quad t \in [a - r, a),$$

$$(2,13)\quad y'(t) = \int_t^b y'(s)\, A(s)\, ds + \begin{cases} \int_{t+r}^b y'(s)\, B(s)\, ds, & t \leqq b - r \\ 0 & , t > b - r \end{cases} - \int_a^b y'(s)\, G(s, t)\, ds + (N^*\lambda)(t) \quad \text{for} \quad t \in (a, b),$$

while γ' *is given by* (2,10).

2,6. Definition. The problem (P*) of finding $y \in \mathscr{BV}_n(a, b)$ right continuous on (a, b) and $\lambda \in \Lambda^*$ such that (2,12) and (2.13) hold is called the *conjugate* problem to (P).

(In virtue of Theorem 2,5 the adjoint problem (2,5) to (P) and the problem (P*) conjugate to (P) are equivalent.)

2,7. Corollary. *The problem* (P) *has a solution only if*

$$(2,14)\quad \int_a^b y'(s) f(s)\, ds + \langle l, \lambda \rangle_\Lambda = 0$$

for all solutions (y', λ) *of the conjugate problem* (P*). *If the operator* U *defined by* (2,4) *has a closed range* $\operatorname{Im}(U)$ *in* $\mathscr{L}_n(a, b) \times \Lambda \times \mathscr{R}_n$, *then the condition* (2,14) *is also sufficient for the existence of a solution to the problem* (P).

(The proof follows from Theorem 2,5 and from the fundamental "alternative" theorem concerning linear equations in B-spaces ([4], VI § 6).)

2,8. Remark. Let $\mathscr{X}$, $\mathscr{Y}$ be B-spaces and let $L : \mathscr{X} \to \mathscr{Y}$ be linear and bounded. A set $\mathscr{Y}^+ \subset \mathscr{Y}^*$ of linear continuous functionals on $\mathscr{Y}$ is said to be total in $\mathscr{Y}^*$ if $\langle y, g \rangle_{\mathscr{Y}} = 0$ for all $g \in \mathscr{Y}^+$ implies $y = 0$. Furthermore, if $L^+ : \mathscr{Y}^+ \to \mathscr{X}^*$ is a linear operator such that $\langle Lx, g \rangle_{\mathscr{Y}} = \langle x, L^+ g \rangle_{\mathscr{X}}$ for all $x \in \mathscr{X}$ and $g \in \mathscr{Y}^+$, we shall say that L^+ is a conjugate operator to L with respect to $\mathscr{Y}^+$. Clearly, L^+ is a restriction of the adjoint operator L^* to L on $\mathscr{Y}^+$. Hence $\operatorname{Ker}(L^+) \subset \operatorname{Ker}(L^*)$. (For some more details concerning conjugate operators see [11].)

Now, let $\mathscr{V}_n(a, b)$ be the space of row n-vector functions of bounded variation on $[a, b]$ and right continuous on (a, b). Then $\mathscr{V}_n(a, b)$ is a total subset in $\mathscr{L}_n^\infty(a, b)$. (In fact, let $f \in \mathscr{L}_n(a, b)$ and

$$0 = \int_a^b y'(t) f(t) \, dt \quad \text{for all} \quad y' \in \mathscr{V}_n(a, b) .$$

Then

$$0 = \int_a^b y'(t) \, dg(t) \quad \text{for all} \quad y' \in \mathscr{V}_n(a, b) , \tag{2,15}$$

where $g \in \mathscr{AC}_n(a, b)$ is an indefinite integral of f on $[a, b]$. Let $g_i(t_1) \neq g_i(t_2)$ for a component g_i of the vector $g = (g_1, g_2, \ldots, g_n)'$ and for some $t_1, t_2 \in [a, b]$, $t_1 < t_2$. Analogously to the second part of the proof of Lemma 5,1 in [10] we put $y'(t) = (y_1(t), y_2(t), \ldots, y_n(t))$, where $y_j(t) = 0$ on $[a, b]$ for $j \neq i$, $y_i(t) = 0$ for $t \in [a, t_1)$, $y_i(t) = 1$ for $t \in [t_1, t_2)$ and $y_i(t) = 0$ for $t \in [t_2, b]$. Then $y' \in \mathscr{V}_n(a, b)$ and

$$\int_a^b y'(t) \, dg(t) = \sum_{j=1}^n \int_a^b y_j(t) \, dg_j(t) = \int_a^b y_i(t) \, dg_i(t) = \int_{t_1}^{t_2} dg_i(t) = g_i(t_2) - g_i(t_1) \neq 0$$

which contradicts (2,15). Hence $g(t) = $ const. on $[a, b]$ and $f(t) = 0$ a.e. on $[a, b]$.)

The operator $D : x \in \mathscr{AC}_n(a, b) \to \dot{x} \in \mathscr{L}_n(a, b)$ is linear and bounded. It is easy to verify that its conjugate operator D^+ with respect to $\mathscr{V}_n(a, b)$ is given by

$$D^+ : y' \in \mathscr{V}_n(a, b) \to \begin{Bmatrix} 0, & t = a \\ -y'(t), & t \in (a, b) \\ 0, & t = b \end{Bmatrix} \in \mathscr{V}_n^0(a, b) .$$

Let us put $\mathscr{Y}^+ = \mathscr{V}_n(a, b) \times \Lambda^* \times \mathscr{R}_n^*$. Then $\mathscr{Y}^+$ is a total subset in $\mathscr{Y}^* = \mathscr{L}_n^\infty(a, b) \times \Lambda^* \times \mathscr{R}_n^*$ and the conjugate operator U^+ to U with respect to $\mathscr{Y}^+$ is given by

$$U^+ : (y', \lambda, \gamma') \in \mathscr{Y}^+ \to (\xi'(t), \eta'(t)) \in \mathscr{V}_n^0(a, b) \times \mathscr{V}_n^0(a - r, a) ,$$

where

$$\xi'(t) = \begin{Bmatrix} 0, & t = a \\ -y'(t), & t \in (a, b) \\ 0, & t = b \end{Bmatrix} + \int_t^b y'(s) A(s) \, ds + \begin{Bmatrix} \int_{t+r}^b y'(s) B(s) \, ds, & t \leq b - r \\ 0 & , t > b - r \end{Bmatrix} -$$

$$- \int_a^b y'(s) G(s, t) \, ds + (N^* \lambda)(t) + \begin{Bmatrix} \gamma', & t = a \\ 0, & t > a \end{Bmatrix} \quad \text{for} \quad t \in [a, b] ,$$

$$\eta'(t) = \int_{t+r}^{a+r} y'(s) B(s) \, ds - \int_a^b y'(s) (G(s, t) - G(s, a)) \, ds + (M^* \lambda)(t) - (N^* \lambda)(a) -$$

$$- \begin{Bmatrix} \gamma', & t < a \\ 0, & t = a \end{Bmatrix} \quad \text{for} \quad t \in [a - r, a] .$$

The equation $U^+(y`, \lambda, \gamma`) = 0$ is identical with the system of equations (2,8), (2,10), (2,13) and hence it is equivalent also with the problem (P*) introduced in Definition 2,6. In Section 2,4 we proved actually that $\mathrm{Ker}\,(U^*) \subset \mathscr{Y}^+$ and hence $\mathrm{Ker}\,(U^+) = \mathrm{Ker}\,(U^*)$.

2,9. Remark. The above procedure can be also applied to the case of initial functions of bounded variation on $[a - r, a]$. This means that instead of $u \in \mathscr{C}_n(a - r, a)$ we are looking for $u \in \mathscr{BV}_n(a - r, a)$. The adjoint problem is again equivalent to the system of the form (2,12), (2,13). Only we have to suppose in addition that $\mathrm{Im}\,(M^*) \subset \mathscr{V}^0_n(a - r, a)$.

2,10. Remark. Some examples of spaces Λ and operators M, N fulfilling Assumptions 2,1 are given in the following § 3. Some conditions on the closedness of $\mathrm{Im}\,(U)$ are given in § 5.

2,11. Remark. The couple $(y`, \lambda)$ being a solution to (P*), the values $y`(a), y`(b)$ may be arbitrary. We can require e.g. $y`(a) = y`(b) = 0$ or $y`(a+) = y`(a), y`(b-) = y`(b)$. In the latter case we add to the system (2,12), (2,13) the conditions

$$(2,16) \qquad y`(a) = -\int_a^b y`(s)\,(G(s, a+) - G(s, a-))\,\mathrm{d}s + (N^*\lambda)(a+) - (M^*\lambda)(a-),$$

$$y`(b) = \int_a^b y`(s)\,G(s, b-)\,\mathrm{d}s - (N^*\lambda)(b-).$$

(Indeed, by (2,12)

$$\int_a^b y`(s)\,A(s)\,\mathrm{d}s + \int_{a+r}^b y`(s)\,B(s)\,\mathrm{d}s = \int_a^b y`(s)\,G(s, a-)\,\mathrm{d}s - (M^*\lambda)(a-).)$$

2,12. Remark. $\mathscr{AC}^*_n(a, b)$ is isometrically isomorphic with $\mathscr{L}^\infty_n(a, b) \times \mathscr{R}^*_n$. Given $g \in \mathscr{AC}^*_n(a, b)$, there exist uniquely determined $\beta` \in \mathscr{R}^*_n$ and $y`(t) \in \mathscr{L}^\infty_n(a, b)$ such that

$$\langle x, g\rangle_{\mathscr{AC}} = \beta` x(a) + \int_a^b y`(t)\dot{x}(t)\,\mathrm{d}t$$

for all $x \in \mathscr{AC}_n(a,b)$. (See [4] IV, 13, 29.) By a similar argument as in 2,4 we could derive the analytic form of the adjoint problem also in the case that N is a general linear bounded operator $\mathscr{AC}_n(a, b) \to \Lambda$ without supposing $\mathrm{Im}(N^*) \subset \mathscr{V}^0_n(a, b)$.

If $N^* : \lambda \in \Lambda^* \to (N^*\lambda, \tilde{N}^*\lambda) \in \mathscr{L}^\infty_n(a, b) \times \mathscr{R}^*_n$ and $(M^*\lambda)(t) - (\tilde{N}^*\lambda) \in \mathscr{V}^0_n(a - r, a)$ for any $\lambda \in \Lambda^*$, then the problem of finding $(y`(t), \lambda) \in \mathscr{L}^\infty_n(a, b) \times \Lambda^*$ such that (2,12) *holds on $[a - r, a)$ and* (2,13) *holds a.e. on $[a, b]$ is equivalent to the adjoint of the given problem* (P).

Let us mention some special cases of the given problem (P) which arise by a special choice of the boundary operators M, N and of the terminal space Λ.

3,1. The case $\Lambda = \mathscr{L}_m(c, d)$. Let $\Lambda = \mathscr{L}_m(c, d)$ $(-\infty < c < d < +\infty)$ and

$$(3,1) \qquad M : u \in \mathscr{C}_n(a - r, a) \to \int_{a-r}^{a} [\mathrm{d}_s M(\alpha, s)] \, u(s) \in \Lambda \, ,$$

$$(3,2) \qquad N : x \in \mathscr{AC}_n(a, b) \to \int_a^b [\mathrm{d}_s N(\alpha, s)] \, x(s) \in \Lambda \, ,$$

where $M(\alpha, s)$ is a Borel measurable in $(\alpha, s) \in [c, d] \times [a - r, a]$ $m \times n$-matrix function such that $\operatorname{var}_{a-r}^{a} M(\alpha, \cdot) < \infty$ for any $\alpha \in [c, d]$ and

$$\int_c^d (\|M(\alpha, a)\| + \operatorname{var}_{a-r}^{a} M(\alpha, \cdot)) \, \mathrm{d}\alpha < \infty$$

and $N(\alpha, s)$ is a Borel measurable in $(\alpha, s) \in [c, d] \times [a, b]$ $m \times n$-matrix function such that $\operatorname{var}_a^b N(\alpha, \cdot) < \infty$ for any $\alpha \in [c, d]$ and

$$\int_c^d (\|N(\alpha, b)\| + \operatorname{var}_a^b N(\alpha, \cdot)) \, \mathrm{d}\alpha < \infty \, .$$

Without any loss of generality we may assume that for any $\alpha \in [c, d]$, $M(\alpha, \cdot)$ is right continuous on $(a - r, a)$, $N(\alpha, \cdot)$ is right continuous on (a, b), $M(\alpha, a) = N(\alpha, a)$ and $N(\alpha, b) = 0$.

Let $x \in AC_n(a, b)$, $u \in C_n(a - r, a)$, $\lambda` \in \mathscr{L}_m^{\infty}(c, d)$. Then by the unsymmetric Fubini theorem ([2])

$$\langle Mu, \lambda` \rangle_{\mathscr{L}} = \int_c^d \lambda`(\alpha) \left(\int_{a-r}^{a} [\mathrm{d}_s M(\alpha, s)] \, u(s) \right) \mathrm{d}\alpha =$$
$$= \int_{a-r}^{a} \left[\mathrm{d}_s \int_c^d \lambda`(\alpha) \, (M(\alpha, s) - M(\alpha, a)) \, \mathrm{d}\alpha \right] u(s)$$

and

$$\langle Nx, \lambda` \rangle_{\mathscr{L}} = \int_c^d \lambda`(\alpha) \left(\int_a^b [\mathrm{d}_s N(\alpha, s)] \, x(s) \right) \mathrm{d}\alpha = \int_a^b \left[\mathrm{d}_s \int_c^d \lambda`(\alpha) \, N(\alpha, s) \, \mathrm{d}\alpha \right] x(s) \, ,$$

where

$$(3,3) \qquad (N^* \lambda`)(t) = \int_c^d \lambda`(\alpha) \, N(\alpha, t) \, \mathrm{d}\alpha \in \mathscr{V}_n^0(a, b)$$

and

$$(3,4) \qquad (M^*\lambda')(t) - (M^*\lambda')(a) =$$
$$= \int_c^d \lambda'(\alpha)(M(\alpha, t) - M(\alpha, a))\, d\alpha \in \mathscr{V}_n^0(a - r, a).$$

Hence in this case the adjoint problem is equivalent to the system (2,12), (2,13), where M^* and N^* have the special form (3,4) and (3,3), respectively.

3,2. The case $\Lambda = \mathscr{C}_m(c, d)$. Similarly we can treat the case of $\Lambda = \mathscr{C}_m(c, d)$ $(-\infty < c < d < +\infty)$ with the operators M, N given by (3,1), (3,2), where $M(\cdot, s)$ and $N(\cdot, \sigma)$ are continuous on $[c, d]$ for any $s \in [a - r, a]$ and $\sigma \in [a, b]$. (Let us note that in this case any linear bounded operator $M : \mathscr{C}_n(a - r, a) \to \Lambda$ can be expressed in the form (3,1), where $M(\alpha, s)$ fulfils all our assumptions.) Analogously as in 3,1 we obtain

$$M^* : \lambda' \in \mathscr{V}_m^0(c, d) \to \int_c^d [d\lambda'(\alpha)]\, M(\alpha, t) \in \mathscr{V}_n(a - r, a),$$

$$N^* : \lambda' \in \mathscr{V}_m^0(c, d) \to \int_c^d [d\lambda'(\alpha)]\, N(\alpha, t) \in \mathscr{V}_n^0(a, b).$$

3,3. Finite dimensional terminal space. Let $\Lambda = \mathscr{R}_m$ and

$$M : u \in \mathscr{C}_n(a - r, a) \to \int_{a-r}^a [dM(s)]\, u(s) \in \mathscr{R}_m,$$

$$N : x \in \mathscr{AC}_n(a, b) \to \int_a^b [dN(s)]\, x(s) \in \mathscr{R}_m,$$

where $M(t)$ and $N(t)$ are $m \times n$-matrix functions of bounded variation on $[a - r, a]$ and $[a, b]$, respectively. We may assume also M right continuous on $(a - r, a)$, N right continuous on (a, b), $M(a) = N(a)$ and $N(b) = 0$.

Let $x \in \mathscr{AC}_n(a, b)$, $u \in \mathscr{C}_n(a - r, a)$ and $\lambda' \in \mathscr{R}_m^*$, then

$$\langle Mu, \lambda' \rangle_{\mathscr{R}} = \lambda'(Mu) = \int_{a-r}^a [d\{\lambda'(M(s) - M(a))\}]\, u(s)$$

and

$$\langle Nx, \lambda' \rangle_{\mathscr{R}} = \lambda'(Nx) = \int_a^b [d(\lambda' N(s))]\, x(s),$$

where $(M^*\lambda')(t) - (M^*\lambda')(a) = \lambda'(M(t) - M(a)) \in \mathscr{V}_n^0(a - r, a)$ and $(N^*\lambda')(t) = \lambda' N(t) \in \mathscr{V}_n^0(a, b)$.

The adjoint problem is equivalent to the conjugate problem (P*) given by (2,12), (2,13) with M^* and N^* defined above. Moreover, we may write it in the form more similar to the adjoint of the boundary value problem for ordinary integro-differential equation ([12]). Let us put for $t \in [a, b]$

$$\tilde{M} = N(a+) - M(a-), \quad \tilde{N} = -N(b-),$$
$$C(t) = G(t, a+) - G(t, a-), \quad D(t) = -G(t, b-),$$

$$L(s) = \begin{cases} N(a+) & \text{for } s = a, \\ N(s) & \text{for } a < s < b, \\ N(b-) & \text{for } s = b, \end{cases} \qquad G_0(t, s) = \begin{cases} G(t, a+) & \text{for } s = a, \\ G(t, s) & \text{for } a < s < b, \\ G(t, b-) & \text{for } s = b. \end{cases}$$

Then, requiring $y'(a+) = y'(a)$, $y'(b-) = y'(b)$ (cf. Remark 2,11) we obtain the conjugate problem (P*) to (P) in the following form:

$$\int_a^b y'(s)\, A(s)\, ds + \int_{t+r}^b y'(s)\, B(s)\, ds - \int_a^b y'(s)\, \dot{G}(s, t)\, ds + \lambda' M(t) = 0,$$
$$\text{on } [a - r, a),$$

$$y'(t) = y'(b) + \int_t^b y'(s)\, A(s)\, ds + \begin{Bmatrix} \int_{t+r}^b y'(s)\, B(s)\, ds, & t \leqq b - r \\ 0 & , t > b - r \end{Bmatrix} -$$
$$- \int_a^b y'(s)\, (G_0(s, t) - G_0(s, b))\, ds + \lambda'(L(t) - L(b)) \quad \text{on } [a, b],$$

$$y'(a) = \lambda' \tilde{M} - \int_a^b y'(s)\, C(s)\, ds, \quad y'(b) = -\lambda' \tilde{N} + \int_a^b y'(s)\, D(s)\, ds.$$

3,4. Boundary value type problems for functional-differential equations of retarded type. In this section we shall deal with boundary value problems for standard functional-differential equation

(3,5) $$\dot{x}(t) = \int_{-r}^0 [d_\vartheta P(t, \vartheta)]\, x(t + \vartheta) + f(t) \quad \text{a.e. on } [a, b],$$

(3,6) $$x(t) = u(t) \quad \text{on } [a - r, a],$$

(3,7) $$Mu + Nx = l \in \Lambda,$$

where the initial functions $u(t)$ are continuous on $[a - r, a]$ and the following assumptions are fulfilled:

$P(t, \vartheta)$ is a Borel measurable in $(t, \vartheta) \in [a, b] \times (-\infty, +\infty)$ $n \times n$-matrix function such that $P(t, \vartheta) = P(t, -r)$ for $\vartheta \leqq -r$, $P(t, \vartheta) = P(t, 0)$ for $\vartheta \geqq 0$, $\text{var}_{-r}^0 P(t, \cdot) < \infty$ for all $t \in [a, b]$ and

$$\int_a^b (\|P(t, 0)\| + \text{var}_{-r}^0 P(t, \cdot))\, dt < \infty.$$

Λ is a B-space and the operators $M : \mathscr{C}_n(a-r, a) \to \Lambda$ and $N : \mathscr{AC}_n(a, b) \to \Lambda$ are linear and bounded, while $\operatorname{Im}(N^*) \subset \mathscr{V}_n^0(a, b)$. Furthermore, $l \in \Lambda$ and $f(t) \in \mathscr{L}_n(a, b)$. We may also assume that $P(t, \cdot)$ is right continuous on $(-r, 0)$ and $P(t, 0) = 0$ for any $t \in [a, b]$.

Let us put for $t \in [a, b]$

$$B(t) = P(t, -r+) - P(t, -r), \quad G(t, s) = \begin{cases} P(t, -r+) & \text{if} \quad s \leqq t - r, \\ P(t, s-t) & \text{if} \quad t - r \leqq s \leqq t, \\ P(t, 0) = 0 & \text{if} \quad s \geqq t. \end{cases}$$

Then $B(t)$ and $G(t, s)$ fulfil Assuptions 2,1. Moreover, given $t \in [a, b]$, $G(t, \cdot)$ is right continuous on $(a - r, b)$, $G(t, b) = 0$ and

$$\int_{-r}^{0} [\mathrm{d}_\vartheta P(t, \vartheta)]\, x(t + \vartheta) = \int_{t-r}^{t} [\mathrm{d}_s P(t, s - t)]\, x(s) =$$
$$= B(t)\, x(t - r) + \int_{a-r}^{b} [\mathrm{d}_s G(t, s)]\, x(s)\,.$$

The problem (3,5)–(3,7) is reduced to the problem of the type (P). Furthermore, for $t \in [a - r, a]$

$$\int_{t+r}^{b} y'(s)\, B(s)\,\mathrm{d}s - \int_a^b y'(s)\, G(s, t)\,\mathrm{d}s = \int_{t+r}^{b} y'(s)\,(P(s, -r+) - P(s, -r))\,\mathrm{d}s -$$
$$- \int_a^{t+r} y'(s)\, P(s, t - s)\,\mathrm{d}s - \int_{t+r}^{b} y'(s)\, P(s, -r+)\,\mathrm{d}s = - \int_a^b y'(s)\, P(s, t - s)\,\mathrm{d}s\,.$$

Analogously for $t \in (a, b - r)$

$$\int_{t+r}^{b} y'(s)\, B(s)\,\mathrm{d}s - \int_a^b y'(s)\, G(s, t)\,\mathrm{d}s = - \int_t^b y'(s)\, P(s, t - s)\,\mathrm{d}s$$

and

$$- \int_a^b y'(s)\, G(s, t)\,\mathrm{d}s = - \int_t^b y'(s)\, P(s, t - s)\,\mathrm{d}s \quad \text{for} \quad t \in [b - r, b]\,.$$

The following theorem is now a direct consequence of Theorem 2,5.

3,5. Theorem. *The problem of finding $y \in \mathscr{BV}_n(a, b)$ right continuous on (a, b) (the values $y(a)$, $y(b)$ may be arbitrary) and $\lambda \in \Lambda^*$ such that*

$$(3,8) \qquad - \int_a^b y'(s)\, P(s, t - s)\,\mathrm{d}s + (M^*\lambda)(t) = 0 \quad \textit{on} \quad [a - r, a)\,,$$

$$(3,9) \qquad y'(t) = - \int_t^b y'(s)\, P(s, t - s)\,\mathrm{d}s + (N^*\lambda)(t) \quad \textit{on} \quad (a, b)$$

is equivalent to the adjoint problem to the problem (3,5)–(3,7).

(The functions $(M^*\lambda)(t)$ and $(N^*\lambda)(t)$ are again such that for any $\lambda \in \Lambda^*$ $(M^*\lambda)(t) - (N^*\lambda)(a) \in \mathscr{V}_n^0(a - r, a)$, $(N^*\lambda)(t) \in \mathscr{V}_n^0(a, b)$ and

$$\langle Mu, \lambda\rangle_\Lambda = \int_{a-r}^{a} [\mathrm{d}\{(M^*\lambda)(t) - (M^*\lambda)(a)\}]\, u(t)\,,$$

$$\langle Nx, \lambda\rangle_\Lambda = \int_a^b [\mathrm{d}(N^*\lambda)(t)]\, x(t)$$

for all $u \in \mathscr{C}_n(a - r, a)$, $x \in \mathscr{A}\mathscr{C}_n(a, b)$ and $\lambda \in \Lambda^*$.)

3,6. Two-point boundary value type problem. Let us consider the "two-point" boundary value type problem given by the system (3,5), (3,6) and

$$\text{(3,10)} \qquad Mu + N_b x = l \in \Lambda\,,$$

where the functions $P(t, \vartheta)$, $f(t)$ and the operator M satisfy the corresponding assumptions of Section 3,4. Given $\lambda \in \Lambda^*$, let $(M^*\lambda)(t)$ denote now a function from $\mathscr{V}_n^0(a - r, a)$ such that

$$\langle Mu, \lambda\rangle_\Lambda = \int_{a-r}^{a} [\mathrm{d}(M^*\lambda)(t)]\, u(t)$$

for all $u \in \mathscr{C}_n(a - r, a)$ and $\lambda \in \Lambda^*$. The operator $N_b = NS_b : \mathscr{A}\mathscr{C}_n(a, b) \to \Lambda$ is the composition of a linear bounded operator $N : \mathscr{C}_n(b - r, b) \to \Lambda$ and of a shift operator $S_b : x \in \mathscr{A}\mathscr{C}_n(a, b) \to x/[b - r, b] \in \mathscr{C}_n(b - r, b)$ (which is also linear and bounded). Let $0 < r \leqq b - a$.

Let $x \in \mathscr{A}\mathscr{C}_n(a, b)$ and $\lambda \in \Lambda^*$. Then

$$\langle N_b x, \lambda\rangle_\Lambda = \langle S_b x, N\lambda\rangle_{\mathscr{C}} = \int_{b-r}^{b} [\mathrm{d}(N^*\lambda)(t)]\, x(t)$$

where $(N^*\lambda)(t) \in \mathscr{V}_n^0(b - r, b)$, and putting

$$(\tilde{N}^*\lambda)(t) = \begin{cases} (N^*\lambda)(b - r+) & \text{for} \quad t = b - r\,, \\ (N^*\lambda)(t) & \text{for} \quad b - r < t \leqq b \end{cases}$$

and

$$(N_b^*\lambda)(t) = \left.\begin{cases} (N^*\lambda)(b - r) & \text{for} \quad a \leqq t < b - r \\ (\tilde{N}^*\lambda)(t) & \text{for} \quad b - r \leqq t \leqq b \end{cases}\right\} \in \mathscr{V}_n^0(a, b)\,,$$

we get finally

$$\langle N_b x, \lambda\rangle_\Lambda = \int_a^b [\mathrm{d}(N_b^*\lambda)(t)]\, x(t)\,.$$

Since all the assumptions of Section 3,4 are satisfied, the following assertion is an immediate consequence of Theorem 3,5.

3,7. Corollary. *The problem of finding* $y \in \mathscr{BV}_n(a, b)$ *right continuous on* (a, b) *(the values* $y(a)$, $y(b)$ *may be arbitrary) and* $\lambda \in \Lambda^*$ *such that*

$$(3{,}11) \qquad \int_a^b y'(s)\, P(s, t - s)\, \mathrm{d}s - (M^*\lambda)(t) = (N^*\lambda)(b - r) \quad \text{on} \quad [a - r, a)\,,$$

$$(3{,}12) \qquad y'(t) + \int_t^b y'(s)\, P(s, t - s)\, \mathrm{d}s = (N^*\lambda)(b - r) \quad \text{on} \quad (a, b - r)\,,$$

$$(3{,}13) \qquad y'(t) + \int_t^b y'(s)\, P(s, t - s)\, \mathrm{d}s - (\tilde{N}^*\lambda)(t) = 0 \quad \text{on} \quad [b - r, b)$$

is equivalent to the adjoint problem to the two-point problem (3,5), (3,6), (3,10).

3,8. Relationship with the adjoint of D. Henry. Let us continue the investigation of the two-point boundary value type problem (3,5), (3,6), (3,10). We shall show that the adjoint problem (3,11) derived in 3,6 can be reduced to the form of D. Henry [8]. Let us put for $\vartheta \in [-r, 0]$ $P(t, \vartheta) = P(t + b - a, \vartheta)$ if $t \in [a - r, a)$. Given a function $z(t)$ defined on $[a - r, b]$ and $t \in [a, b]$, we put

$$z_t^0(\alpha) = \begin{cases} z(t + \alpha) & \text{if} \quad \alpha \in [-r, 0)\,, \\ 0 & \text{if} \quad \alpha = 0\,. \end{cases}$$

Let $\mathscr{V}_n(-r, 0)$ be the space of all row n-vector functions of bounded variation on $[-r, 0]$ and right continuous on $(-r, 0)$. Let $R(\beta, \alpha)$ be the resolvent kernel for the Volterra integral equation

$$z'(\alpha) + \int_\alpha^0 z'(\beta)\, P(b + \beta, \alpha - \beta)\, \mathrm{d}\beta = 0\,, \quad \alpha \in [-r, 0]\,.$$

Gronwall's inequality applied to the „resolvent equation"

$$R(\beta, \alpha) + \int_\beta^\alpha R(\beta, \gamma)\, P(b + \gamma, \alpha - \gamma)\, \mathrm{d}\gamma = P(b + \beta, \alpha - \beta)\,; \quad \alpha, \beta \in [-r, 0]$$

yields analogously as in the proof of Lemma 1 in [14] that $\operatorname{var}_{-r}^0 R(\beta, \cdot) < \infty$ for any $\beta \in [-r, 0]$, while the function $r(\beta) = \operatorname{var}_{-r}^0 R(\beta, \cdot)$ is bounded on $[-r, 0]$. Hence the resolvent operator

$$R : w'(\alpha) \in V_n(-r, 0) \to \int_\alpha^0 w'(\beta)\, R(\beta, \alpha)\, \mathrm{d}\beta \in \mathscr{V}_n(-r, 0)$$

is linear and bounded and for any $w' \in \mathscr{V}_n(-r, 0)$, the unique solution $z'(\alpha)$ on $[-r, 0]$ to

$$z'(\alpha) + \int_\alpha^0 z'(\beta)\, P(b + \beta,\ \alpha - \beta)\, \mathrm{d}\beta = w'(\alpha)$$

is given by

$$z' = w' - Rw' = (I - R)\,w',$$

where I denotes the identity operator.

Now, let (y', λ) be a solution to (3,11)–(3,13). Let us extend the function $y'(t)$ on the interval $[a - r, a]$ in such a way that

$$\text{(3,15)} \qquad y'(t) + \int_t^a y'(s)\,P(s, t - s)\,ds = -(M^*\lambda)(t) \quad \text{for} \quad t \in [a - r, a)$$

and

$$\text{(3,16)} \qquad y'(a) + \int_a^b y'(s)\,P(s, a - s)\,ds = (N^*\lambda)(b - r).$$

Since

$$\int_t^a y'(s)\,P(s, t - s)\,ds = \int_{t-a}^0 y'(a + \beta)\,P(a + \beta,\ t - a - \beta)\,d\beta =$$

$$= \int_\alpha^0 y'(a + \beta)\,P(b + \beta,\ \alpha - \beta)\,d\beta, \quad \text{where} \quad \alpha = t - a,$$

(3,15) yields

$$\text{(3,17)} \qquad y'^0_a = -(I - R)(M^*\lambda).$$

The last equation (3,13) in our conjugate system is obviously equivalent to the condition

$$\text{(3,18)} \qquad y'^0_b = (I - R)(\tilde{N}^*\lambda).$$

Finally, owing to (3,15) and (3,16) the equations (3,11) and (3,12) can be replaced by the single equation

$$\text{(3,19)} \quad y'(t) + \int_t^b y'(s)\,P(s, t - s)\,ds = (N^*\lambda)(b - r) \quad \text{on} \quad [a - r,\ b - r).$$

The system (3,17)–(3,19) is just the adjoint problem of D. Henry from [8]. (Only we have the expression depending on λ instead of an arbitrary constant on the right hand side of the Volterra integral equation on $[a - r, b - r)$.)

Obviously the couple (y', λ) being a solution to the system (3,17)–(3,19), it is a solution to (3,11)–(3,13).

3,9. Periodic solutions. Let $a = 0$, $b = T < \infty$ $(r \leqq T)$. Let $P(\cdot, \vartheta)$ be for any $\vartheta \in [-r, 0]$ a T-periodic function on $(-\infty, +\infty)$. Let us consider the periodic problem consisting of the equations (3,5), (3,6) and

$$\text{(3,20)} \qquad u(t) - x(T + t) = 0 \quad \text{for} \quad t \in [-r, 0]$$

(i.e., in (3,10) we have $A = \mathscr{C}_n(-r, 0)$, $l = 0$, $M = I$, $N_T = NS_T$, $N : z(t) \in \in \mathscr{C}_n(T - r, T) \to -z(T + s) \in \mathscr{C}_n(-r, 0)$.)

By Corollary 3,7 the adjoint problem is equivalent to the system of equations for $y^\backprime(t)$ of bounded variation on $[-r, T]$ and right continuous on $(-r, T - r) \cup \cup (T - r, T)$ and for $\lambda^\backprime(t) \in \mathscr{V}_n^0(-r, 0)$,

$$y^\backprime(t) + \int_t^T y^\backprime(s)\, P(s, t - s)\, \mathrm{d}s = -\lambda^\backprime(-r) \quad \text{on} \quad [-r, T - r], \tag{3,21}$$

$$y^\backprime(t) + \int_t^0 y^\backprime(s)\, P(s, t - s)\, \mathrm{d}s = -\lambda^\backprime(t) \quad \text{on} \quad [-r, 0), \tag{3,22}$$

$$y^\backprime(t) + \int_t^T y^\backprime(s)\, P(s, t - s)\, \mathrm{d}s = -\lambda^\backprime(t - T) \quad \text{on} \quad [T - r, T). \tag{3,23}$$

Indeed, since actually we are looking for $y^\backprime(t)$ in the space $\mathscr{L}_n^\infty(-r, T)$, we may change the values of $y^\backprime$ on a set of measure zero in $[-r, T]$. Hence we may put

$$y^\backprime(0) + \int_0^T y^\backprime(s)\, P(s, -s)\, \mathrm{d}s = -\lambda^\backprime(-r)$$

and

$$y^\backprime(T - r) + \int_{T-r}^T y^\backprime(s)\, P(s, T - r - s)\, \mathrm{d}s = -\lambda^\backprime(-r).$$

($P(s, -s+) = P(s, -s)$ for any $s \neq r$ and thus $y^\backprime(0+) = y^\backprime(0)$.)

Furthermore, since by the periodicity assumption on $P(\cdot, \vartheta)$

$$\int_t^T y^\backprime(s)\, P(s, t - s)\, \mathrm{d}s = \int_{t-T}^0 y^\backprime(T + \beta)\, P(\beta, t - T - \beta)\, \mathrm{d}\beta \quad \text{for} \quad t \in [T - r, T],$$

the system (3,22), (3,23) is equivalent to the condition

$$y^\backprime(t) = y^\backprime(T + t) \quad \text{for} \quad t \in [-r, 0). \tag{3,24}$$

3,10. Corollary. *The adjoint to the periodic problem* (3,5), (3,6), (3,20) *is equivalent to the problem of finding* $y(t) \in \mathscr{BV}_n(-r, T)$ *right continuous on* $(-r, T - r) \cup \cup (T - r, T)$ *which satisfies* (3,21) *and* (3,24), *where* $\lambda^\backprime(-r)$ *stands for an arbitrary constant n-vector.*

(*In other words, the problem of finding T-periodic solutions to the equation*

$$y^\backprime(t) + \int_t^T y^\backprime(s)\, P(s, t - s)\, \mathrm{d}s = \text{const}$$

is a well posed adjoint problem to the problem of finding T-periodic solutions to the equation (3,5).)

4. BOUNDARY VALUE TYPE PROBLEMS FOR HEREDITARY DIFFERENTIAL EQUATIONS OF THE DELFOUR-MITTER TYPE

4,1. Notation. Let $-\infty < \alpha < \beta < +\infty$. $\mathscr{L}_n^2(\alpha, \beta)$ is the Hilbert space of square integrable (column) n-vector functions on $[\alpha, \beta]$ with the inner product

$$u, v \in \mathscr{L}_n^2(\alpha, \beta) \to (u, v)_{\mathscr{L}} = \int_\alpha^\beta u^*(s)\, v(s)\, ds = \int_\alpha^\beta v^*(s)\, u(s)\, ds\,.$$

(The corresponding norm on $\mathscr{L}_n^2(\alpha, \beta)$ is given by

$$u \in \mathscr{L}_n^2(\alpha, \beta) \to \|u\|_{\mathscr{L}^2} = \left(\int_\alpha^\beta \|u(s)\|^2\, ds\right)^{1/2}.$$

$\mathscr{W}_n^{1,2}(\alpha, \beta)$ is the Hilbert space of functions $x : [\alpha, \beta] \to \mathscr{R}_n$ which are absolutely continuous on $[\alpha, \beta]$ and whose derivatives Dx are square integrable on $[\alpha, \beta]$. The inner product and the corresponding norm are on $\mathscr{W}_n^{1,2}(\alpha, \beta)$ given by

$$x, y \in \mathscr{W}_n^{1,2}(\alpha, \beta) \to (x, y)_{\mathscr{W}} = (Dx, Dy)_{\mathscr{L}} + (x, y)_{\mathscr{L}}$$

and

$$x \in \mathscr{W}_n^{1,2}(\alpha, \beta) \to \|x\|_{\mathscr{W}} = (\|Dx\|_{\mathscr{L}^2}^2 + \|x\|_{\mathscr{L}^2}^2)^{1/2}\,.$$

The corresponding spaces of row vector functions will be denoted also by $\mathscr{L}_n^2(\alpha, \beta)$ and $\mathscr{W}_n^{1,2}(\alpha, \beta)$. No misunderstanding may arise.

4,2. Assumptions. Let $-\infty < a < b < +\infty$ and $r > 0$. Let $A(t)$ and $B(t)$ be $n \times n$-matrix functions essentially bounded on $[a, b]$ and $f(t) \in \mathscr{L}_n^2(a, b)$, let M and N be constant $m \times n$-matrices and $l \in R_m$. Let Λ be an arbitrary B-space, $w \in \Lambda$ and let $P : \mathscr{L}_n^2(a - r, a) \to \Lambda$ and $Q : \mathscr{W}_n^{1,2}(a, b) \to \Lambda$ be linear and bounded operators.

4,3. Problem (π). The subject of this paragraph is the following boundary value type problem (π)

Determine $x \in \mathscr{W}_n^{1,2}(a, b)$, $\xi \in \mathscr{R}_n$ and $u \in \mathscr{L}_n^2(a - r, a)$ in such a way that

$$\dot{x}(t) - A(t)\, x(t) - \begin{cases} B(t)\, u(t - r), & t < a + r \\ B(t)\, x(t - r), & t \geqq a + r \end{cases} = f(t) \quad \text{a.e. on} \quad [a, b]\,, \tag{4,1}$$

$$Pu + Qx = w\,, \tag{4,2}$$

$$M\xi + N\, x(b) = l\,, \tag{4,3}$$

$$x(a) - \xi = 0\,. \tag{4,4}$$

Let $\mathscr{W} = \mathscr{W}_n^{1,2}(a, b) \times \mathscr{R}_n \times \mathscr{L}_n^2(a - r, a)$, $\mathscr{Z} = \mathscr{L}_n^2(a, b) \times \Lambda \times \mathscr{R}_m \times \mathscr{R}_n$ and let the operators $D, A, B_1 : \mathscr{W}_n^{1,2}(a, b) \to \mathscr{L}_n^2(a, b)$ and $B_2 : \mathscr{L}_n^2(a - r, a) \to$ $\to \mathscr{L}_n^2(a, b)$ be defined analogously as in 2,3 and

$$U \begin{bmatrix} x \\ \xi \\ u \end{bmatrix} \in \mathscr{W} \to \begin{bmatrix} Dx - Ax - B_1 x - B_2 u \\ Pu + Qx \\ M\xi + N\,x(b) \\ x(a) - \xi \end{bmatrix} \in \mathscr{Z}\,.$$

The operator U is clearly linear and bounded and the given problem (π) is equivalent to the operator equation

$$U \begin{bmatrix} x \\ \xi \\ u \end{bmatrix} = \begin{bmatrix} f \\ w \\ l \\ 0 \end{bmatrix}.$$

4,4. Remark. The corresponding initial value problem (4,1) and (4,4) (with $u \in \in \mathscr{L}_n^2(a - r, a)$ and $\xi \in \mathscr{R}_n$ fixed) was studied in [3].

4,5. Theorem. *Let $\eta' \in \mathscr{L}_n^2(a, b)$, $\lambda \in \Lambda^*$, $\gamma' \in \mathscr{R}_m^*$ and $\delta' \in \mathscr{R}_n^*$. Then $(\eta', \lambda, \gamma', \delta') \in \in \operatorname{Ker}(U^*)$ iff there exists $y' \in \mathscr{L}_n^2(a, b)$ such that $y' + (\mathrm{d}/\mathrm{d}t)(Q^*\lambda) \in \mathscr{AC}_n(a, b)$, $y(t) = \eta(t)$ a.e. on $[a, b]$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[y' + \frac{\mathrm{d}}{\mathrm{d}t}(Q^*\lambda)\right](t) = -y'(t)\,A(t) - \begin{Bmatrix} y'(t + r)\,B(t + r), & t < b - r \\ 0, & t > b - r \end{Bmatrix} +$$

$$+ (Q^*\lambda)(t) \quad \text{a.e. on} \quad [a, b]\,,$$

$$\left[y' + \frac{\mathrm{d}}{\mathrm{d}t}(Q^*\lambda)\right](a) = \gamma' M\,, \quad \left[y' + \frac{\mathrm{d}}{\mathrm{d}t}(Q^*\lambda)\right](b) = -\gamma' N\,,$$

$$y'(t + r)\,B(t + r) - (P^*\lambda)(t) = 0 \quad \text{a.e. on} \quad [a - r, a]\,,$$

while $\delta' = \gamma' M$ ($P^* : \Lambda^* \to \mathscr{L}_n^2(a - r, a)$ and $Q^* : \Lambda^* \to \mathscr{W}_n^{1,2}(a, b)$ are the adjoints to P and Q).

Proof. Let $\eta' \in \mathscr{L}_n^2(a, b)$, $\lambda \in \Lambda^*$, $\gamma' \in \mathscr{R}_m^*$ and $\delta' \in \mathscr{R}_n^*$. Then $(\eta', \lambda, \gamma', \delta') \in \in \operatorname{Ker}(U^*)$ iff for any $(x, \xi, u) \in \mathscr{W}$

$$0 = \left(\begin{bmatrix} x \\ \xi \\ u \end{bmatrix}, U^*(\eta', \lambda, \gamma', \delta')\right)_{\mathscr{W}} = \left(U \begin{bmatrix} x \\ \xi \\ u \end{bmatrix}, (\eta', \lambda, \gamma', \delta')\right)_{\mathscr{Z}} =$$

$$= \int_a^b \eta'(t)\,\dot{x}(t)\,\mathrm{d}t - \int_a^b \eta'(t)\,A(t)\,x(t)\,\mathrm{d}t - \int_a^{b-r} \eta'(t + r)\,B(t + r)\,x(t)\,\mathrm{d}t -$$

$$- \int_{a-r}^a \eta'(t + r)\,B(t + r)\,u(t)\,\mathrm{d}t + \gamma'(M\xi + N\,x(b)) + \delta'(x(a) - \xi) +$$

$$+ (u, P^*\lambda)_{\mathscr{L}} + (x, Q^*\lambda)_{\mathscr{W}} =$$

$$= \int_a^b \eta'(t)\,\dot{x}(t)\,\mathrm{d}t + \int_a^b \left[\frac{\mathrm{d}}{\mathrm{d}t}(Q^*\lambda)(t)\right]\dot{x}(t)\,\mathrm{d}t - \int_a^b p'(t)\,x(t)\,\mathrm{d}t +$$

$$+ \gamma' N\,x(b) + \delta'\,x(a) - \int_{a-r}^a q'(t)\,u(t)\,\mathrm{d}t + (\gamma' M - \delta')\,\xi\,,$$

where

$$p'(t) = \eta'(t)\,A(t) + \begin{Bmatrix} \eta'(t+r)\,B(t+r), & t < b - r \\ 0, & t \geqq b - r \end{Bmatrix} - (Q^*\lambda)(t) \quad \text{on} \quad [a, b]$$

and

$$q'(t) = \eta'(t+r)\,B(t+r) - (P^*\lambda)(t) \quad \text{on} \quad [a-r, a]\,.$$

In particular, putting $\xi = 0$ and $x(t) = 0$ on $[a, b]$, we get

(4,5) $$\eta'(t+r)\,B(t+r) - (P^*\lambda)(t) = 0 \quad \text{a.e. on} \quad [a-r, a]\,.$$

Furthermore, putting $x(t) = 0$ on $[a, b]$ and $u(t) = 0$ on $[a - r, a]$, we get

(4,6) $$\gamma' M - \delta' = 0\,.$$

Let us put

$$g'(t) = \begin{Bmatrix} \int_t^b p'(s)\,\mathrm{d}s - \gamma' N - \delta', & t = a \\ \int_t^b p'(s)\,\mathrm{d}s - \gamma' N & , a < t < b \\ 0 & , t = b \end{Bmatrix} \in \mathscr{V}_n^0(a, b)\,.$$

Then, in virtue of the integration-by-parts formula,

$$0 = \int_a^b \left\{\eta'(t) + \left[\frac{\mathrm{d}}{\mathrm{d}t}(Q^*\lambda)(t)\right]\right\}\dot{x}(t)\,\mathrm{d}t + \int_a^b [\mathrm{d}g'(t)]\,x(t) =$$

$$= \int_a^b \left\{\eta'(t) + \left[\frac{\mathrm{d}}{\mathrm{d}t}(Q^*\lambda)(t)\right] - g'(t)\right\}\dot{x}(t)\,\mathrm{d}t - g'(a)\,x(a)$$

for all $x \in \mathscr{AC}_n(a, b)$. Again, we deduce that

(4,7) $$g'(a) = \int_a^b y'(s)\,A(s)\,\mathrm{d}s + \int_{a+r}^b y'(s)\,B(s)\,\mathrm{d}s - \int_a^b (Q^*\lambda)(s)\,\mathrm{d}s - \gamma' N - \delta' = 0$$

and

(4,8) $$y'(t) + \left[\frac{\mathrm{d}}{\mathrm{d}t}(Q^*\lambda)(t)\right] = \int_t^b y'(s)\,A(s)\,\mathrm{d}s - \int_t^b (Q^*\lambda)(s)\,\mathrm{d}s - \gamma' N +$$

$$+\begin{cases}\int_{t+r}^{b} y`(s)\, B(s)\,\mathrm{d}s, & t < b - r \\ 0 & , t \geqq b - r\end{cases} \quad \text{on} \quad (a, b)$$

for some $y` \in \mathscr{L}_n^2(a, b)$, $y`(t) = \eta`(t)$ a.e. on $[a, b]$.

By (4,8), $[y` + (\mathrm{d}/\mathrm{d}t)(Q^*\lambda)](a+)$ and $[y` + (\mathrm{d}/\mathrm{d}t)(Q^*\lambda)](b-)$ exist,

$$\left[y` + \frac{\mathrm{d}}{\mathrm{d}t}(Q^*\lambda)\right](b-) = -\gamma`N$$

and according to (4,6) and (4,7)

$$\left[y` + \frac{\mathrm{d}}{\mathrm{d}t}(Q^*\lambda)\right](a+) = \delta` = \gamma`M\,.$$

The theorem easily follows.

4,6. Corollary. *Let the operator Q in (4,2) be a linear and bounded mapping of $\mathscr{L}_n^2(a, b)$ into Λ. Then $(\eta`, \lambda, \gamma`, \delta`) \in \operatorname{Ker}(U^*)$ iff there is $y` \in \mathscr{AC}_n(a, b)$ such that $y`(t) = \eta`(t)$ a.e. on $[a, b]$ and*

$$\dot{y}`(t) = -y`(t)\,A(t) - \begin{cases} y`(t + r)\, B(t + r), & t < b - r \\ 0 & , t > b - r\end{cases} + (Q^*\lambda)(t) \quad \text{a.e. on} \quad [a, b]\,,$$

$$y`(a) = \gamma`M\,, \quad y`(b) = -\gamma`N\,,$$

$$-y`(t + r)\, B(t + r) + (P^*\lambda)(t) = 0 \quad \text{a.e. on} \quad [a - r, a]$$

($P^* : \Lambda^* \to \mathscr{L}_n^2(a - r, a)$ and $Q^* : \Lambda^* \to \mathscr{L}_n^2(a, b)$ are adjoints of P and Q).

Proof. Since for all $x \in \mathscr{L}_n^2(a, b)$ and $\lambda \in \Lambda^*$

$$\langle Qx, \lambda\rangle_\Lambda = (x, Q^*\lambda)_{\mathscr{L}} = \int_a^b (Q^*\lambda)(t)\, x(t)\,\mathrm{d}t\,,$$

the term $[(\mathrm{d}/\mathrm{d}t)(Q^*\lambda)]$ does not appear in the formula (4,8).

4,7. Remark. Let $Q : \mathscr{L}_n^2(a, b) \to \Lambda$ be linear and bounded. Then Q is also bounded as an operator $\mathscr{W}_n^{1,2}(a, b) \to \Lambda$ and apparently we have two possible adjoint problems, defined in Theorem 4,5 and Corollary 4,6, respectively. We must take into account that in this case we should write $\tilde{Q}^*$ instead of Q^* in the former adjoint, where $\tilde{Q} = QE$ and $E : x \in \mathscr{W}_n^{1,2}(a, b) \to x \in \mathscr{L}_n^2(a, b)$ is a continuous imbedding of $\mathscr{W}_n^{1,2}(a, b)$ into $\mathscr{L}_n^2(a, b)$. (Given $\lambda \in \Lambda^*$ and $x \in \mathscr{W}_n^{1,2}(a, b)$),

$$\int_a^b (Q^*\lambda)(t)\, x(t)\,\mathrm{d}t = \int_a^b \left\{\left[\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{Q}^*\lambda)(t)\right]\dot{x}(t) + (\tilde{Q}^*\lambda)(t)\, x(t)\right\}\mathrm{d}t\,.$$

All the boundary value type problems which occur in paragraphs 2 and 3 of this paper may be formulated as operator equations of the type

$$U\xi = \eta ,$$

where U is a linear bounded mapping of either $\mathscr{X}_c = \mathscr{AC}_n(a, b) \times \mathscr{C}_n(a - r, a)$ or $\mathscr{X}_v = \mathscr{AC}_n(a, b) \times \mathscr{BV}_n(a - r, a)$ into $\mathscr{Y} = \mathscr{L}_n(a, b) \times \Lambda \times \mathscr{R}_n$ and Λ is a B-space. The aim of this paragraph is to characterize in some special cases the range Im (U) of the operator U and, in particular, to find some conditions guaranteeing the closedness of Im (U).

Let $(\mathscr{C} + \mathscr{BV})_n\,(a - r, a)$ denote the set of all functions $w : [a - r, a] \to \mathscr{R}_n$ for which there exist functions $u \in \mathscr{C}_n(a - r, a)$ and $v \in \mathscr{BV}_n(a - r, a)$ such that $w(t) = u(t) + v(t)$ on $[a - r, a]$.

In what follows we make use of the following lemma which is a slight modification of the variation-of-constants formula due to H. T. Banks [1].

5,1. Lemma. *Let the* $n \times n$*-matrix function* $P(t, \vartheta)$ *fulfil the corresponding assumptions from* Sec. 3,4. *Given* $f \in \mathscr{L}_n(a, b)$ *and* $u \in (\mathscr{C} + \mathscr{BV})_n\,(a - r, a)$, *there is just one solution to the initial value problem* ((3,5), (3,6))

$$\dot{x}(t) = \int_{-r}^{0} [\mathrm{d}_\vartheta P(t, \vartheta)]\, x(t + \vartheta) + f(t) \quad \text{a.e. on} \quad [a, b] ,$$

$$x(t) = u(t) \quad \text{on} \quad [a - r, a] .$$

There exist a linear operator $\Phi : (\mathscr{C} + \mathscr{BV})_n\,(a - r, a) \to \mathscr{AC}_n(a, b)$ *and a linear bounded operator* $\Psi : \mathscr{L}_n(a, b) \to \mathscr{AC}_n(a, b)$ *such that this solution is given by*

$$x = \Phi u + \Psi f . \tag{5,1}$$

The operator Φ *as a mapping* $\mathscr{BV}_n(a - r, a) \to \mathscr{AC}_n(a, b)$ *is completely continuous and as a mapping* $\mathscr{C}_n(a - r, a) \to \mathscr{AC}_n(a, b)$ *bounded. Moreover, if* $b - r \geqq a$ *and if* $S_b : x \in \mathscr{AC}_n(a, b) \to x/[b - r, b] \in \mathscr{C}_n(b - r, b)$, *then the operator* $T = S_b\Phi : \mathscr{C}_n(a - r, a) \to \mathscr{C}_n(b - r. b)$ *is completely continuous.*

(The compactness of $\Phi : \mathscr{BV}_n(a - r, a) \to \mathscr{AC}_n(a, b)$ was shown in [13] and the proof of the compactness of T can be find in [7], Remark 8,9.)

5,2. Remark. It follows from the special form of the operator Φ (cf. [1]) that for any $u \in (\mathscr{C} + \mathscr{BV})_n\,(a - r, a)$

$$\Phi u = \Phi^0\, u(a) + \Phi^1 u , \tag{5,2}$$

where $\Phi^0 : \mathscr{R}_n \to \mathscr{AC}_n(a, b)$ is linear and bounded and $\Phi^1 : (\mathscr{C} + \mathscr{BV})_n\,(a - r, a) \to \mathscr{AC}_n(a, b)$ is linear and completely continuous as an operator $\mathscr{BV}_n(a - r, a) \to$

$\to \mathcal{AC}_n(a, b)$ and bounded as an operator $\mathcal{C}_n(a - r, a) \to \mathcal{AC}_n(a, b)$. Moreover, if v is a simple jump function $v(t) = 0$ on $[a - r, a)$ and $v(a) = d$, then $\Phi^1 v = 0$.

5,3. Problem (3,5)–(3,7). Let us turn back to the problem (3,5)–(3,7) whose adjoint was derived in Sec. 3,4. Let Λ be an arbitrary B-space and let the operators $M : \mathcal{C}_n(a - r, a) \to \Lambda$ and $N : \mathcal{AC}_n(a, b) \to \Lambda$ and the $n \times n$-matrix function $P(t, \vartheta)$ fulfil the assumptions of Sec. 3,4. Let $f \in \mathcal{L}_n(a, b)$ and $l \in \Lambda$. Let us put $\mathcal{X}_c = \mathcal{AC}_n(a, b) \times \mathcal{C}_n(a - r, a)$, $\mathcal{Y} = \mathcal{L}_n(a, b) \times \Lambda \times \mathcal{R}_n$,

$$P_1 : x \in \mathcal{AC}_n(a, b) \to \int_{\max(-r, a-t)}^{0} [\mathrm{d}_\vartheta P(t, \vartheta)]\, x(t + \vartheta) \in \mathcal{L}_n(a, b)\,,$$

$$P_2 : u \in C_n(a - r, a) \to \int_{-r}^{\max(-r, a-t)} [\mathrm{d}_\vartheta P(t, \vartheta)]\, u(t + \vartheta), \in \mathcal{L}_n(a, b)\,,$$

and

(5,3)
$$U : \begin{pmatrix} x \\ u \end{pmatrix} \in \mathcal{X}_c \to \begin{bmatrix} Dx - P_1 x - P_2 u \\ Mu + Nx \\ u(a) - x(a) \end{bmatrix} \in \mathcal{Y}$$

(where again $D : x \in \mathcal{AC}_n(a, b) \to \dot{x} \in \mathcal{L}_n(a, b)$). The system (3,5)–(3,7) is equivalent to the operator equation

(5,4)
$$U \begin{pmatrix} x \\ u \end{pmatrix} = \begin{bmatrix} f \\ l \\ 0 \end{bmatrix}.$$

5,3,1. Theorem. *Let* Im $(M + N\Phi)$ *be closed in* Λ, *then the operator* U *defined by* (5,3) *has closed range* Im (U) *in* $\mathcal{Y}$.

Proof. Let $(f, l, d) \in \mathcal{Y}$. According to the variation-of-constants formula (5,1) a couple $\begin{pmatrix} x \\ u \end{pmatrix} \in \mathcal{X}_c$ is a solution to the equation

$$U \begin{pmatrix} x \\ u \end{pmatrix} = \begin{bmatrix} f \\ l \\ d \end{bmatrix}$$

iff

$$x = \Phi\tilde{u} + \Psi f = \Phi^0(u(a) + d) + \Phi^1 u + \Psi f = \Phi^0 d + \Phi u + \Psi f\,,$$

where $\tilde{u} = u + u_d$, $u_d(t) = 0$ on $[a - r, a)$, $u_d(a) = d(\Phi^1 u_d = 0$, cf. Remark 5,2) and $u \in \mathcal{C}_n(a - r, a)$ is a solution to the operator equation

$$[M + N\Phi]\, u = -N\Psi f + l - N\Phi^0 d\,.$$

Let us denote

$$S : \begin{bmatrix} f \\ l \\ d \end{bmatrix} \in \mathscr{Y} \to -N\Psi f + l - N\Phi^0 d \in \Lambda \,.$$

Then $S(\mathrm{Im}\,(U)) = \mathrm{Im}\,(M + N\Phi)$ and since the operator S is linear and bounded, our assertion readily follows.

5,3,2. Corollary. *If* $\Lambda = \mathscr{R}_m$, *then* Im (U) *is closed in* $\mathscr{Y}$.
(In this case Im $(M + N\Phi)$ is a k – dimencional $(0 \leqq k \leqq m)$ linear subspace of $\mathscr{R}_m$.)

5,3,3. Corollary. *Let* $0 \leqq r \leqq b - a$, $S_b : x \in \mathscr{A}\mathscr{C}_n(a, b) \to x \mid [b - r, b] \in \in \mathscr{C}_n(b - r, b)$ *and let* $\tilde{N} : \mathscr{C}_n(b - r, b) \to \Lambda$ *be linear and bounded. Let the operator U be given by* (5,3), *where N is replaced by* $N_b = \tilde{N}S_b$. *Then, if the operator M posseses a bounded inverse* $M^{-1} : \Lambda \to \mathscr{C}_n(a - r, a)$, *the range* Im (U) *of U is closed in* $\mathscr{Y}$.

Proof. By Theorem 5,3,1 Im (U) is closed in $\mathscr{Y}$ if the range of the operator

$$M + \tilde{N}S_b\Phi = M + \tilde{N}T : \mathscr{C}_n(a - r, a) \to \Lambda$$

is closed. Since by Lemma 5,1 the operator $T = S_b\Phi : \mathscr{C}_n(a - r, a) \to \mathscr{C}_n(b - r, b)$ is completely continuous, the existence of a bounded M^{-1} implies the closedness of Im $(M + \tilde{N}T)$ and hence also of Im (U).

5,3,4. Remark. Our restriction to two-point boundary value type problems in Corollary 5,3,3 does not mean an essential loss of generality (cf. [8]).

5,3,5. Corollary. *The T-periodic problem* (3,5), (3,6), (3,20) (cf. Sec. 3,9) *has a solution iff*

$$\int_0^T y^{`}(s) f(s)\,\mathrm{d}s = 0$$

for all T-periodic solutions $y^{`}(t)$ (i.e., $y^{`}(t) = y^{`}(T + t)$ *on* $[-r, 0)$) *of the equation*

$$y^{`}(t) + \int_t^b y^{`}(s)\,P(s, t - s)\,\mathrm{d}s = \text{const.} \quad \text{on} \quad [-r, T - r]\,.$$

(Proof follows from Corollaries 3,10 and 5,3,3.)

5,3,6. Remark. Let Λ_1 be a B-space and let the operators $M_1 : \mathscr{C}_n(a - r, a) \to \Lambda_1$ and $N_1 : \mathscr{A}\mathscr{C}_n(a, b) \to \Lambda_1$ be linear and bounded. If $\Lambda = \mathscr{C}_n(a - r, a) \times \Lambda_1$ and

$$M : u \in \mathscr{C}_n(a - r, a) \to \begin{bmatrix} u \\ M_1 u \end{bmatrix} \in \Lambda\,, \quad N : x \in \mathscr{A}\mathscr{C}_n(a, b) \to \begin{bmatrix} 0 \\ N_1 x \end{bmatrix} \in \Lambda\,,$$

then the operator U given by (5,3) has closed range $\mathrm{Im}(U)$ in $\mathscr{Y} = \mathscr{L}_n(a, b) \times \times \mathscr{C}_n(a - r, a) \times \Lambda_1 \times \mathscr{R}_n$. (Indeed, according to Lemma 5,1 an element (f, h, l, d) of $\mathscr{Y}$ belongs to $\mathrm{Im}(U)$ iff

$$F(f, h, l, d) = N_1 \Psi f + (M_1 + N_1 \Phi) h - l + N_1 \Phi^0 d = 0 .$$

It is easy to see that the operator $F : \mathscr{Y} \to \Lambda_1$ is linear and bounded. Consequently, the set $\mathrm{Im}(U) = \mathrm{Ker}(F)$ is closed in $\mathscr{Y}$.)

5,3,7. Remark. All the assertions of this section will remain true if we replace the initial space $\mathscr{C}_n(a - r, a)$ by $\mathscr{BV}_n(a - r, a)$. Moreover, Corollary 5,3,3 could be now formulated directly for a general linear bounded operator $N : \mathscr{AC}_n(a, b) \to \Lambda$. ($N$ need not be of the two-point character $N = \tilde{N}S_b$.) This is possible in virtue of the compactness of the operator $\Phi : \mathscr{BV}_n(a - r, a) \to \mathscr{AC}_n(a, b)$ in the variation-of-constants formula (5,1) (cf. Lemma 5,1).

5,4. Problem (2,1)–(2,3). The subject of this section is the general problem of finding $x \in \mathscr{AC}_n(a, b)$ and $u \in \mathscr{BV}_n(a - r, a)$ which satisfy (2,1)–(2,3). Let Assumptions 2,1 be fulfilled. We make use of the notation introduced in Sec. 2,3. (Only $\mathscr{C}_n(a - r, a)$ should be replaced everywhere by $\mathscr{BV}_n(a - r, a)$.)

5,4,1. Lemma. *Let $-\infty < c < d < +\infty$ and let $K(t, s)$ be an $n \times n$-matrix function defined and Borel measurable in (t, s) on $[a, b] \times [c, d]$ and such that $\mathrm{var}_c^d K(t, \cdot) < \infty$ for any $t \in [a, b]$, while*

$$\int_a^b (\mathrm{var}_c^d K(t, \cdot) + \|K(t, d)\|) \, \mathrm{d}t < \infty .$$

Then the operator

$$K : u \in \mathscr{BV}_n(c, d) \to \int_c^d [\mathrm{d}_s K(t, s)] \, u(s) \in \mathscr{L}_n(a, b)$$

is completely continuous.

Proof. The operator K is surely linear and bounded.

Let $\{u^j\}_{j=1}^{\infty} \subset \mathscr{BV}_n(c, d)$ and $\|u^j\|_{\mathscr{BV}} < 1$ $(j = 1, 2, \ldots)$. Then by Helly's Choice Theorem there exists a subsequence $\{u^{j_l}\} \subset \{u^j\}$ and $u^0 \in \mathscr{BV}_n(c, d)$ such that

$$\lim_{l \to \infty} u^{j_l}(s) = u^0(s) \quad \text{for all} \quad s \in [c, d] .$$

Let us put for $s \in [c, d]$ and $l = 1, 2, \ldots$

$$v^l(s) = \|u^{j_l}(s) - u^0(s)\|$$

and for $t, s \in [a, b] \times [c, d]$

$$k(t, s) = \mathrm{var}_c^s K(t, \cdot) .$$

Then $\|v^l(s)\| \leqq \|u^0\|_{\mathscr{BV}} + 1$ on $[c, d]$ for any $l = 1, 2, \ldots$, $\operatorname{var}_c^d k(t, \cdot) = \operatorname{var}_c^d K(t, \cdot)$ for any $t \in [a, b]$ and by the unsymmetric Fubini theorem

$$\int_a^b \left\| \int_c^d [\mathrm{d}_s K(t, s)] (u^{j_l}(s) - u^0(s)) \right\| \mathrm{d}t \leqq \int_a^b \left(\int_c^d [\mathrm{d}_s k(t, s)]\, v^l(s) \right) \mathrm{d}t =$$
$$= \int_c^d \left[\mathrm{d}_s \int_a^b k(t, s)\, \mathrm{d}t \right] v^l(s)\,.$$

Given a subdivision $\{c = s_0 < s_1 \ldots < s_m = d\}$ of $[c, d]$,

$$\sum_{i=1}^m \left\| \int_a^b (k(t, s_i) - k(t, s_{i-1}))\, \mathrm{d}t \right\| \leqq \int_a^b \left(\sum_{i=1}^m \|k(t, s_i) - k(t, s_{i-1})\| \right) \mathrm{d}t \leqq$$
$$\leqq \int_a^b (\operatorname{var}_c^d k(t, \cdot))\, \mathrm{d}t < \infty\,.$$

Thus

$$\operatorname{var}_c^d \left(\int_a^b k(t, \cdot)\, \mathrm{d}t \right) < \infty$$

and according to the dominated convergence theorem for Perron-Stieltjes integrals

$$\lim_{l \to \infty} \int_c^d \left[\mathrm{d}_s \int_a^b k(t, s)\, \mathrm{d}t \right] v^l(s) = 0$$

or

$$\lim_{l \to \infty} \|K u^{j_l} - K u^0\|_{\mathscr{L}} = \lim_{l \to \infty} \int_a^b \left\| \int_c^d [\mathrm{d}_s K(t, s)] (u^{j_l}(s) - u^0(s)) \right\| \mathrm{d}t = 0$$

which completes the proof.

5,4,2. Remark. The operator

$$u \in \mathscr{C}_n(c, d) \to \int_c^d [\mathrm{d}_s K(t, s)]\, u(s) \in \mathscr{L}_n(a, b)$$

(with $K(t, s)$ fulfilling the assumptions of Lemma 5,4,1) need not be generally completely continuous.

5,4,3. Theorem. *If the operator $M : \mathscr{BV}_n(a - r, a) \to A$ has a bounded inverse M^{-1}, then the operator U given by (2,4) (with $\mathscr{C}_n(a - r, a)$ replaced by $\mathscr{BV}_n(a - r, a)$) has closed range in $\mathscr{Y}$.*

Proof. By Lemma 5,1 applied to initial value problems of the type

$$\dot{x}(t) = A(t)\, x(t) + B(t)\, x(t - r) + g(t) \quad \text{a.e. on} \quad [a, b]\,,$$
$$x(t) = u(t) \quad \text{on} \quad [a - r, a]\,,$$

the triple $(f, l, d) \in \mathscr{Y}$ belongs to $\operatorname{Im}(U)$ iff there is a solution $\begin{pmatrix} x \\ u \end{pmatrix} \in \mathscr{X}_v = \mathscr{AC}_n(a, b) \times \times \mathscr{BV}_n(a - r, a)$ to the system of operator equations

$$(5,5) \qquad x - \Psi G_1 x - \Phi u - \Psi G_2 u = \Psi f + \Phi^0 d\,,$$
$$Mu + Nx = l\,,$$

where the operator $\Phi : \mathscr{BV}_n(a - r, a) \to \mathscr{AC}_n(a, b)$ is linear and completely continuous and the operators $\Phi^0 : \mathscr{R}_n \to \mathscr{AC}_n(a, b)$ and $\Psi : \mathscr{L}_n(a, b) \to \mathscr{AC}_n(a, b)$ are linear and bounded. Since there exists a bounded inverse M^{-1} of M, the latter equation in (5,5) yields $u = M^{-1}l - M^{-1}Nx$, while the former becomes

$$x - \{\Phi M^{-1}N + \Psi G_1 + \Psi G_2 M^{-1}N\}\, u = \Psi f + (\Phi + \Psi G_2)\, M^{-1}l + \Phi^0 d\,.$$

Let us put $K = \Phi M^{-1}N + \Psi G_1 + \Psi G_2 M^{-1}N$, $S(f, l, d) = \Psi f + (\Phi + \Psi G_2) \,.\, M^{-1}l + \Phi^0 d$ and let I denote the identity operator on $\mathscr{AC}_n(a, b)$. Then $S(\operatorname{Im}(U)) = \operatorname{Im}(I - K)$ and since $S : \mathscr{Y} \to \mathscr{AC}_n(a, b)$ is linear and bounded, $\operatorname{Im}(U)$ is closed if $\operatorname{Im}(I - K)$ is closed. The operators G_1, G_2 are completely continuous by Lemma 5,4,1 and since the operators M^{-1}, N and Ψ are bounded the operator K is also completely continuous and $\operatorname{Im}(I - K)$ is closed.

5,4,4. Remark. As an easy consequence of Theorem 5,4,3 we obtain that in the case of the T-periodic problem (i.e. $a = 0$, $b = T$, $r < T$, $\Lambda = \mathscr{AC}_n(-r, 0)$, $M = I$, $N : x \in \mathscr{AC}_n(0, T) \to x_T(s) = x(T + s) \in \mathscr{AC}_n(-r, 0)$ and $l = 0$) the range $\operatorname{Im}(U)$ of U is closed in $\mathscr{Y}$.

5,5. Boundary value problems for ordinary integrodifferential equations. If $r = 0$ and $\Lambda = \mathscr{R}_m$, then the given problem (2,1)–(2,3) reduces to the boundary value problem for an ordinary integrodifferential equation of the form

$$(5,6) \qquad \dot{x}(t) = A(t)\, x(t) + \int_a^b [\mathrm{d}_s G(t, s)]\, x(s) + f(t) \quad \text{a.e. on} \quad [a, b]\,,$$

$$(5,7) \qquad Nx = l\,,$$

where the $n \times n$-matrix function $A(t)$ is $\mathscr{L}$-integrable on $[a, b]$, $\operatorname{var}_a^b G(t, \cdot) < \infty$ for any $t \in [a, b]$,

$$\int_a^b (\operatorname{var}_a^b G(t, \cdot) + \|G(t, b)\|)\, \mathrm{d}t < \infty\,,$$

$f \in \mathscr{L}_n(a, b)$, $l \in \mathscr{R}_m$ and the operator $N : \mathscr{AC}_n(a, b) \to \mathscr{R}_m$ is linear and bounded. (The initial space reduces to $\mathscr{R}_n$.)

Let us reformulate the problem (5,6), (5,7) as the operator equation

$$Ux = \begin{pmatrix} f \\ l \end{pmatrix},$$

where

$$(5,8) \qquad U : x \in \mathscr{AC}_n(a, b) \to \begin{pmatrix} Dx - Ax - Gx \\ Nx \end{pmatrix} \in \mathscr{L}_n(a, b) \times \mathscr{R}_m$$

and the symbols D, A, G have the obvious menaning.

5,5,1. Theorem. *The operator U defined by (5,8) has closed range in $\mathscr{L}_n(a, b) \times \times \mathscr{R}_m$.*

Proof. There exist linear and bounded operators $\Phi^0 : \mathscr{R}_n \to \mathscr{AC}_n(a, b)$ and $\Psi : : \mathscr{L}_n(a, b) \to \mathscr{AC}_n(a, b)$ such that an n-vector function $x(t)$ is a solution to the given problem iff

$$x = \Phi^0 c + \Psi h + \Psi f,$$

where the couple $(h, c) \in \mathscr{L}_n(a, b) \times \mathscr{R}_n$ $(h = Gx)$ is a solution to the system

$$(5,9) \qquad \begin{aligned} h - (G\Phi^0)\, c - (G\Psi)\, h &= (G\Psi) f, \\ (N\Phi^0)\, c + (N\Psi)\, h &= l - (N\Psi) f. \end{aligned}$$

$(N\Phi^0)$ is a constant $m \times n$-matrix. Let e.g. $m < n$. Putting

$$Q = I_n - \begin{bmatrix} N\Phi^0 \\ 0_{n-m,n} \end{bmatrix}, \quad \tilde{l} = \begin{bmatrix} l \\ 0_{n-m,1} \end{bmatrix} \in \mathscr{R}_n,$$

$$R : h \in \mathscr{L}_n(a, b) \to \begin{bmatrix} (N\Psi)\, h \\ 0_{n-m,1} \end{bmatrix} \in \mathscr{R}_n$$

($0_{p,q}$ denotes the zero $p \times q$-matrix and I_n is the identity $n \times n$-matrix),

$$K : \begin{pmatrix} h \\ c \end{pmatrix} \in \mathscr{L}_n(a, b) \times \mathscr{R}_n \to \begin{bmatrix} (G\Phi^0)\, c + (G\Psi)\, h \\ Qc - Rh \end{bmatrix} \in \mathscr{L}_n(a, b) \times \mathscr{R}_n$$

and

$$S : \begin{pmatrix} f \\ l \end{pmatrix} \in \mathscr{L}_n(a, b) \times \mathscr{R}_m \to \begin{bmatrix} (G\Psi) f \\ \tilde{l} - Rf \end{bmatrix} \in \mathscr{L}_n(a, b) \times \mathscr{R}_n,$$

the system (5,9) becomes

$$(I - K) \begin{pmatrix} h \\ c \end{pmatrix} = S \begin{pmatrix} f \\ l \end{pmatrix}$$

and $S(\operatorname{Im}(U)) = \operatorname{Im}(I - K)$. Since by Lemma 5,4,1 the operator G is completely continuous, it is easy to verify that the operator K is completely continuous. It means that $\operatorname{Im}(I - K)$ is closed and taking into account that the operator S is linear and bounded we complete the proof. The case $m > n$ can be treated analogously.

Let $N(t)$ be an $m \times n$-matrix function of bounded variation on $[a, b]$ and let the operator N be given by

$$(5,10) \qquad N : x \in \mathscr{AC}_n(a, b) \to \int_a^b [\mathrm{d}N(s)]\, x(s) \in \mathscr{R}_m .$$

Without any loss of generality we may assume that for any $t \in [a, b]$ the functions $G(t, \cdot)$ and N are right-continuous on (a, b). Let us put for $t \in [a, b]$

$$C(t) = G(t, a+) - G(t, a), \quad D(t) = G(t, b) - G(t, b-),$$

$$G_0(t, s) = \begin{cases} G(t, a+) & \text{for } s = a, \\ G(t, s) & \text{for } a < s < b, \\ G(t, b-) & \text{for } s = b, \end{cases} \qquad L(s) = \begin{cases} N(a+) & \text{for } s = a, \\ N(s) & \text{for } a < s < b, \\ N(b-) & \text{for } s = b, \end{cases}$$

$$M = N(a+) - N(a), \quad N = N(b) - N(b-).$$

Then similarly as in Sec. 3,3 we obtain that the adjoint problem to (5,6), (5,7) is equivalent to the problem of finding $y \in \mathscr{BV}_n(a, b)$, right-continuous on $[a, b)$ and left-continuous at b and $\lambda \in \mathscr{R}_n$ such that

$$(5,11) \quad y'(t) = y'(b) + \int_t^b y'(s)\, A(s)\, \mathrm{d}s - \int_a^b y'(s)\,(G_0(s, t) - G_0(s, b))\, \mathrm{d}s + \\ + \lambda'(L(t) - L(b)) \quad \text{on} \quad [a, b],$$

$$(5,12) \quad y'(a) = \lambda' M - \int_a^b y'(s)\, C(s)\, \mathrm{d}s, \quad y'(b) = -\lambda' N + \int_a^b y'(s)\, D(s)\, \mathrm{d}s .$$

The following theorem is then a direct corollary of Theorem 5,5,1.

5,5,2. Theorem. *The problem* (5,6), (5,7) *possesses a solution iff*

$$\int_a^b y'(s) f(s)\, \mathrm{d}s + \lambda' l = 0$$

for any solution $(y'(t), \lambda')$ *of the adjoint problem* (5,11), (5,12).

Theorem 5,5,2 generalizes Theorem 3,1 from [12].

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[1] *Banks H. T.:* Representations for Solutions fo Linear Functional Differential Equations, Journ. Diff. Eq. *5* (1969), 399—409.

[2] *Cameron R. H.* and *Martin W. T.,* An unsymmetric Fubini theorem, Bull. A.M.S. *47* (1941), 121—125.

[3] *Delfour M. C.* and *Mitter S. K.,* Hereditary Differential Systems with Constant Delays. II — A class of affine systems and the adjoint problem, to appear in Journ. Diff. Eq.

[4] *Dunford N.* and *Schwartz T.,* Linear operators, part I, Interscience Publishers, New York—London, 1958.

[5] *Halanay A.,* Periodic solutions of linear systems with time lag (in Russian), Rev. Math. pures appl. (Acad. R.P.R.) *6* (1961), 141—158.

[6] *Halanay A.,* On a Boundary Value Problem for Linear Systems with Time Lag, Journ. Diff. Eq. *2* (1966), 47—56.

[7] *Hale J. K.,* Functional Differential Equations, Applied Mathematical Sciences, vol. 3, Springer-Verlag, New York, 1971.

[8] *Henry D.,* The Adjoint of a Linear Functional Differential Equation and Boundary Value Problems, Journ. Diff. Eq. *9* (1971), 55—66.

[9] *Lifšic E. A.,* Fredholm alternative for the problem of periodic solutions to differential equations with retarded argument (in Russian), Probl. mat. analiza složnych sistem *1* (1967), 53—57.

[10] *Schwabik Št.,* On an integral operator in the space of functions with bounded variation, Čas. pěst. mat. *97* (1972), 297—330.

[11] *Schwabik Št.,* Remark on linear equations in Banach space, Čas. pěst. mat., *99* (1974), 115—122.

[12] *Tvrdý M.* and *Vejvoda O.,* General boundary value problem for an integro-differential system and its adjoint, Čas. pěst. mat. *97* (1972), 399—419 and *98* (1973), 26—42.

[13] *Tvrdý M.,* Note on functional-differential equations with initial functions of bounded variation, Czech. Math. J. *25* (*100*) (1975), 67—70.

[14] *Wexler D.,* On boundary value problems for an ordinary linear differential system, Ann. Mat. pura appl. *80* (1968), 123—134.

Remark 2,12 was added in the proofs. Its assertion was proved in [15] (Theorem 4,4). The results of sec. 5,5 were shown in another way also in [16].

[15] *Tvrdý M.*, Linear functional-differential operators: normal solvability and adjoints, to appear in the Proceedings of the Colloquium on Differential Equations (Keszthely, Hungary — September 1974),

[16] *Maksimov V. P.*, General boundary value problem for linear functional-differential equation is noetherian (in Russian), Diff. urav. *10* (1974), 2288—2291.

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