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NOTE ON FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH INITIAL FUNCTIONS OF BOUNDED VARIATION

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In this note we shall deal with the standard functional-differential equation of retarded type

$$\dot{x}(t) = \int_{-r}^{0} [\mathrm{d}_\vartheta P(t, \vartheta)]\, x(t + \vartheta) + f(t) \quad \text{a.e. on} \quad [a, b]\,, \tag{1}$$

$$x(t) = u(t) \quad \text{on} \quad [a - r, a]\,, \tag{2}$$

where $-\infty < a < b < +\infty$ and the initial functions $u(t)$ are of bounded variation on $[a - r, a]$. We assume that $P(t, \vartheta)$ is a Borel measurable in $(t, \vartheta) \in [a, b] \times \times (-\infty, +\infty)$ $n \times n$-matrix function such that $p(t) = \mathrm{var}_{-r}^{0} P(t, \cdot) < \infty$ for all $t \in [a, b]$ and

$$\int_a^b p\,(t)\,\mathrm{d}t < \infty\,,$$

$f(t)$ is an n-vector function Lebesgue integrable on $[a, b]$ $(f(t) \in \mathscr{L}_n(a, b))$. We shall suppose also $P(t, \vartheta) = P(t, -r)$ for $\vartheta \leqq -r$ and $P(t, \vartheta) = P(t, 0)$ for $\vartheta \geqq 0$. Without any loss of generality we may suppose furthermore that $P(t, \cdot)$ is right continuous on $(-r, 0)$ and $P(t, 0) = 0$ for all $t \in [a, b]$.

Let $\mathscr{BV}_n(a - r, a)$ denote the space of (column) n-vector functions with bounded variation on $[a - r, a]$. $\mathscr{AC}_n(a, b)$ is the space of n-vector functions which are absolutely continuous on $[a, b]$. The introduced spaces are equipped with the usual norms

$$u \in \mathscr{BV}_n(a - r, a) \to \|u\|_{\mathscr{BV}} = \|u(a)\| + \mathrm{var}_{a-r}^{a}\, u\,,$$

$$x \in \mathscr{AC}_n(a, b) \quad \to \|x\|_{\mathscr{AC}} = \|x(a)\| + \mathrm{var}_a^b\, x\,,$$

$$f \in \mathscr{L}_n(a, b) \quad \to \|f\|_{\mathscr{L}} = \int_a^b \|f(t)\|\,\mathrm{d}t\,.$$

Proposition 1. *There exists a unique $n \times n$-matrix function $Y(t, s)$ defined on $[a, b] \times [a, b]$ and such that*

$$(3) \qquad Y(t, s) = \begin{cases} I - \int_s^t Y(t, \sigma)\, P(\sigma, s - \sigma)\, d\sigma & \text{for} \quad a \leqq t \leqq b\,, \quad a \leqq s \leqq t\,, \\ I & \text{for} \quad a \leqq t \leqq b\,, \quad t \leqq s \leqq b\,, \end{cases}$$

where I is the identity $n \times n$-matrix. Given $t \in [a, b]$, $Y(t, \cdot)$ is of bounded variation on $[a, b]$ and given $s \in [a, b]$, $Y(\cdot, s)$ is absolutely continuous on $[a, b]$.

(For the proof of a slightly modified assertion see J. K. Hale [2], Theorem 32,2.)

The following representation of solutions of the system (1), (2) is well known (cf. H. T. Banks [1] or J. K. Hale [2], Theorems 16,1 and 32,2):

Proposition 2. *Given $u \in \mathscr{BV}_n(a - r, a)$, there exists a unique n-vector function $x(t)$ defined on $[a - r, b]$ and absolutely continuous on $[a, b]$ and such that (1) and (2) hold. This function $x(t)$ is on $[a, b]$ given by*

$$(4) \qquad x = \Phi u + \Psi f\,,$$

where

$$\Phi : u \in \mathscr{BV}_n(a - r, a) \to Y(t, a)\, u(a) + \int_{a-r}^{a} \left[d_s \int_a^t Y(t, \sigma)\, P(\sigma, s - \sigma)\, d\sigma \right] u(s) \in \in \mathscr{AC}_n(a, b)\,,$$

$$\Psi : f \in \mathscr{L}_n(a, b) \to \int_a^t Y(t, s) f(s)\, ds \in \mathscr{AC}_n(a, b)$$

and $Y(t, s)$ is defined by Proposition 1.

The operators Φ, Ψ in (4) are obviously linear and bounded. The aim of this note is to show that Φ is even completely continuous. By Theorem 3,1 of Št. Schwabik [5] it suffices to show that the function

$$(5) \qquad K(t, s) = \int_a^t Y(t, \sigma)\, P(\sigma, s - \sigma)\, d\sigma\,, \quad (t, s) \in [a, b] \times [a - r, a]$$

is of bounded two-dimensional variation (according to Vitali) on $[a, b] \times [a - r, a]$ ($v(K) < \infty$) and $\operatorname{var}_{a-r}^{a} K(a, \cdot) + \operatorname{var}_a^b K(\cdot, a) < \infty$. Such functions are said to be of strongly bounded variation on $[a, b] \times [a - r, a]$. (For the definition and basic properties of functions of bounded two-dimensional variation see T. H. Hildebrandt [4].)

Lemma 1. *The fundamental matrix solution $Y(t, s)$ defined by Proposition* 1 *is of strongly bounded variation on $[a, b] \times [a, b]$.*

Proof. Analogously to J. K. Hale in the proof of Theorem 32,2 in [2] we shall introduce the function $W(t, s)$ fulfilling the matrix Volterra integral equation

$$W(t, s) = \begin{cases} - P(t, s - t) - \int_s^t W(t, \sigma)\, P(\sigma, s - \sigma)\, d\sigma & \text{for } a \leqq t \leqq b,\ a \leqq s \leqq t\,, \\ 0 & \text{for } a \leqq t \leqq b,\ t \leqq s \leqq b\,. \end{cases}$$

The existence of such a function $W(t, s)$ follows from the contraction mapping principle. Moreover, given $t \in [a, b]$, the function $W(t, \cdot)$ is of bounded variation on $[a, b]$. Now, let $s, t \in [a, b]$, $s \leqq t$ and let $\{s = s_0 < s_1 < \ldots < s_m = t\}$ be an arbitrary subdivision of the interval $[s, t]$. Then

$$\sum_{j=1}^{m} \|W(t, s_j) - W(t, s_{j-1})\| \leqq \sum_{j=1}^{m} \|P(t, s_j - t) - P(t, s_{j-1} - t)\| +$$

$$+ \sum_{j=1}^{m} \left\{ \int_{s_j}^{t} \|W(t, \sigma)\| \, \|P(\sigma, s_j - \sigma) - P(\sigma, s_{j-1} - \sigma)\| \, d\sigma + \right.$$

$$\left. + \int_{s_{j-1}}^{s_j} \|W(t, \sigma) P(\sigma, s_{j-1} - \sigma)\| \, d\sigma \right\} \leqq p(t) + 2 \int_{s}^{t} (\mathrm{var}_{\sigma}^{t} W(t, \cdot)) \, p(\sigma) \, d\sigma ,$$

where $p(t) = \mathrm{var}_{-r}^{0} P(t, \cdot)$ for $t \in [a, b]$. Gronwall's inequality yields

$$\|W(t, s)\| \leqq \mathrm{var}_{s}^{t} W(t, \cdot) \leqq p(t) \exp\left(2 \int_{s}^{t} p(\sigma) \, d\sigma\right) < \infty \tag{6}$$

for all $t, s \in [a, b]$, $t \geqq s$. It is easy to verify (cf. [2], proof of Theorem 32,2) that for all $t, s \in [a, b]$

$$Y(t, s) = I + \int_{s}^{t} W(\tau, s) \, d\tau .$$

Furthermore, let $v = \{a = t_0 < t_1 < \ldots < t_p = b;\ a = s_0 < s_1 < \ldots < s_q = b\}$ be an arbitrary net type subdivision of $[a, b] \times [a, b]$. Then according to (6)

$$\sum_{j=1}^{p} \sum_{k=1}^{q} \Delta\Delta_{j,k} Y = \sum_{j=1}^{p} \sum_{k=1}^{q} \|Y(t_j, s_k) - Y(t_{j-1}, s_k) - Y(t_j, s_{k-1}) + Y(t_{j-1}, s_{k-1})\| \leqq$$

$$\leqq \sum_{j=1}^{p} \sum_{k=1}^{q} \left\| \int_{t_{j-1}}^{t_j} (W(\tau, s_k) - W(\tau, s_{k-1})) \, d\tau \right\| \leqq \int_{a}^{b} \sum_{k=1}^{q} \|W(\tau, s_k) - W(\tau, s_{k-1})\| \, d\tau \leqq$$

$$\leqq \int_{a}^{b} \mathrm{var}_{a}^{\tau} W(\tau, \cdot) \, d\tau = \int_{a}^{b} p(\tau) \exp\left(2 \int_{a}^{\tau} p(\sigma) \, d\sigma\right) d\tau = M < \infty .$$

Thus

$$\mathrm{v}(Y) = \sup \sum_{j=1}^{p} \sum_{k=1}^{q} \Delta\Delta_{j,k} Y \leqq M < \infty$$

which completes the proof.

Corollary 1. *There exists $M < \infty$ such that for all $t, s \in [a, b]$*

$$\|Y(t, s)\| + \mathrm{var}_{a}^{b} Y(t, \cdot) + \mathrm{var}_{a}^{b} Y(\cdot, s) + \mathrm{v}(Y) \leqq M .$$

Lemma 2. *The function $K(t, s)$ defined by (5) is of strongly bounded variation on $[a, b] \times [a - r, a]$.*

Proof. a) $K(a, \cdot) = 0$ on $[a - r, a]$.

b) Let $\{a = t_0 < t_1 < \ldots < t_m = b\}$ be an arbitrary subdivision of $[a, b]$. Then by Corollary 1

$$\sum_{j=1}^{m} \|K(t_j, a) - K(t_{j-1}, a)\| = \sum_{j=1}^{m} \left\| \int_{t_{j-1}}^{t_j} Y(t_j, \sigma) P(\sigma, a - \sigma) \, d\sigma + \right.$$

$$\left. + \int_a^{t_{j-1}} (Y(t_j, \sigma) - Y(t_{j-1}, \sigma)) P(\sigma, a - \sigma) \, d\sigma \right\| \leqq M \int_a^b p(\sigma) \, d\sigma < \infty .$$

Hence $\mathrm{var}_a^b K(\cdot, a) < \infty$.

c) Given a net type subdivision $\{a = t_0 < t_1 < \ldots < t_p = b;\ a - r = s_0 < < s_1 < \ldots < s_q = a\}$ of $[a, b] \times [a - r, a]$, we have by Corollary 1

$$\sum_{j=1}^{p} \sum_{k=1}^{q} \|K(t_j, s_k) - K(t_{j-1}, s_k) - K(t_j, s_{k-1}) + K(t_{j-1}, s_{k-1})\| =$$

$$= \sum_{j=1}^{p} \sum_{k=1}^{q} \left\| \int_a^{t_{j-1}} (Y(t_j, \sigma) - Y(t_{j-1}, \sigma)) (P(\sigma, s_k - \sigma) - P(\sigma, s_{k-1} - \sigma)) \, d\sigma + \right.$$

$$\left. + \int_{t_{j-1}}^{t_j} Y(t_j, \sigma) (P(\sigma, s_k - \sigma) - P(\sigma, s_{k-1} - \sigma)) \, d\sigma \right\| \leqq$$

$$\leqq \int_a^b (\mathrm{var}_a^b Y(\cdot, \sigma) + \sup_{\tau \in [a,b]} \|Y(\tau, \sigma)\|) \, \mathrm{var}_{-r}^0 P(\sigma, \cdot) \, d\sigma \leqq M \int_a^b p(\sigma) \, d\sigma < \infty .$$

Consequently, $\mathrm{v}(K) < \infty$ and this completes the proof of Lemma 2.

The following theorem is a direct consequence of Theorem 3,1 from [5] and of Lemma 2.

Theorem. *The Cauchy operator Φ in the variation – of – constants formula* (4) *is completely continuous.*

References

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