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ON OPTIMAL CONTROL OF SYSTEMS WITH INTERFACE SIDE CONDITIONS

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Let $0 < \tau < 1$. Denote by D_n the space of functions $x : [0,1] \to R_n$ which are absolutely continuous on $[0,\tau]$ and on $(\tau,1]$ and such that their derivatives $\dot{x}$ are square integrable on $[0,1]$ ($\dot{x} \in L^2_n$). We want to establish necessary conditions for a local extremum of the functional of the type

$$F : (x,u) \in D_n \times L^2_m \to g_0\big(x(0)\big) + g_\tau\big(x(\tau+)\big) + g_1\big(x(1)\big) + \int_0^1 h\big(s,x(s),u(s)\big)\, ds \in R \tag{0.1}$$

subject to the constraints

$$x(t) - A(t)x(t) - B(t)u(t) = 0 \quad \text{a.e. on} \quad [0,1] \tag{0.2}$$

and

$$Mx(0) + Nx(\tau+) + \int_0^1 K(s)\, \dot{x}(s)\, ds = 0 \; . \tag{0.3}$$

1. Preliminaries

Throughout the paper the elements in R_n are considered to be column n-vectors. Given a $c \in R_n$, c^* denotes its transposition. Given a Banach space X , $||\cdot||_X$ and X^* denote the norm on X and the dual of X , respectively. For any $x \in X$ and $\phi \in X^*$, the value of the functional ϕ on x is denoted by $\langle x,\phi\rangle_X$. If Y is also a Banach space, then $L(X,Y)$ denotes the space of linear continuous mappings of X into Y . For $A \in L(X,Y)$, $N(A)$, $R(A)$ and A^* denote its null space, range and adjoint, respectively.

Furthermore, L^2_n denotes the space of functions $x : [0,1] \to R$ square integrable on $[0,1]$, equipped with its usual norm denoted by $||\cdot||_L$. The norm on D_n is defined by $x \in D_n \to ||x||_D = |x(0)| + |x(\tau + \;)| + ||\dot{x}||_L$. Obviously D_n is isometrically isomorphic with

$L_n^2 \times R_{2n}$. Its dual will be identified with $L_n^2 \times R_{2n}$, while

$$< x,\phi >_D = a^*x(0) + b^*x(\tau+) + < \dot{x},w >_L = $$
$$= a^*x(0) + b^*x(\tau+) + \int_0^1 w^*(s)\ \dot{x}(s)\ ds$$

for any $x \in D_n$ and $\phi = (w,a,b) \in L_n^2 \times R_n \times R_n$.

We shall keep the following assumptions.

ASSUMPTIONS. $A(t)$, $B(t)$ and $K(t)$ are square integrable on $[0,1]$ matrix valued functions of the types $n \times n$, $n \times m$ and $k \times n$, respectively, M and N are $k \times n$-matrices. The functions $g_0(x)$, $g_\tau(x)$, $g_1(x)$ and $h(t,x,u)$ are continuous and continuously differentiable with respect to x and u .

2. Lagrange Multiplier Theorem

Let us define

$$A : x \in D_n \to \begin{bmatrix} \dot{x}(t) - A(t).x(t) \\ Mx(0) + Nx(\tau+) + \int_0^1 K(s)\ \dot{x}(s)\ ds \end{bmatrix},$$

$$B : u \in L_m^2 \to \begin{bmatrix} B(t)u(t) \\ 0 \end{bmatrix}$$

and

$$T : (x,u) \in D_n \times L_m^2 \to Ax - Bu .$$

Then $A \in L(D_n, L_n^2 \times R_k)$, $B \in L(L_m^2, L_n^2 \times R_k)$ and $T \in L(D_n \times L_m^2, L_n^2 \times R_k)$ and the constraints (0.2), (0.3) may be replaced by the operator equation for $(x,u) \in D_n \times L_m^2$

$$T(x,u) = 0 . \tag{2.1}$$

The operator A is related to interface boundary value problems. It is known (cf. [1]) that under our assumptions A is normally solvable, i.e. $(f,r) \in L_n^2 \times R_k$ belongs to its range iff $< y,f >_L + \gamma r = 0$ for all $(y,\gamma) \in N(A^*)$ ($N(A^*) \subset L_n^2 \times R_k$). It was also shown in [1] that $N(A^*)$ consists of all $(y,\gamma) \in L_n^2 \times R_k$ for which there exists a $z \in D_n$ such that $z^*(t) = y^*(t) + \gamma^* K(t)$ a.e. on $[0,1]$ and

$$- \dot{z}^*(t) - z^*(t)A(t) + \gamma^*K(t)A(t) = 0 \quad \text{a.e. on} \quad [0,1] , \tag{2.2}$$

$$- z^*(0) + \gamma^*M = 0 , \quad z^*(\tau-) = 0 , \tag{2.3}$$

$$- z^*(\tau+) + \gamma^*N = 0 , \quad z^*(1) = 0 . \tag{2.4}$$

It is easy to see that $0 \leqq \dim N(A) + \dim N(A^*) < \infty$. Hence we may apply Proposition 1.2 of [6] to obtain necessary and sufficient conditions for the complete controllability of the system (0.2), (0.3).

PROPOSITION. $R(T) = L_n^2 \times R_k$ *iff the only couple* $(z,\gamma) \in D_n \times R_k$ *fulfilling* (2.2) - (2.4) *together with*

$$- z^*(t)B(t) + \gamma^*K(t)B(t) = 0 \quad a.e.\ on \quad [0,1] \tag{2.5}$$

is the trivial one: $z(t) = 0$ *on* $[0,1]$ *and* $\gamma = 0$.

Let us suppose that $R(T) = L_n^2 \times R_k$ and let $(x_0,u_0) \in D_n \times L_m^2$ be such that $T(x_0,u_0) = 0$. From the abstract Lagrange Multiplier Theorem (cf. [4] 9.3, Theorem 1) we obtain that if (x_0,u_0) is a local extremum on $N(T)$ of the functional F defined by (0.1) then there exists a couple $(y,\gamma) \in L_n^2 \times R_k$ such that each $(x,u) \in D_n \times L_m^2$ satisfies

$$[F'(x_0,u_0)](x,u) = \langle T(x,u),(y,\gamma) \rangle_{L_n^2 \times R_k} , \tag{2.6}$$

where $F'(x_0,u_0)$ stands for the Frechet derivative of F at the point (x_0,u_0) with respect to (x,u) ($F'(x_0,u_0) \in L(D_n \times L_m^2, R)$). Inserting the explicit form (0.1) of F into (2.6), applying the integration by parts formula and taking into account that

$$(x,u) \in X \to a^*x(0) + b^*x(\tau+) + \int_0^1 w^*(s)\,\dot{x}(s)\,ds + \int_0^1 v^*(s)\,u(s)\,ds \in R$$

is the zero functional on $D_n \times L_m^2$ iff $a = b = 0$, $w(s) = 0$ and $v(s) = 0$ a.e. on $[0,1]$ we obtain the following result.

THEOREM (Lagrange Multipliers). *Let* $R(T) = L_n^2 \times R_k$. *Then* $(x_0,u_0) \in D_n \times L_m^2$ *is a local extremum of* F *on* $N(T)$ *only if*

$$\dot{x}_0(t) - A(t)x_0(t) - B(t)u_0(t) = 0 \quad a.e.\ on \quad [0,1] , \tag{2.7}$$

$$Mx_0(0) + Nx_0(\tau+) + \int_0^1 K(s)\,\dot{x}_0(s)\,ds = 0 \tag{2.8}$$

and there exist $z \in D_n$ *and* $\gamma \in R_k$ *such that*

$$- \dot{z}^*(t) - z^*(t)A(t) + \gamma^*K(t)A(t) = \left(\frac{\partial h}{\partial x}(t,x_0(t),u_0(t))\right)^* \quad a.e.\ on\ [0,1]\ , \tag{2.9}$$

$$- z^*(0) + \gamma^*M = \left(\frac{\partial g_0}{\partial x}(x_0(0))\right)^* , \quad z^*(\tau-) = 0\ , \tag{2.10}$$

$$- z^*(\tau+) + \gamma^*N = \frac{\partial g_\tau}{\partial x}(x_0(\tau+)))^* , \quad z^*(1) = \left(\frac{\partial g_1}{\partial x}(x_0(1))\right)^* , \tag{2.11}$$

$$- z^*(t)B(t) + \gamma^*K(t)B(t) = \left(\frac{\partial h}{\partial u}(t,x_0(t),u_0(t))\right)^* , \quad a.e.\ on\ [0,1]\ . \tag{2.12}$$

REMARK. Related topics were treated e.g. in [2], [3], [5].

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