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GENERAL BOUNDARY VALUE PROBLEM FOR AN INTEGRODIFFERENTIAL SYSTEM AND ITS ADJOINT

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0. INTRODUCTION

In [17] we studied the boundary value problem

$$\dot{x} = A(t)\,x + \dot{K}(t)\,(M_c\,x(a) + N_c\,x(b)) + \dot{G}(t)\int_a^b [\mathrm{d}H(s)]\,x(s) + \dot{f}(t)\,, \tag{0,1}$$

$$M\,x(a) + N\,x(b) + \int_a^b [\mathrm{d}L_1(s)]\,x(s) = l_1\,, \tag{0,2}$$

$$\int_a^b [\mathrm{d}L_2(s)]\,x(s) = l_2\,, \tag{0,3}$$

where A is an $n \times n$-matrix function L-integrable on the compact interval $[a, b]$, f is a regular n-vector function of bounded variation on $[a, b]$; K, G, H, L_1 and L_2 are regular matrix functions of bounded variation on $[a, b]$; M, N and M_c, N_c are constant matrices of the type $m \times n$ and $(2n - m) \times n$, respectively, such that

$$\det \begin{pmatrix} M, & N \\ M_c, & N_c \end{pmatrix} \neq 0\,; \tag{0,4}$$

l_1 and l_2 are constant vectors.

The equation (0,1) was considered as a generalized integrodifferential equation, i.e., an n-vector function x is its solution iff for any $t \in [a, b]$

$$x(t) = x(a) + \int_a^t A(s)\,x(s)\,\mathrm{d}s + (K(t) - K(a))\,(M_c\,x(a) + N_c\,x(b)) + \\ + (G(t) - G(a))\int_a^b [\mathrm{d}H(s)]\,x(s) + f(t) - f(a)\,.$$

*) The last paragraph (§ 5) was added.

We derived the adjoint problem and Green's function for the problem (0,1), (0,2), (0,3) and, moreover, we established a relation between the number of linearly independent solutions to the given and the adjoint homogeneous problems. All these results could be easily transferred to the case that absolutely continuous solutions to the equation (0,1) are looked for.

In this paper we continue the investigation of such boundary value problems. The assumption (0,4) is omitted and the kernel of the integral operator in (0,1) is now general, not necessarily degenerate. Solutions are sought as functions absolutely continuous on $[a, b]$.

Paragraphs 1 and 2 have a preparatory character. In § 3 we treat the boundary value problem

$$\dot{x} = A(t)\,x + \int_a^b [\mathrm{d}_s G(t, s)]\, x(s) + f(t)\,, \tag{0,5}$$

$$\int_a^b [\mathrm{d}L(s)]\, x(s) = l\,, \tag{0,6}$$

where A and f are L-integrable on $[a, b]$, L is of bounded variation on $[a, b]$, l is a constant vector, G is defined and measurable on $[a, b] \times [a, b]$ and such that $\operatorname{var}_a^b G(t, \cdot) < \infty$ for any $t \in [a, b]$ and $\|G(t, a)\| + \operatorname{var}_a^b G(t, \cdot)$ is square integrable on $[a, b]$ (i.e. $G \in \mathscr{L}^2[\mathscr{BV}]$, see § 2). To this problem an adjoint ("in the sense of differential equations") is derived (Definition 3,1) in such a way that the usual Fredholm alternative holds. Furthermore a relation between the number of linearly independent solutions to the given and the adjoint homogeneous problems is established. Applying the general method of D. WEXLER [20] we show in the last paragraph (§ 5) that the adjoint boundary value problem derived in § 3 is equivalent to the true adjoint problem ("in the sense of functional analysis").

In § 4 we give some existence results for the weakly nonlinear boundary value problem

$$\dot{x} = A(t)\,x + \int_a^b [\mathrm{d}_s G(t, s)]\, x(s) + \varepsilon\, \Phi(\varepsilon)\,(x)\,(t)\,, \tag{0,7}$$

$$\int_a^b [\mathrm{d}L(s)]\, x(s) + \varepsilon\, \Lambda(\varepsilon)\,(x) = 0\,, \tag{0,8}$$

where $\varepsilon \geqq 0$ is a small parameter, $\Phi(\varepsilon)$ and $\Lambda(\varepsilon)$ for ε fixed are mappings of the space $\mathscr{AC}$ of n-vector functions absolutely continuous on $[a, b]$ into itself and of $\mathscr{AC}$ into the n-dimensional Euclidean space $\mathscr{R}_n$, respectively, subjected to some smoothness conditions. Similar boundary value problems were investigated by M. URABE [14],

[15], [16] and M. KWAPISZ [11]. The essential difference between their problems and ours is in the presence of the integral operator

$$\int_a^b [d_s G(t, s)]\, x(s)\,.$$

Moreover, in [14] and [15] only the less general multipoint boundary value problem was treated while, on the other hand, [11] and [16] deal only with the more simple noncritical case.

For other references to boundary value problems related to (0,5), (0,6) or (0,7), (0,8) see for instance [3], [9], [17] or [11], [18], respectively.

1. PRELIMINARIES

Let $-\infty < a < b < \infty$. The closed interval $a \leqq t \leqq b$ is denoted by $J = [a, b]$ and the open interval $a < t < b$ is denoted by (a, b). All quantities are considered as real.

Given an arbitrary $p \times q$-matrix $M = (M_{i,j})$ $(i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, q)$, rank (M) denotes its rank, M' its transpose and

$$\|M\| = \max_{i=1,2,\ldots,p} \sum_{j=1}^{q} |M_{i,j}|\,.$$

Considering p-vectors as $p \times 1$-matrices, it is for an arbitrary p-vector $x = (x_1, x_2, \ldots, x_p)'$

$$\|x\| = \max_{i=1,2,\ldots,p} |x_i|\,, \quad \|x'\| = \sum_{j=1}^{p} |x_j| \leqq p\|x\|\,.$$

The space of all p-vectors equipped with this norm is denoted by $\mathscr{R}_p$. The space of all row p-vectors (i.e. of $1 \times p$-matrices) is denoted by $\mathscr{R}_p^*$. Whenever a product of matrices occurs their types are assumed such that the multiplication is feasible. (This concerns such "products" as $F\, dG$, $F[dG]\, H$, too.) I_p denotes the identity $p \times p$-matrix and $O_{p,q}$ the zero $p \times q$-matrix. (Usually we write briefly I and O.)

Given an arbitrary matrix function F defined on J and an arbitrary subdivision $\sigma = \{a = t_0 < t_1 < \ldots < t_m = b\}$ of the interval J, let us denote

$$v(F; \sigma) = \sum_{j=1}^{m} \|F(t_j) - F(t_{j-1})\| \quad \text{and} \quad \operatorname{var}_a^b F = \sup_\sigma v(F; \sigma)\,.$$

A matrix function F is said to be of bounded variation on J if $\operatorname{var}_a^b F < \infty$. The space of all $p \times q$-matrix functions of bounded variation on J is denoted by $\mathscr{BV}_{p,q}$, or briefly $\mathscr{BV}$, if no misunderstanding can arise. Any $\mathscr{BV}_{p,q}$ is a Banach space (B-space) with the norm $\|F\|_{\mathscr{BV}} = \|F(a)\| + \operatorname{var}_a^b F$.

$\mathcal{AC}$ denotes the B-space of all column n-vector functions absolutely continuous on J equipped with the norm

$$\|f\|_{\mathcal{AC}} = \|f\|_{\mathcal{BV}} = \|f(a)\| + \operatorname{var}_a^b f = \|f(a)\| + \int_a^b \|\dot{f}(t)\| \, \mathrm{d}t .$$

Symbols $\int_a^b F(t)\,\mathrm{d}t$ and $\int_a^b F(t)\,\mathrm{d}G(t)$ stand for the Lebesgue and the σ-Young integral, respectively. For the definition of the σ-Young integral and its basic properties (relation to the Riemann-Stieltjes integral, substitution theorem, integration by parts formula, indefinite integrals etc.) see T. H. Hildebrandt [7] II 19,3. (The list of properties of the σ-Young integral is given in [17] as well.) Let us recall here only Dirichlet's formula in a special form.

Lemma 1,1. *Let $F, G, X, Y \in \mathcal{BV}$, G being continuous on $[a, b]$. Then*

$$\int_a^b [\mathrm{d}F(t)] \left(X(t) \int_a^t Y(s)\,\mathrm{d}G(s) \right) = \int_a^b \left(\left(\int_s^b [\mathrm{d}F(t)]\, X(t) \right) Y(s)\,\mathrm{d}G(s) \right).$$

Given an arbitrary matrix function $G(t, s)$ defined on $J \times J$ and an arbitrary net subdivision $\sigma' \times \sigma'' = \{a = t_0 < t_1 < \ldots < t_{m'} = b, a = s_0 < s_1 < \ldots < s_{m''} = b\}$ of the interval $J \times J$, let us put

$$w(G; \sigma' \times \sigma'') = \sum_{j=1}^{m'} \sum_{k=1}^{m''} \|G(t_j, s_k) - G(t_{j-1}, s_k) - G(t_j, s_{k-1}) + G(t_{j-1}, s_{k-1})\|$$

and $\operatorname{Var}_{J \times J} G = \sup_{\sigma' \times \sigma''} w(G; \sigma' \times \sigma'')$ (the least upper bound is taken over all possible net subdivisions $\sigma' \times \sigma''$ of $J \times J$). A matrix function $G(t, s)$ is said to be of strongly bounded variation on $J \times J$ if

$$\operatorname{Var}_{J \times J} G + \operatorname{var}_a^b G(a, \cdot) + \operatorname{var}_a^b G(\cdot, a) < \infty .$$

The space of all $n \times n$-matrix functions of strongly bounded variation on $J \times J$ is denoted by $\mathcal{SBV}$. If $G \in \mathcal{SBV}$ then $\operatorname{var}_a^b G(t, .) + \operatorname{var}_a^b G(\cdot, s) \leqq M < \infty$ for any $t, s \in J$ and G is bounded on $J \times J$. (See [7] III 4,8.)

Given a natural number k, the space of all $p \times q$-matrix functions F defined on J and such that

$$\int_a^b \|F(t)\|^k \,\mathrm{d}t < \infty$$

is denoted by $\mathcal{L}_{p,q}^k$ (or briefly $\mathcal{L}^k$). Any $\mathcal{L}_{p,q}^k$ is a B-space with the norm $\|F\|_k = (\int_a^b \|F(t)\|^k \,\mathrm{d}t)^{1/k}$.

The space of all $n \times n$-matrix functions $K(t, s)$ defined and measurable on $J \times J$ and such that

$$\iint_{J \times J} \|K(\tau, \sigma)\|^2 \, d\tau \, d\sigma < \infty , \quad \int_a^b \|K(t, \sigma)\|^2 \, d\sigma < \infty , \quad \int_a^b \|K(\tau, s)\|^2 \, d\tau < \infty$$

$$\text{for any} \quad t, s \in J$$

is denoted by $\mathscr{L}_2$. For $K \in \mathscr{L}_2$ we denote $|||K||| = (\iint_{J \times J} \|K(\tau, \sigma)\|^2 \, d\tau \, d\sigma)^{1/2}$.

In the following, besides some well-known theorems as those of Fubini, Lebesgue etc., we make also use of the Tonelli-Hobson theorem.

Lemma 1,2. (Tonelli-Hobson). *If $K(t, s)$ is defined and measurable on $J \times J$ and if any one of the three integrals*

$$\iint_{J \times J} \|K(t, s)\| \, dt \, ds , \quad \int_a^b \left(\int_a^b \|K(t, s)\| \, dt \right) ds , \quad \int_a^b \left(\int_a^b \|K(t, s)\| \, ds \right) dt$$

exists, then the integrals

$$\iint_{J \times J} K(t, s) \, dt \, ds , \quad \int_a^b \left(\int_a^b K(t, s) \, dt \right) ds , \quad \int_a^b \left(\int_a^b K(t, s) \, ds \right) dt$$

all exist and are equal to one another.

(For the proof see [1], pp. 63–64.)

2. SPACE $\mathscr{L}^2[\mathscr{BV}]$

The space $\mathscr{L}^2[\mathscr{BV}]$ is formed by all $n \times n$-matrix functions $G(t, s)$ defined and measurable on $J \times J$ and such that

$$\operatorname{var}_a^b G(t, \cdot) < \infty \quad \textit{for any} \quad t \in J$$

and

$$\|G(t, \cdot)\|_{\mathscr{BV}} = (\|G(t, a)\| + \operatorname{var}_a^b G(t, \cdot)) \in \mathscr{L}^2 .$$

The properties of functions from $\mathscr{L}^2[\mathscr{BV}]$ which are needed in the sequel are stated in the following lemmas. (The assertion of the first one – Lemma 2,1 – is evident.)

Lemma 2,1. a) *If $G \in \mathscr{L}^2[\mathscr{BV}]$, then $G(\cdot, s) \in \mathscr{L}^2$ for any $s \in J$.* b) $\mathscr{S}\mathscr{BV} \subset \mathscr{L}^2[\mathscr{BV}]$.

Lemma 2,2. *If $H \in \mathscr{L}_2$, then*

$$G(t, s) = \int_a^s H(t, \sigma) \, d\sigma \in \mathscr{L}^2[\mathscr{BV}] .$$

Proof. For an arbitrary subdivision $\sigma = \{a = s_0 < s_1 < \ldots < s_m = b\}$ of the interval J and for any $t \in J$,

$$(v(G(t,\cdot);\sigma))^2 \leqq \left(\sum_{j=1}^{m}\int_{s_{j-1}}^{s_j}\|H(t,\tau)\|\,\mathrm{d}\tau\right)^2 = \left(\int_a^b \|H(t,\tau)\|\,\mathrm{d}\tau\right)^2 \leqq$$

$$\leqq (b-a)\int_a^b \|H(t,\tau)\|^2\,\mathrm{d}\tau < \infty\,.$$

Hence for any $t \in J$

$$\|G(t,\cdot)\|^2_{\mathscr{BV}} = (\operatorname{var}_a^b G(t,\cdot))^2 \leqq (b-a)\int_a^b \|H(t,\tau)\|^2\,\mathrm{d}\tau \in \mathscr{L}^1\,,$$

wherefrom the assertion of the lemma immediately follows.

Lemma 2,3. *Let $X \in \mathscr{BV}$ and let $G \in \mathscr{L}^2[\mathscr{BV}]$. Then*

$$F(t) = \int_a^b [\mathrm{d}_s G(t,s)]\,X(s) \in \mathscr{L}^2\,.$$

Proof follows readily from the estimate

$$\|F(t)\|^2 \leqq (\sup_{\tau\in J}\|X(\tau)\|)^2\,(\operatorname{var}_a^b G(t,\cdot))^2 \quad \text{on} \quad J\,.$$

Lemma 2,4. *Let $X, Y \in \mathscr{BV}$ and let $G \in \mathscr{L}^2[\mathscr{BV}]$. Then*

$$H(t,s) = \int_s^b [\mathrm{d}_\sigma G(t,\sigma)]\,X(\sigma)\,Y(s) \in \mathscr{L}_2\,.$$

Proof. Since for any $t, s \in J$

$$\|H(t,s)\|^2 \leqq (\sup_{\tau\in J}\|X(\tau)\|)^2\,(\sup_{\tau\in J}\|Y(\tau)\|)^2\,(\operatorname{var}_a^b G(t,\cdot))^2\,,$$

the right-hand side being L-integrable on J,

$$\int_a^b\left(\int_a^b \|H(t,s)\|^2\,\mathrm{d}s\right)\mathrm{d}t < \infty\,.$$

Hence our assertion follows by the Tonelli-Hobson theorem (Lemma 1,2).

Lemma 2,5. *Let $X \in \mathscr{BV}$, $Y \in \mathscr{L}^2$ and $G \in \mathscr{L}^2[\mathscr{BV}]$. Then*

$$F(t) = \int_a^b Y(s)\left(\int_t^b [\mathrm{d}_\sigma G(s,\sigma)]\,X(\sigma)\right)\mathrm{d}s$$

is bounded on J.

Proof. By Lemma 2,4 $F(t)$ is defined for all $t \in J$. Moreover for any $t \in J$

$$\|F(t)\| \leqq \left(\int_a^b \|Y(s)\|^2 \,\mathrm{d}s\right)^{1/2} \left(\int_a^b \left\|\int_t^b [\mathrm{d}_\sigma G(s,\sigma)]\, X(\sigma)\right\|^2 \mathrm{d}s\right)^{1/2} \leqq$$

$$\leqq \left(\int_a^b \|Y(s)\|^2 \,\mathrm{d}s\right)^{1/2} (\sup_{s\in J} \|X(s)\|) \left(\int_a^b (\mathrm{var}_a^b\, G(s,\cdot))^2 \,\mathrm{d}s\right)^{1/2} < \infty\,.$$

Lemma 2,6. *Let $Y \in \mathscr{L}^2$ and $G \in \mathscr{L}^2[\mathscr{BV}]$. Then*

$$F(t) = \int_a^b Y(s)\, G(s,t)\,\mathrm{d}s \in \mathscr{BV}\,.$$

Proof. For any subdivision $\sigma = \{a = t_0 < t_1 < \ldots < t_m = b\}$ of the interval J,

$$v(F;\sigma) \leqq \int_a^b \|Y(s)\| \left(\sum_{j=1}^m \|G(s,t_j) - G(s,t_{j-1})\|\right) \mathrm{d}s \leqq$$

$$\leqq \left(\int_a^b \|Y(s)\|^2 \,\mathrm{d}s\right)^{1/2} \left(\int_a^b (\mathrm{var}_a^b\, G(s,\cdot))^2 \,\mathrm{d}s\right)^{1/2} < \infty\,.$$

Lemma 2,7. *Let X be absolutely continuous on J, let $Y \in \mathscr{L}^2$ and $G \in \mathscr{L}^2[\mathscr{BV}]$. Then*

$$\int_a^b \left[\mathrm{d}_t \int_a^b Y(s)\, G(s,t)\,\mathrm{d}s\right] X(t) = \int_a^b Y(t) \left(\int_a^b [\mathrm{d}_s G(t,s)]\, X(s)\right) \mathrm{d}t\,.$$

Proof. Both the integrals exist by lemmas 2,3 and 2,6. By double use of integration by parts and of Fubini's theorem we obtain succesively

$$\int_a^b \left[\mathrm{d}_t \int_a^b Y(s)\, G(s,t)\,\mathrm{d}s\right] X(t) =$$

$$= \int_a^b Y(s)\, G(s,b)\, X(b)\,\mathrm{d}s - \int_a^b Y(s)\, G(s,a)\, X(a)\,\mathrm{d}s - \int_a^b \left(\int_a^b Y(s)\, G(s,t)\,\mathrm{d}s\right) \dot{X}(t)\,\mathrm{d}t =$$

$$= \int_a^b Y(s)\, G(s,b)\, X(b)\,\mathrm{d}s - \int_a^b Y(s)\, G(s,a)\, X(a)\,\mathrm{d}s - \int_a^b Y(t) \left(\int_a^b G(t,s)\, \dot{X}(s)\,\mathrm{d}s\right) \mathrm{d}t =$$

$$= \int_a^b Y(t) \left(\int_a^b [\mathrm{d}_s G(t,s)]\, X(s)\right) \mathrm{d}t\,.$$

(This is a special case of the unsymmetric Fubini theorem due to R. H. Cameron and W. T. Martin [2].)

In this paragraph we investigate the existence of a solution to the boundary value problem $(\mathscr{P})$ given by

$$(3,1) \qquad \dot{x} = A(t)\,x + C(t)\,x(a) + D(t)\,x(b) + \int_a^b [\mathrm{d}_s G(t,s)]\,x(s) + f(t)\,,$$

$$(3,2) \qquad M\,x(a) + N\,x(b) + \int_a^b [\mathrm{d}L(s)]\,x(s) = l\,,$$

where $A \in \mathscr{L}^1_{n,n}$, $f \in \mathscr{L}^1 = \mathscr{L}^1_{n,1}$, C and $D \in \mathscr{L}^2_{n,n}$, $G \in \mathscr{L}^2[\mathscr{BV}]$, $L \in \mathscr{BV}_{m,n}$, M and N are constant $m \times n$-matrices and $l \in \mathscr{R}_m$, L and $G(t,\cdot)$ are for any $t \in J$ continuous at a from the right and at b from the left.

Hereafter we write $\mathscr{L}^k$ instead of $\mathscr{L}^k_{n,1}$.

The assumption of the continuity of $G(t,\cdot)$ and L at a and at b does not mean any loss of generality. In fact, let us suppose $G \in \mathscr{L}^2[\mathscr{BV}]$. Then $G(t, a+\tau) \in \mathscr{L}^2_{n,n}$ for any $\tau \in (0, b-a)$. Since $\|G(t, a+\tau)\|^2 \leqq \|G(t,\cdot)\|^2_{\mathscr{BV}} \in \mathscr{L}^2_{n,n}$ for any $t \in J$, we have by Lebesgue's theorem $C_0(t) = C(t) + G(t, a+) - G(t,a) \in \mathscr{L}^2_{n,n}$. Analogously $D_0(t) = D(t) + G(t,b) - G(t, b-) \in \mathscr{L}^2_{n,n}$. Defining $G_0(t,s) = G(t,s)$ for $t \in J$ and $a < s < b$, $G_0(t,a) = G(t, a+)$ and $G_0(t,b) = G(t, b-)$ for $t \in J$, we get

$$C(t)\,x(a) + D(t)\,x(b) + \int_a^b [\mathrm{d}_s G(t,s)]\,x(s) =$$

$$= C_0(t)\,x(a) + D_0(t)\,x(b) + \int_a^b [\mathrm{d}_s G_0(t,s)]\,x(s)\,,$$

with $G_0 \in \mathscr{L}^2[\mathscr{BV}]$ and $G_0(t,\cdot)$ continuous at a and at b. On the other hand, it is clear that our problem $(\mathscr{P})$ can be always written as

$$\dot{x} = A(t)\,x + \int_a^b [\mathrm{d}_s G_1(t,s)]\,x(s) + f(t)\,, \quad \int_a^b [\mathrm{d}L_1(s)]\,x(s) = l\,,$$

with some $L_1 \in \mathscr{BV}_{m,n}$ and $G_1 \in \mathscr{L}^2[\mathscr{BV}]$, not necessarily continuous at a and at b.

The homogeneous boundary value problem corresponding to $(\mathscr{P})$ (i.e. the problem $(\mathscr{P})$ with $f = 0$ a.e. on J and $l = 0$) is denoted by $(\mathscr{P}_0)$.

By a fundamental theorem there exists an $n \times n$-matrix function X absolutely continuous on J and such that for any $t, s \in J$

$$X(t) = X(s) + \int_s^t A(\tau)\,X(\tau)\,\mathrm{d}\tau\,, \quad X(a) = I \quad \text{and} \quad \det X(t) \neq 0\,.$$

Further

$$X^{-1}(t) = X^{-1}(s) - \int_s^t X^{-1}(\tau) A(\tau) \, d\tau$$

and hence

$$(3,3) \quad X(t) X^{-1}(s) = I + X(t) \int_s^t X^{-1}(\tau) A(\tau) \, d\tau = I + \left(\int_s^t A(\tau) X(\tau) \, d\tau \right) X^{-1}(s)$$

$$\text{for} \quad t, s \in J .$$

Given an arbitrary $c \in \mathscr{R}_n$ and $g \in \mathscr{L}^1$, there exists a unique solution x to the equation $\dot{x} = A(t) x + g(t)$ on J such that $x(a) = c$. This solution is given by

$$x(t) = X(t) c + X(t) \int_a^t X^{-1}(s) g(s) \, ds .$$

Therefore x is a solution to $(\mathscr{P})$ iff

$$(3,4) \qquad x(t) = X(t) c + X(t) \int_a^t X^{-1}(s) h(s) \, ds + X(t) \int_a^t X^{-1}(s) f(s) \, ds ,$$

where $c \in R_n$ and $h \in \mathscr{L}^2$ satisfy the linear system of "integro-algebraic" equations

$$(3,5) \qquad -h(t) + \left\{ C(t) + D(t) X(b) + \int_a^b [d_s G(t, s)] X(s) \right\} c +$$

$$+ \int_a^b \left\{ D(t) X(b) + \int_s^b [d_\sigma G(t, \sigma)] X(\sigma) \right\} X^{-1}(s) h(s) \, ds =$$

$$= - \int_a^b \left\{ D(t) X(b) + \int_s^b [d_\sigma G(t, \sigma)] X(\sigma) \right\} X^{-1}(s) f(s) \, ds ,$$

$$\left\{ M + N X(b) + \int_a^b [dL(s)] X(s) \right\} c +$$

$$+ \int_a^b \left\{ N X(b) + \int_s^b [dL(\sigma)] X(\sigma) \right\} X^{-1}(s) h(s) \, ds =$$

$$= l - \int_a^b \left\{ N X(b) + \int_s^b [dL(\sigma)] X(\sigma) \right\} X^{-1}(s) f(s) \, ds .$$

The system (3,5) is obtained by inserting (3,4) into (3,2) and into the condition

$$h(t) = C(t) x(a) + D(t) x(b) + \int_a^b [d_s G(t, s)] x(s) .$$

(By Lemmas 1,1, 2,3 and 2,4

$$(3,6) \qquad u(t) = \int_a^b [\mathrm{d}_s G(t, s)]\, X(s) \left(\int_a^s X^{-1}(\sigma) f(\sigma)\, \mathrm{d}\sigma \right) =$$

$$= \int_a^b \left(\int_s^b [\mathrm{d}_\sigma G(t, \sigma)]\, X(\sigma) \right) X^{-1}(s) f(s)\, \mathrm{d}s \in \mathscr{L}^2 ,$$

$$v = \int_a^b [\mathrm{d}L(s)]\, X(s) \left(\int_a^s X^{-1}(\sigma) f(\sigma)\, \mathrm{d}\sigma \right) =$$

$$= \int_a^b \left(\int_s^b [\mathrm{d}L(\sigma)]\, X(\sigma) \right) X^{-1}(s) f(s)\, \mathrm{d}s \in \mathscr{R}_m ,$$

$$H_1(t) = C(t) + D(t)\, X(b) + \int_a^b [\mathrm{d}_s G(t, s)]\, X(s) \in \mathscr{L}^2_{n,n} ,$$

$$H_2(t) = \left\{ N\, X(b) + \int_t^b [\mathrm{d}L(s)]\, X(s) \right\} X^{-1}(t) \in \mathscr{L}^2_{m,n} ,$$

$$K(t, s) = \left\{ D(t)\, X(b) + \int_s^b [\mathrm{d}_\sigma G(t, \sigma)]\, X(\sigma) \right\} X^{-1}(s) \in \mathscr{L}_2 ,$$

$$C = \left\{ M + N\, X(b) + \int_a^b [\mathrm{d}L(s)]\, X(s) \right\} \quad \text{is an } m \times n\text{-matrix.})$$

Let us note that if $x \in \mathscr{AC}$, then by Lemma 2,3 $h \in \mathscr{L}^2$ and conversely if $h \in \mathscr{L}^2$, $c \in \mathscr{R}_n$ and x is given by (3,4), then $x \in \mathscr{AC}$.

Lemma 3,1. *Let C be a constant $p \times q$-matrix, let $u \in \mathscr{L}^2$, $H_1 \in \mathscr{L}^2_{n,q}$, $H_2 \in \mathscr{L}^2_{p,n}$, $K \in \mathscr{L}_2$ and $v \in \mathscr{R}_p$. Then the system*

$$(3,7) \qquad -h(t) + H_1(t)\, c + \int_a^b K(t, s)\, h(s)\, \mathrm{d}s = u(t) ,$$

$$Cc + \int_a^b H_2(s)\, h(s)\, \mathrm{d}s = v$$

for a couple $(h, c) \in \mathscr{L}^2 \times \mathscr{R}_q$ has a solution iff

$$\int_a^b \chi'(s)\, u(s)\, \mathrm{d}s + \gamma' v = 0$$

holds for any couple $(\chi, \gamma) \in \mathscr{L}^2 \times \mathscr{R}_p$ *satisfying the system*

$$(3,8) \qquad -\chi'(t) + \gamma' H_2(t) + \int_a^b \chi'(s) K(s, t) \, ds = 0 ,$$

$$\gamma' C + \int_a^b \chi'(s) H_1(s) \, ds = 0 .$$

Proof. Necessity. Let (h, c) be a solution of (3,7) and let (χ, γ) be a solution of (3,8). Then

$$\int_a^b \chi'(s) u(s) \, ds + \gamma' v = \int_a^b \left[-\chi'(t) + \gamma' H_2(t) + \int_a^b \chi'(s) K(s, t) \, ds \right] h(t) \, dt + $$

$$+ \left[\gamma' C + \int_a^b \chi'(s) H_1(s) \, ds \right] c = 0 .$$

Sufficiency will be proved similarly as the analogous theorem for Fredholm integral equations. (See e.g. [12], pp. 41–44 or [6] in a similar situation.)

There exist (cf. Remark 3,1) a natural number n', $K_0 \in \mathscr{L}_2$, $K_1 \in \mathscr{L}^2_{n,n'}$ and $K_2 \in \mathscr{L}^2_{n',n}$ such that

$$(3,9) \qquad K(t, s) \equiv K_0(t, s) + K_1(t) K_2(s) \text{ on } J \times J , \quad |||K_0||| < 1 .$$

(Hence 1 is not an eigenvalue of K_0.) Consequently, the equation $(3,7)_1$ is equivalent to

$$h(t) - \int_a^b K_0(t, s) h(s) \, ds = -u(t) + H_1(t) c + K_1(t) \int_a^b K_2(s) h(s) \, ds .$$

Accordingly

$$h(t) = \left[-u(t) + H_1(t) c + K_1(t) \int_a^b K_2(s) h(s) \, ds \right] +$$

$$+ \int_a^b \Gamma(t, s) \left[-u(s) + H_1(s) c + K_1(s) \int_a^b K_2(\sigma) h(\sigma) \, d\sigma \right] ds ,$$

where Γ is the resolvent kernel of K_0. Denoting for $t \in J$

$$(3,10) \qquad \tilde{H}_1(t) = H_1(t) + \int_a^b \Gamma(t, s) H_1(s) \, ds , \quad \tilde{K}_1(t) = K_1(t) + \int_a^b \Gamma(t, s) K_1(s) \, ds ,$$

$$\tilde{u}(t) = u(t) + \int_a^b \Gamma(t, s) u(s) \, ds ,$$

we obtain that (3,7) is equivalent to

$$(3,11)\qquad -h(t) + \tilde{H}_1(t)\, c + \tilde{K}_1(t) \int_a^b K_2(s)\, h(s)\, \mathrm{d}s = \tilde{u}(t)\,,$$

$$Cc + \int_a^b H_2(s)\, h(s)\, \mathrm{d}s = v\,.$$

Let us denote

$$(3,12)\qquad B_{1,1} = \int_a^b K_2(s)\, \tilde{H}_1(s)\, \mathrm{d}s\,, \qquad B_{1,2} = \int_a^b K_2(s)\, \tilde{K}_1(s)\, \mathrm{d}s - I\,,$$

$$B_{2,1} = \int_a^b H_2(s)\, \tilde{H}_1(s)\, \mathrm{d}s + C\,, \qquad B_{2,2} = \int_a^b H_2(s)\, \tilde{K}_1(s)\, \mathrm{d}s\,,$$

$$w_1 = \int_a^b K_2(s)\, \tilde{u}(s)\, \mathrm{d}s\,, \qquad w_2 = v + \int_a^b H_2(s)\, \tilde{u}(s)\, \mathrm{d}s\,,$$

$$d = \int_a^b K_2(s)\, h(s)\, \mathrm{d}s\,.$$

The equation $(3,11)_1$ now becomes

$$(3,13)\qquad h(t) = -\tilde{u}(t) + \tilde{H}_1(t)\, c + \tilde{K}_1(t)\, d\,.$$

Multiplying (3,13) by K_2 from the left and integrating over J, we get

$$B_{1,1}c + B_{1,2}d = w_1\,.$$

Further we have by $(3,11)_2$ and (3,13)

$$B_{2,1}c + B_{2,2}d = w_2\,.$$

Therefore (3,7) is equivalent to

$$(3,14)\qquad Bb = w\,,$$

where

$$B = \begin{pmatrix} B_{1,1}, & B_{1,2} \\ B_{2,1}, & B_{2,2} \end{pmatrix}, \quad b = \begin{pmatrix} c \\ d \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

(If $b = (c', d')'$ is a solution to (3,14), $\tilde{u}$, $\tilde{H}_1$ and $\tilde{K}_1$ are given by (3,10) and h is given by (3,13), then (h, c) is a solution to (3,7). If (h, c) is a solution to (3,7) and d is given by (3,12), then $b = (c', d')'$ is a solution to (3,14).)

Analogously, (3,8) is equivalent to

$$(3,15)\qquad \beta' B = 0$$

(If $\beta' = (\gamma', \delta')$ is a solution to (3,15) and χ is given by

$$\chi'(t) = \gamma'\left[H_2(t) + \int_a^b H_2(s)\,\Gamma(s, t)\,\mathrm{d}s\right] + \delta'\left[K_2(t) + \int_a^b K_2(s)\,\Gamma(s, t)\,\mathrm{d}s\right],$$

then (χ, γ) is a solution to (3,8). If (χ, γ) is a solution to (3,8) and

$$\delta' = \int_a^b \chi'(s)\,K_1(s)\,\mathrm{d}s\,,$$

then $\beta' = (\gamma', \delta')$ is a solution to (3,15).)

Let $\beta' = (\gamma', \delta')$ be an arbitrary solution of (3,15) and let (χ, γ) be the corresponding solution of (3,8). Then by (3,10) and (3,12)

$$\begin{aligned}
\delta' w_1 + \gamma' w_2 &= \gamma' v + \int_a^b [\gamma' H_2(s) + \delta' K_2(s)]\,\tilde{u}(s)\,\mathrm{d}s = \\
&= \gamma' v + \int_a^b [\gamma' H_2(s) + \delta' K_2(s)]\left[u(s) + \int_a^b \Gamma(s, \sigma)\,u(\sigma)\,\mathrm{d}\sigma\right]\mathrm{d}s = \\
&= \gamma' v + \int_a^b \left\{\gamma'\left[H_2(s) + \int_a^b H_2(\sigma)\,\Gamma(\sigma, s)\,\mathrm{d}\sigma\right] + \right. \\
&\quad \left. + \delta'\left[K_2(s) + \int_a^b K_2(\sigma)\,\Gamma(\sigma, s)\,\mathrm{d}\sigma\right]\right\} u(s)\,\mathrm{d}s = \gamma' v + \int_a^b \chi'(s)\,u(s)\,\mathrm{d}s\,.
\end{aligned}$$

From this and from the relation between the systems (3,7) and (3,14) the assertion of the lemma immediately follows.

Remark 3,1. The decomposition of a square integrable or continuous kernel into small and degenerate parts is well known for scalar kernels. Here we show that the matrix decomposition (3,9) is true. For the sake of brevity let us suppose $n = 2$. The extension to the general case is trivial. Let

$$K(t, s) = P(t, s) + Q(t, s)\,, \quad Q(t, s) = \left(\sum_{k=1}^{m_{ij}} A_{i,j,k}(t)\,B_{i,j,k}(s)\right)_{i,j=1,2}\,.$$

Let us denote $n' = m_{1,1} + m_{1,2} + m_{2,1} + m_{2,2}$. Let $K_1(t)$ be the $2 \times n'$-matrix function with the rows

$$(A_{1,1,1}(t), \ldots, A_{1,1,m_{1,1}}(t), A_{1,2,1}(t), \ldots, A_{1,2,m_{1,2}}(t), \underbrace{0, \ldots, 0}_{m_{2,1}+m_{2,2}})$$

and

$$(\underbrace{0, \ldots, 0}_{m_{1,1}+m_{1,2}}, A_{2,1,1}(t), \ldots, A_{2,1,m_{2,1}}(t), A_{2,2,1}(t), \ldots, A_{2,2,m_{2,2}}(t))\,.$$

Let $K_2(s)$ be the $n' \times 2$-matrix function with the columns

$$(B_{1,1,1}(s), \ldots, B_{1,1,m_{1,1}}(s), \underbrace{0, \ldots, 0}_{m_{1,2}}, B_{2,1,1}(s), \ldots, B_{2,1,m_{2,1}}(s), \underbrace{0, \ldots, 0}_{m_{2,2}})'$$

and

$$(\underbrace{0, \ldots, 0}_{m_{1,1}}, B_{1,2,1}(s), \ldots, B_{1,2,m_{1,2}}(s), \underbrace{0, \ldots, 0}_{m_{2,1}}, B_{2,2,1}(s), \ldots, B_{2,2,m_{2,2}}s))' .$$

It is easy to verify that $K_1(t)\,K_2(s) = Q(t, s)$.

Definition 3,1. The problem $(\mathscr{P}^*)$ to find an n-vector function y of bounded variation on J and an m-vector γ such that for any $t \in J$

(3,16)
$$y'(t) = y'(a) - \int_a^t y'(s)\,A(s)\,\mathrm{d}s - \gamma'(L(t) - L(a)) - \int_a^b y'(s)\,(G(s, t) - G(s, a))\,\mathrm{d}s$$

and

$$(3,17)\quad y'(a) + \gamma' M + \int_a^b y'(s)\,C(s)\,\mathrm{d}s = 0\,, \quad y'(b) - \gamma' N - \int_a^b y'(s)\,D(s)\,\mathrm{d}s = 0\,,$$

is called *the boundary value problem adjoint to* $(\mathscr{P})$.

Remark 3,2. The equation (3,16) is a generalized differential equation in the sense of J. Kurzweil [10]. A boundary value problem of the type $(\mathscr{P})$ for such equations is treated in [13].

Remark 3,3. It is easy to see that any function $y \in \mathscr{L}^2$ fulfilling (3,16) on J has a bounded variation on J and is continuous at a from the right and at b from the left.

The following theorem shows that Definition 3,1 is reasonable

Theorem 3,1. *The boundary value problem* $(\mathscr{P})$ *has a solution if and only if*

$$(3,18)\qquad \int_a^b y'(s)\,f(s)\,\mathrm{d}s = \gamma' l$$

for any solution (y, γ) *of the adjoint boundary value problem* $(\mathscr{P}^*)$.

Proof. It suffices to show that (3,5) has a solution iff (3,18) holds. By lemmas 2,3, 2,4 and 3,1, (3,5) has a solution iff

(3,19)
$$\int_a^b \Big\{\gamma' N\,X(b) + \gamma' \int_t^b [\mathrm{d}L(s)]\,X(s) + \\ + \int_a^b \chi'(s)\Big(D(s)\,X(b) + \int_t^b [\mathrm{d}_\sigma G(s, \sigma)]\,X(\sigma)\Big)\,\mathrm{d}s\Big\}\,X^{-1}(t)\,f(t)\,\mathrm{d}t = \gamma' l$$

for any solution (χ, γ) of

$$(3,20)\qquad -\chi'(t) + \gamma' \left\{N\, X(b) + \int_t^b [\mathrm{d}L(s)]\, X(s)\right\} X^{-1}(t) + \\ + \int_a^b \chi'(s) \left\{D(s)\, X(b) + \int_t^b [\mathrm{d}_\sigma G(s, \sigma)]\, X(\sigma)\right\} X^{-1}(t)\, \mathrm{d}s = 0\,,$$

$$\gamma' \left\{M + N\, X(b) + \int_a^b [\mathrm{d}L(s)]\, X(s)\right\} + \\ + \int_a^b \chi'(s) \left\{C(s) + D(s)\, X(b) + \int_a^b [\mathrm{d}_\sigma G(s, \sigma)]\, X(\sigma)\right\} \mathrm{d}s = 0\,.$$

According to $(3,20)_1$, (3,19) means

$$(3,19')\qquad \int_a^b \chi'(t)\, f(t)\, \mathrm{d}t = \gamma' l\,.$$

By Lemma 2,5 the function

$$\int_a^b \varkappa'(s) \left(\int_t^b [\mathrm{d}_\sigma G(s, \sigma)]\, X(\sigma)\right) \mathrm{d}s$$

is bounded on J for any $\varkappa \in \mathscr{L}^2$. Hence

$$(3,21)\qquad \chi'(t) = \gamma' N\, X(b)\, X^{-1}(t) + \gamma' \int_t^b [\mathrm{d}L(s)]\, X(s)\, X^{-1}(t) + \\ + \int_a^b \chi'(s) \left\{D(s)\, X(b) + \int_t^b [\mathrm{d}_\sigma G(s, \sigma)]\, X(\sigma)\right\} X^{-1}(t)\, \mathrm{d}s$$

is bounded on J as well. Let us insert this expression for χ' into

$$\int_t^b \chi'(s)\, A(s)\, \mathrm{d}s\,.$$

Then Lemma 1,1 and Fubini's theorem imply

$$\int_t^b \chi'(s)\, A(s)\, \mathrm{d}s = \gamma' N\, X(b) \int_t^b X^{-1}(s)\, A(s)\, \mathrm{d}s + \\ + \gamma' \int_t^b [\mathrm{d}L(s)]\, X(s) \left(\int_t^s X^{-1}(\sigma)\, A(\sigma)\, \mathrm{d}\sigma\right) + \\ + \int_a^b \chi'(s)\, D(s)\, X(b) \left(\int_t^b X^{-1}(\sigma)\, A(\sigma)\, \mathrm{d}\sigma\right) \mathrm{d}s + \\ + \int_a^b \chi'(s) \left(\int_t^b \left(\int_\sigma^b [\mathrm{d}_\varrho G(s, \varrho)]\, X(\varrho)\right) X^{-1}(\sigma)\, A(\sigma)\, \mathrm{d}\sigma\right) \mathrm{d}s\,.$$

Taking into account (3,3) and the relation

$$\int_t^b \left(\left(\int_\sigma^b [\mathrm{d}_\varrho G(s, \varrho)] \, X(\varrho) \right) X^{-1}(\sigma) \, A(\sigma) \, \mathrm{d}\sigma = \int_t^b [\mathrm{d}_\sigma G(s, \sigma)] \left(X(\sigma) \int_t^\sigma X^{-1}(\varrho) \, A(\varrho) \, \mathrm{d}\varrho \right) \right.$$

which follows from Lemma 1,1,

$$\int_t^b \chi'(s) \, A(s) \, \mathrm{d}s = \gamma' N \, X(b) \, X^{-1}(t) - \gamma' N + \gamma' \int_t^b [\mathrm{d}L(s)] \, X(s) \, X^{-1}(t) -$$

$$- \gamma'(L(b) - L(t)) + \int_a^b \chi'(s) \, D(s) \, X(b) \, X^{-1}(t) \, \mathrm{d}s - \int_a^b \chi'(s) \, D(s) \, \mathrm{d}s +$$

$$+ \int_a^b \chi'(s) \left(\int_a^b [\mathrm{d}_\sigma G(s, \sigma)] \, X(\sigma) \right) X^{-1}(t) \, \mathrm{d}s - \int_a^b \chi'(s) \left(\int_t^b [\mathrm{d}_\sigma G(s, \sigma)] \right) \mathrm{d}s =$$

$$= \chi'(t) - \chi'(b) - \gamma'(L(b) - L(t)) - \int_a^b \chi'(s) \, (G(s, b) - G(s, t)) \, \mathrm{d}s \, .$$

This implies that if (χ, γ) is a solution to (3,20), then (χ, γ) is a solution to (3,16) as well. Moreover

$$\chi'(a) = - \gamma' M - \int_a^b \chi'(s) \, C(s) \, \mathrm{d}s \, , \quad \chi'(b) = \gamma' N + \int_a^b \chi'(s) \, D(s) \, \mathrm{d}s \, .$$

Therefore the couple (χ, γ) satisfies (3,17). The sufficiency of the condition (3,18) for the existence of a solution to $(\mathscr{P})$ immediately follows.

On the other hand, x and (y, γ) being solutions to $(\mathscr{P})$ and $(\mathscr{P}^*)$, respectively,

$$\int_a^b y'(s) f(s) \, \mathrm{d}s = \int_a^b y'(s) \left\{ \dot{x}(s) - A(s) \, x(s) - C(s) \, x(a) - D(s) \, x(b) - \right.$$

$$\left. - \int_a^b [\mathrm{d}_\sigma G(s, \sigma)] \, X(\sigma) \right\} \mathrm{d}s + \int_a^b \left[\mathrm{d} \left\{ y'(s) + \int_a^s y'(\sigma) \, A(\sigma) \, \mathrm{d}\sigma + \gamma' L(s) + \right. \right.$$

$$\left. \left. + \int_a^b y'(\sigma) \, G(\sigma, s) \, \mathrm{d}\sigma \right\} \right] x(s) = \left(y'(b) - \int_a^b y'(s) \, D(s) \, \mathrm{d}s \right) x(b) -$$

$$- \left(y'(a) + \int_a^b y'(s) \, C(s) \, \mathrm{d}s \right) x(a) + \gamma' \int_a^b [\mathrm{d}L(s)] \, x(s) =$$

$$= \gamma' \left\{ M \, x(a) + N \, x(b) + \int_a^b [\mathrm{d}L(s)] \, x(s) \right\} = \gamma' l \, ,$$

by the substitution theorem, the integration by parts formula and Lemma 2,7. This completes the proof.

The following two assertions follow readily from the proofs of Lemma 3,1 and of Theorem 3,1.

Corollary 1. *The homogeneous boundary value problem* $(\mathscr{P}_0)$ *has only the trivial solution iff rank* $(B) = n + n'$ *for the matrix B defined by* (3,6), (3,9), (3,10) *and* (3,12).

Corollary 2. *A couple* (y, γ) *is a solution to* $(\mathscr{P}^*)$ *iff it is a solution to* (3,20).

Remark 3,4. By Lemmas 2,1 and 2,2 the assumptions of this paragraph on G are fulfilled e.g. if $G \in \mathscr{SBV}$, $G(t, \cdot)$ being for any $t \in J$ continuous at a and at b, or if

$$G(t, s) = \int_a^s H(t, \sigma)\, d\sigma ,$$

where $H \in \mathscr{L}_2$. If the latter is the case and if L is absolutely continuous on J, then the adjoint equation to (3,1) is an ordinary integrodifferential equation

$$\dot{y}' = -y' A(t) - \gamma' \dot{L}(t) - \int_a^b y'(s)\, H(s, t)\, ds ,$$
$$\dot{\gamma}' = 0 ,$$

Remark 3,5. The additional condition (3,2) can be written in the form

$$M\, x(a) + N\, x(b) + \int_a^b [dL_1(s)]\, x(s) = l_1 , \quad \int_a^b [dL_2(s)]\, x(s) = l_2 ,$$

where M, N are $m_1 \times n$-matrices, $L_j(t)$ are $m_j \times n$-matrices, l_j is an m_j-vector $(j = 1, 2)$, while $m_1 + m_2 = n$, $0 \leqq m_1 \leqq 2n$, rank $(M, N) = m_1$, L_1 and L_2 are functions of bounded variation on J and continuous at a and at b. Furthermore, let us suppose

$$C(t)\, x(a) + D(t)\, x(b) = K(t)\, (M_c\, x(a) + N_c\, x(b)) ,$$

where $K \in \mathscr{L}^2_{n,2n-m_1}$ and M_c, N_c are constant $(2n - m_1) \times n$-matrices such that

$$\text{rank} \begin{pmatrix} M , N \\ M_c, N_c \end{pmatrix} = 2n .$$

(The case of $K \in \mathscr{L}^2_{n,p}$ and M_c, N_c being constant $p \times n$-matrices such that

$$\text{rank} \begin{pmatrix} M , N \\ M_c, N_c \end{pmatrix} = m_1 + p , \quad 0 \leqq m_1 + p \leqq 2n$$

is only seemingly more general, cf. [17] Remark 6,2a.)

Analogously as in the proof of Theorem 3,1 the problem $(\mathscr{P})$ has a solution iff

$$\int_a^b \chi'(s)\, f(s)\, ds = \gamma_1' l_1 + \gamma_2' l_2$$

for any solution $(\chi, \gamma_1, \gamma_2)$ to the system

(3,22)

$$-\chi'(t) + \gamma_1'\left\{N X(b) + \int_t^b [\mathrm{d}L_1(s)]\, X(s)\right\} X^{-1}(t) + \gamma_2' \left\{\int_t^b [\mathrm{d}L_2(s)]\, X(s)\right\} X^{-1}(t) +$$

$$+ \int_a^b \chi'(s) \left\{K(s)\, N_c\, X(b) + \int_t^b [\mathrm{d}_\sigma G(s, \sigma)]\, X(\sigma)\right\} X^{-1}(t)\, \mathrm{d}s = 0\,,$$

$$\gamma_1' \left\{M + N\, X(b) + \int_a^b [\mathrm{d}L_1(s)]\, X(s)\right\} + \gamma_2' \int_a^b [\mathrm{d}L_2(s)]\, X(s) +$$

$$+ \int_a^b \chi'(s) \left\{K(s)\, (M_c + N_c\, X(b)) + \int_a^b [\mathrm{d}_\sigma G(s, \sigma)]\, X(\sigma)\right\} \mathrm{d}s = 0\,.$$

Let P, Q and P_c, Q_c be respectively adjoint and complementary adjoint matrices to (M, N, M_c, N_c), i.e. P, Q are $n \times (2n - m_1)$ – matrices and P, Q are $n \times m_1$-matrices such that

(3,23)
$$\begin{pmatrix} -M\,, N \\ -M_c, N_c \end{pmatrix} \begin{pmatrix} P_c, P \\ Q_c, Q \end{pmatrix} = I_{2n}\,.$$

It follows from (3,22) that

$$\chi'(t) = \chi'(a) - \int_a^t \chi'(s)\, A(s)\, \mathrm{d}s - \gamma_1'(L_1(t) - L_1(a)) - \gamma_2' (L_2(t) - L_2(a)) -$$

$$- \int_a^b \chi'(s)\, (G(s, t) - G(s, a))\, \mathrm{d}s\,,$$

$$\chi'(a) = -\gamma_1' M - \left(\int_a^b \chi'(s)\, K(s)\, \mathrm{d}s\right) M_c\,, \quad \chi'(b) = \gamma_1' N + \left(\int_a^b \chi'(s)\, K(s)\, \mathrm{d}s\right) N_c$$

or by (3,23)

$$\chi'(a)\, P + \chi'(b)\, Q = \gamma_1'(-MP + NQ) + \int_a^b \chi'(s)\, K(s)\, \mathrm{d}s\, (-M_c P + N_c Q) =$$

$$= \int_a^b \chi'(s)\, K(s)\, \mathrm{d}s\,,$$

$$\chi'(a)\, P_c + \chi'(b)\, Q_c = \gamma_1'(-MP_c + NQ_c) + \int_a^b \chi'(s)\, K(s)\, \mathrm{d}s(-M_c P_c + N_c Q_c) = \gamma_1'\,.$$

Similarly as in the proof of Theorem 3,1 we can complete the proof of the following assertion.

Under the assumptions of this remark the boundary value problem ($\mathscr{P}$) *has a solution if and only if*

$$\int_a^b y'(s) f(s)\, ds = (y'(a) P_c + y'(b) Q_c)\, l_1 + \delta' l_2$$

for any solution (y, δ) *of the adjoint boundary value problem*

$$y'(t) = y'(a) - \int_a^t y'(s) A(s)\, ds - (y'(a) P_c + y'(b) Q_c)(L_1(t) - L_1(a)) -$$

$$- \delta'(L_2(t) - L_2(a)) - \int_a^b y'(s)(G(s, t) - G(s, a))\, ds\,,$$

$$y'(a) P + y'(b) Q - \int_a^b y'(s) K(s)\, ds = 0\,.$$

Let us complete Theorem 3,1 by deriving the relation between the number of linearly independent solutions to ($\mathscr{P}_0$) and to ($\mathscr{P}^*$).

The couples $(y_j, \gamma_j) \in \mathscr{L}^2 \times \mathscr{R}_m$ $(j = 1, 2, \ldots, r)$ *are said to be linearly dependent on* J *if there exists a nonzero r-vector* $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ *such that* $\lambda_1 y_1(t) + \lambda_2 y_2(t) + \ldots + \lambda_r y_r(t) = 0$ *a.e. on* J *and* $\lambda_1 \gamma_1 + \lambda_2 \gamma_2 + \ldots + \lambda_r \gamma_r = 0$. *They are linearly independet on* J *if they are not linearly dependent on* J.

Lemma 3,2. *Let* C, $H_1(t)$, $H_2(t)$ *and* $K(t, s)$ *fulfil the assumptions of Lemma* 3,1 *and let* $u(t) = 0$ *a.e. on* J *and* $v = 0$. *Let the system* (3,7) *which is now homogeneous have exactly* r *linearly independent solutions on* J, *then its adjoint* (3,8) *has exactly* $r^* = r + p - q$ *linearly independent solutions on* J.

Proof. All symbols used in this proof have the same meaning as in the proof of Lemma 3,1. The system (3,14) is now homogeneous ($w = 0$).

Let (h, c) be a solution of (3,7) and let

$$d = \int_a^b K_2(s)\, h(s)\, ds\,, \quad b = (c', d')'\,.$$

Then b is a solution of (3,14). Let us put

$$\tilde{h}(t) = \tilde{H}_1(t)\, c + \tilde{K}_1(t)\, d\,.$$

Then by the above argument $(\tilde{h}, c)$ is a solution of (3,7) and

$$\tilde{h}(t) = \left[H_1(t) + \int_a^b \Gamma(t,s) H_1(s)\,ds\right] c + \left[K_1(t) + \int_a^b \Gamma(t,s) K_1(s)\,ds\right] \int_a^b K_2(\sigma) h(\sigma)\,d\sigma =$$

$$= H_1(t)\,c + K_1(t) \int_a^b K_2(s)\,h(s)\,ds +$$

$$+ \int_a^b \Gamma(t,s) \left[H_1(s)\,c + \int_a^b K_1(s) K_2(\sigma)\,h(\sigma)\,d\sigma\right] ds =$$

$$= h(t) - \int_a^b K_0(t,s)\,h(s)\,ds + \int_a^b \Gamma(t,s) \left[h(s) - \int_a^b K_0(s,\sigma)\,h(\sigma)\,d\sigma\right] ds =$$

$$= h(t) - \int_a^b \left[K_0(t,s) + \int_a^b \Gamma(t,\sigma) K_0(\sigma,s)\,d\sigma\right] h(s)\,ds + \int_a^b \Gamma(t,s)\,h(s)\,ds = h(t)\,.$$

Let $b = (c', d')'$ be a solution of (3,14) and $h(t) = \tilde{H}_1(t)\,c + \tilde{K}_1(t)\,d$. Then (h, c) is a solution of (3,7). Let us put

$$\tilde{d} = \int_a^b K_2(s)\,h(s)\,ds\,.$$

Then $\tilde{b} = (c', \tilde{d}')'$ is a solution of (3,14) and

$$\tilde{d} = \left(\int_a^b K_2(s)\,\tilde{H}_1(s)\,ds\right) c + \left(\int_a^b K_2(s)\,\tilde{K}_1(s)\,ds\right) d = B_{1,1}c + B_{1,2}d + d = d\,.$$

Thus the system (3,7) has exactly r linearly independent solutions on J iff the system (3,14) has exactly r linearly independent solutions. It means that the rank of the $(p + n') \times (q + n')$-matrix B equals $q + n' - r$. Hence (3,15) has exactly $(p + n') - (q + n' - r) = r + p - q$ linearly independent solutions. Similarly as for the systems (3,7) and (3,14) it could be shown that then the system (3,8) has exactly $r^* = r + p - q$ linearly independent solutions on J.

Theorem 3,2. *If the homogeneous boundary value problem $(\mathscr{P}_0)$ has exactly r linearly independent solutions on J, then its adjoint $(\mathscr{P}^*)$ has exactly $r^* = r + m - n$ linearly independent solutions on J.*

Proof follows readily from the relationship between $(\mathscr{P})$ and (3,5) and between $(\mathscr{P}^*)$ and (3,20) and from Lemma 3,2, where $p = m$ and $q = n$. (See also Corollary 2 of Theorem 3,1.)

Remark 3,6. Lemma 3,2 indicates that the nullity of the matrix B defined by (3,6) (3,9), (3,10) and (3,12) does not depend on the choice of the decomposition (3,9).

Remark 3,7. An astute reader could find the assertion of Theorem 3,2 confusing. If we added to matrices M, N, L and l one zero row, we should obtain an equivalent boundary value problem. Let the corresponding homogeneous boundary value problem have exactly r linearly independent solutions on J. Then by Theorem 3,2 the adjoint should have both $r + m - n$ and $r + m + 1 - n$ linearly independent solutions on J. But here we must take into account that while in the former case the adjoint problem has solutions (y, γ), where γ is an m-vector, in the latter case the adjoint problem has solutions (y, δ), where δ is an $(m + 1)$-vector, the last component of δ being arbitrary. Nevertheless it can be seen that it is reasonable to remove from (3,2) all linearly dependent rows.

(to be continued)