On Partially Orthogonal *hp* Edge Elements for Maxwell's Equations

Tomáš Vejchodský (vejchod@math.cas.cz)

Pavel Šolín (solin@utep.edu)

Svatava Vyvialová (svyvialova@utep.edu)

Mathematical Institute, Academy of Sciences
Žitná 25, 115 67 Prague 1
Czech Republic

Department of Mathematical Sciences
University of Texas at El Paso
El Paso, TX 79968-0514
USA

Time-Harmonic Maxwell's Equations

$$\mathbf{curl}\left(\mu_{\mathrm{r}}^{-1}\,\mathrm{curl}\,\mathbf{E}\right) - \kappa^2 \epsilon_{\mathrm{r}} \mathbf{E} = \mathbf{F} \quad \text{in } \Omega,$$

- $\mathbf{curl} = (\partial/\partial x_2, -\partial/\partial x_1)^\top$
- $\mathrm{curl}\,\mathbf{E} = \partial E_2/\partial x_1 - \partial E_1/\partial x_2$
- $\Omega \subset \mathbb{R}^2$
- $\mu_{\mathrm{r}} = \mu_{\mathrm{r}}(x) \in \mathbb{R}$ relative permeability
- $\epsilon_{\mathrm{r}} = \epsilon_{\mathrm{r}}(x) \in \mathbb{C}^{2\times 2}$ relative permittivity
- $\mathbf{E} = \mathbf{E}(x) \in \mathbb{C}^2$ phaser of the electric field intensity
- $\mathbf{F} = \mathbf{F}(x) \in \mathbb{C}^2$
- $\kappa \in \mathbb{R}$ the wave number

Time-Harmonic Maxwell's Equations

$$\mathbf{curl}\left(\mu_{\mathrm{r}}^{-1}\,\mathrm{curl}\,\mathbf{E}\right)-\kappa^{2}\epsilon_{\mathrm{r}}\mathbf{E}=\mathbf{F}\quad\text{in }\Omega,$$

Perfect conducting boundary conditions:

$$\mathbf{E}\cdot\tau=0,\quad\text{on }\Gamma_{P}.$$

Impedance boundary conditions:

$$\mu_{\mathrm{r}}^{-1}\,\mathrm{curl}\,\mathbf{E}-\mathrm{i}\kappa\lambda\mathbf{E}\cdot\tau=\mathbf{g}\cdot\tau\quad\text{on }\Gamma_{I}.$$

- $\tau=(-\nu_2,\nu_1)^{\top}$ positively oriented unit tangent vector
- $\lambda=\lambda(x)>0$ impedance
- $\mathbf{g}=\mathbf{g}(x)\in\mathbb{C}^2$

Weak and *hp*-FEM Formulations

$$V = \{\mathbf{E} \in \mathbf{H}(\mathrm{curl}, \Omega) : \mathbf{E} \cdot \tau = 0 \text{ on } \Gamma_P\}$$

$$\mathbf{E} \in V : \quad \boxed{a(\mathbf{E}, \mathbf{\Phi}) = \mathcal{F}(\mathbf{\Phi})} \quad \forall \mathbf{\Phi} \in V$$

$$a(\mathbf{E}, \mathbf{\Phi}) = \left(\mu_{\mathrm{r}}^{-1} \,\mathrm{curl}\, \mathbf{E}, \mathrm{curl}\, \mathbf{\Phi}\right) - \kappa^2 \left(\epsilon_{\mathrm{r}} \mathbf{E}, \mathbf{\Phi}\right) - \mathrm{i}\kappa \left\langle \lambda \mathbf{E} \cdot \tau, \mathbf{\Phi} \cdot \tau \right\rangle$$

$$\mathcal{F}(\mathbf{\Phi}) = (\mathbf{F}, \mathbf{\Phi}) + \langle \mathbf{g} \cdot \tau, \mathbf{\Phi} \cdot \tau \rangle$$

Weak and *hp*-FEM Formulations

$$V = \{\mathbf{E} \in \mathbf{H}(\mathrm{curl}, \Omega) : \mathbf{E} \cdot \tau = 0 \text{ on } \Gamma_P\}$$

$$\mathbf{E} \in V : \quad \boxed{a(\mathbf{E}, \mathbf{\Phi}) = \mathcal{F}(\mathbf{\Phi})} \quad \forall \mathbf{\Phi} \in V$$

$$a(\mathbf{E}, \mathbf{\Phi}) = \left(\mu_{\mathrm{r}}^{-1} \,\mathrm{curl}\, \mathbf{E}, \mathrm{curl}\, \mathbf{\Phi}\right) - \kappa^2 \left(\epsilon_{\mathrm{r}} \mathbf{E}, \mathbf{\Phi}\right) - \mathrm{i}\kappa \left\langle \lambda \mathbf{E} \cdot \tau, \mathbf{\Phi} \cdot \tau \right\rangle$$

$$\mathcal{F}(\mathbf{\Phi}) = (\mathbf{F}, \mathbf{\Phi}) + \langle \mathbf{g} \cdot \tau, \mathbf{\Phi} \cdot \tau \rangle$$

$$V_{hp} = \Big\{ \mathbf{E}_{hp} \in V : \mathbf{E}_{hp}|_{K_j} \in \left[P^{p_j}(K_j)\right]^2 \text{ and } \mathbf{E}_{hp} \cdot \tau_k \text{ is continuous on each edge } e_k \Big\}$$

$$\mathbf{E}_{hp} \in V_{hp} : \quad \boxed{a(\mathbf{E}_{hp}, \mathbf{\Phi}_{hp}) = \mathcal{F}(\mathbf{\Phi}_{hp})} \quad \forall \mathbf{\Phi}_{hp} \in V_{hp}$$

Weak and *hp*-FEM Formulations

$$V = \{\mathbf{E} \in \mathbf{H}(\mathrm{curl}, \Omega) : \mathbf{E} \cdot \tau = 0 \text{ on } \Gamma_P\}$$

$$\mathbf{E} \in V : \quad \boxed{a(\mathbf{E}, \mathbf{\Phi}) = \mathcal{F}(\mathbf{\Phi})} \quad \forall \mathbf{\Phi} \in V$$

$$a(\mathbf{E}, \mathbf{\Phi}) = \left(\mu_{\mathrm{r}}^{-1}\,\mathrm{curl}\,\mathbf{E}, \mathrm{curl}\,\mathbf{\Phi}\right) - \kappa^2 \left(\epsilon_{\mathrm{r}}\mathbf{E}, \mathbf{\Phi}\right) - \mathrm{i}\kappa \left\langle \lambda \mathbf{E} \cdot \tau, \mathbf{\Phi} \cdot \tau \right\rangle$$

$$\mathcal{F}(\mathbf{\Phi}) = (\mathbf{F}, \mathbf{\Phi}) + \langle \mathbf{g} \cdot \tau, \mathbf{\Phi} \cdot \tau \rangle$$

$$V_{hp} = \left\{ \mathbf{E}_{hp} \in V : \mathbf{E}_{hp}|_{K_j} \in \left[P^{p_j}(K_j)\right]^2 \text{ and } \mathbf{E}_{hp} \cdot \tau_k \text{ is continuous on each edge } e_k \right\}$$

$$\mathbf{E}_{hp} \in V_{hp} : \quad \boxed{a(\mathbf{E}_{hp}, \mathbf{\Phi}_{hp}) = \mathcal{F}(\mathbf{\Phi}_{hp})} \quad \forall \mathbf{\Phi}_{hp} \in V_{hp}$$

$$\boxed{\mathbf{E}_{hp} = \sum_j^N \underbrace{c_j}_{\in \mathbb{C}} \psi_j} \qquad \psi_j \ \ldots \ \text{a basis of } V_{hp}$$

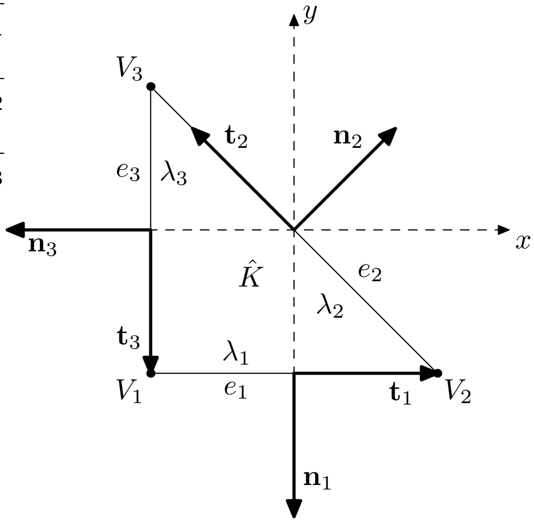
Choice of Basis

Whitney functions:

$$\hat{\psi}_0^{e_1} = \frac{\lambda_3 \mathbf{n}_2}{\mathbf{n}_2 \cdot \mathbf{t}_1} + \frac{\lambda_2 \mathbf{n}_3}{\mathbf{n}_3 \cdot \mathbf{t}_1}$$

$$\hat{\psi}_0^{e_2} = \frac{\lambda_1 \mathbf{n}_3}{\mathbf{n}_3 \cdot \mathbf{t}_2} + \frac{\lambda_3 \mathbf{n}_1}{\mathbf{n}_1 \cdot \mathbf{t}_2}$$

$$\hat{\psi}_0^{e_3} = \frac{\lambda_2 \mathbf{n}_1}{\mathbf{n}_1 \cdot \mathbf{t}_3} + \frac{\lambda_1 \mathbf{n}_2}{\mathbf{n}_2 \cdot \mathbf{t}_3}$$



Choice of Basis

Whitney functions:

$$\hat{\psi}_0^{e_1} = \frac{\lambda_3 \mathbf{n}_2}{\mathbf{n}_2 \cdot \mathbf{t}_1} + \frac{\lambda_2 \mathbf{n}_3}{\mathbf{n}_3 \cdot \mathbf{t}_1}$$

$$\hat{\psi}_0^{e_2} = \frac{\lambda_1 \mathbf{n}_3}{\mathbf{n}_3 \cdot \mathbf{t}_2} + \frac{\lambda_3 \mathbf{n}_1}{\mathbf{n}_1 \cdot \mathbf{t}_2}$$

$$\hat{\psi}_0^{e_3} = \frac{\lambda_2 \mathbf{n}_1}{\mathbf{n}_1 \cdot \mathbf{t}_3} + \frac{\lambda_1 \mathbf{n}_2}{\mathbf{n}_2 \cdot \mathbf{t}_3}$$

First order functions:

$$\hat{\psi}_1^{e_1} = \frac{\lambda_3 \mathbf{n}_2}{\mathbf{n}_2 \cdot \mathbf{t}_1} - \frac{\lambda_2 \mathbf{n}_3}{\mathbf{n}_3 \cdot \mathbf{t}_1}$$

$$\hat{\psi}_1^{e_2} = \frac{\lambda_1 \mathbf{n}_3}{\mathbf{n}_3 \cdot \mathbf{t}_2} - \frac{\lambda_3 \mathbf{n}_1}{\mathbf{n}_1 \cdot \mathbf{t}_2}$$

$$\hat{\psi}_1^{e_3} = \frac{\lambda_2 \mathbf{n}_1}{\mathbf{n}_1 \cdot \mathbf{t}_3} - \frac{\lambda_1 \mathbf{n}_2}{\mathbf{n}_2 \cdot \mathbf{t}_3}$$

Choice of Basis

Whitney functions:

$$\hat{\psi}_0^{e_1} = \frac{\lambda_3 \mathbf{n}_2}{\mathbf{n}_2 \cdot \mathbf{t}_1} + \frac{\lambda_2 \mathbf{n}_3}{\mathbf{n}_3 \cdot \mathbf{t}_1}$$
$$\hat{\psi}_0^{e_2} = \frac{\lambda_1 \mathbf{n}_3}{\mathbf{n}_3 \cdot \mathbf{t}_2} + \frac{\lambda_3 \mathbf{n}_1}{\mathbf{n}_1 \cdot \mathbf{t}_2}$$
$$\hat{\psi}_0^{e_3} = \frac{\lambda_2 \mathbf{n}_1}{\mathbf{n}_1 \cdot \mathbf{t}_3} + \frac{\lambda_1 \mathbf{n}_2}{\mathbf{n}_2 \cdot \mathbf{t}_3}$$

First order functions:

$$\hat{\psi}_1^{e_1} = \frac{\lambda_3 \mathbf{n}_2}{\mathbf{n}_2 \cdot \mathbf{t}_1} - \frac{\lambda_2 \mathbf{n}_3}{\mathbf{n}_3 \cdot \mathbf{t}_1}$$
$$\hat{\psi}_1^{e_2} = \frac{\lambda_1 \mathbf{n}_3}{\mathbf{n}_3 \cdot \mathbf{t}_2} - \frac{\lambda_3 \mathbf{n}_1}{\mathbf{n}_1 \cdot \mathbf{t}_2}$$
$$\hat{\psi}_1^{e_3} = \frac{\lambda_2 \mathbf{n}_1}{\mathbf{n}_1 \cdot \mathbf{t}_3} - \frac{\lambda_1 \mathbf{n}_2}{\mathbf{n}_2 \cdot \mathbf{t}_3}$$

$$\hat{\psi}_k^{e_1} = \frac{2k-1}{k} L_{k-1}(\lambda_3 - \lambda_2)\hat{\psi}_1^{e_1} - \frac{k-1}{k} L_{k-2}(\lambda_3 - \lambda_2)\hat{\psi}_0^{e_1},$$
$$\hat{\psi}_k^{e_2} = \frac{2k-1}{k} L_{k-1}(\lambda_1 - \lambda_3)\hat{\psi}_1^{e_2} - \frac{k-1}{k} L_{k-2}(\lambda_1 - \lambda_3)\hat{\psi}_0^{e_2},$$
$$\hat{\psi}_k^{e_3} = \frac{2k-1}{k} L_{k-1}(\lambda_2 - \lambda_1)\hat{\psi}_1^{e_3} - \frac{k-1}{k} L_{k-2}(\lambda_2 - \lambda_1)\hat{\psi}_0^{e_3}, \quad k = 2, 3, \ldots$$

Bubble Functions – I (Monomial)

Edge based bubbles:

$$\hat{\psi}_k^{b,e_1} = \lambda_3 \lambda_2 L_{k-2}(\lambda_3 - \lambda_2)\mathbf{n}_1,$$
$$\hat{\psi}_k^{b,e_2} = \lambda_1 \lambda_3 L_{k-2}(\lambda_1 - \lambda_3)\mathbf{n}_2,$$
$$\hat{\psi}_k^{b,e_3} = \lambda_2 \lambda_1 L_{k-2}(\lambda_2 - \lambda_1)\mathbf{n}_3, \quad k = 2, 3, \ldots$$

Genuine bubbles:

$$\hat{\psi}_{n_1,n_2}^{b,1} = (\lambda_1)^{n_1} \lambda_2 (\lambda_3)^{n_2} \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$\hat{\psi}_{n_1,n_2}^{b,2} = (\lambda_1)^{n_1} \lambda_2 (\lambda_3)^{n_2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad 1 \leq n_1, n_2$$

Bubble Functions – II (Legendre)

Edge based bubbles:

$$\hat{\psi}_k^{b,e_1} = \lambda_3 \lambda_2 L_{k-2}(\lambda_3 - \lambda_2)\mathbf{n}_1,$$
$$\hat{\psi}_k^{b,e_2} = \lambda_1 \lambda_3 L_{k-2}(\lambda_1 - \lambda_3)\mathbf{n}_2,$$
$$\hat{\psi}_k^{b,e_3} = \lambda_2 \lambda_1 L_{k-2}(\lambda_2 - \lambda_1)\mathbf{n}_3, \quad k = 2, 3, \dots$$

Genuine bubbles:

$$\hat{\psi}_{n_1,n_2}^{b,1} = \lambda_1 \lambda_2 \lambda_3 L_{n_1-1}(\lambda_3 - \lambda_2) L_{n_2-1}(\lambda_2 - \lambda_1) \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$\hat{\psi}_{n_1,n_2}^{b,2} = \lambda_1 \lambda_2 \lambda_3 L_{n_1-1}(\lambda_3 - \lambda_2) L_{n_2-1}(\lambda_2 - \lambda_1) \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad 1 \le n_1, n_2$$

Bubble Functions – III (Gram-Schmidt)

Use scalar product

$$\int_{\hat{K}} \mathrm{curl}\, \psi \, \mathrm{curl}\, \varphi \, \mathrm{d}\xi + \int_{\hat{K}} \psi \cdot \varphi \, \mathrm{d}\xi$$

to orthonormalize the Legendre bubbles.

Bubble Functions – IV (Eigen-Bubbles)

$$\hat{Q}_0(\hat{K}) = \left\{ w \in \left[P^p(\hat{K})\right]^2 : w \cdot \tau = 0 \text{ on } \partial\hat{K} \right\}$$

Solve the eigen-problem: find $\hat{\psi} \in \hat{Q}_0(\hat{K})$ such that

$$\int_{\hat{K}} \operatorname{curl} \hat{\psi} \operatorname{curl} \varphi \, \mathrm{d}\xi = \lambda \int_{\hat{K}} \hat{\psi} \cdot \varphi \, \mathrm{d}\xi \quad \forall \varphi \in \hat{Q}_0(\hat{K}).$$

Bubble Functions – IV (Eigen-Bubbles)

$$\hat{Q}_0(\hat{K}) = \left\{ w \in \left[P^p(\hat{K})\right]^2 : w \cdot \tau = 0 \text{ on } \partial\hat{K} \right\}$$

Solve the eigen-problem: find $\hat{\psi} \in \hat{Q}_0(\hat{K})$ such that

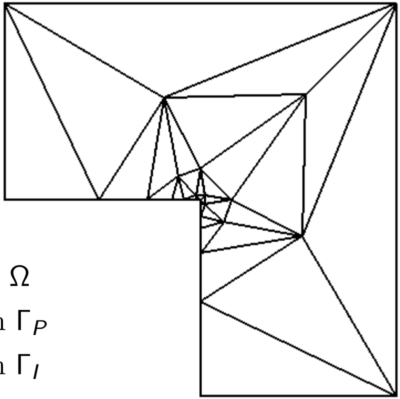
$$\int_{\hat{K}} \operatorname{curl} \hat{\psi} \operatorname{curl} \varphi \, \mathrm{d}\xi = \lambda \int_{\hat{K}} \hat{\psi} \cdot \varphi \, \mathrm{d}\xi \quad \forall \varphi \in \hat{Q}_0(\hat{K}).$$

- If $\hat{\psi}_i$ and $\hat{\psi}_j$ correspond to $\lambda_i \neq \lambda_j$ then

$$\int_{\hat{K}} \operatorname{curl} \hat{\psi}_i \operatorname{curl} \hat{\psi}_j \, \mathrm{d}\xi = \int_{\hat{K}} \hat{\psi}_i \cdot \hat{\psi}_j \, \mathrm{d}\xi = 0$$

- $$\int_{\hat{K}} \operatorname{curl} \hat{\psi}_i \operatorname{curl} \hat{\psi}_i \, \mathrm{d}\xi - \int_{\hat{K}} \hat{\psi}_i \cdot \hat{\psi}_i \, \mathrm{d}\xi = \pm 1$$

Model Problem (L-shape domain)



$$\begin{aligned} \mathbf{curl}\,(\mathrm{curl}\,\mathbf{E}) - \mathbf{E} &= \mathbf{F} && \text{in } \Omega \\ \mathbf{E}\cdot\tau &= 0 && \text{on } \Gamma_P \\ \mathrm{curl}\,\mathbf{E} - \mathrm{i}\mathbf{E}\cdot\tau &= \mathbf{g}\cdot\tau && \text{on } \Gamma_I \end{aligned}$$

$$\mathbf{E} = \frac{2}{3} r^{-\frac{1}{3}} \begin{bmatrix} \cos\left(\frac{\pi}{6} + \frac{\theta}{3}\right) \\ \sin\left(\frac{\pi}{6} + \frac{\theta}{3}\right) \end{bmatrix}$$

Normalization

- 'All the bubbles should have the same size.'
- The natural product

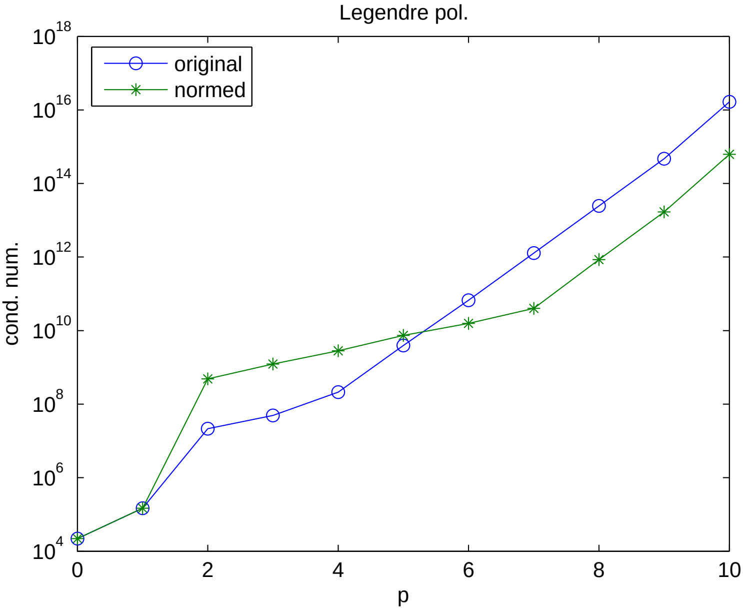
$$(\operatorname{curl}\psi, \operatorname{curl}\varphi) - (\psi, \varphi)$$

  is indefinite.
- Normalization

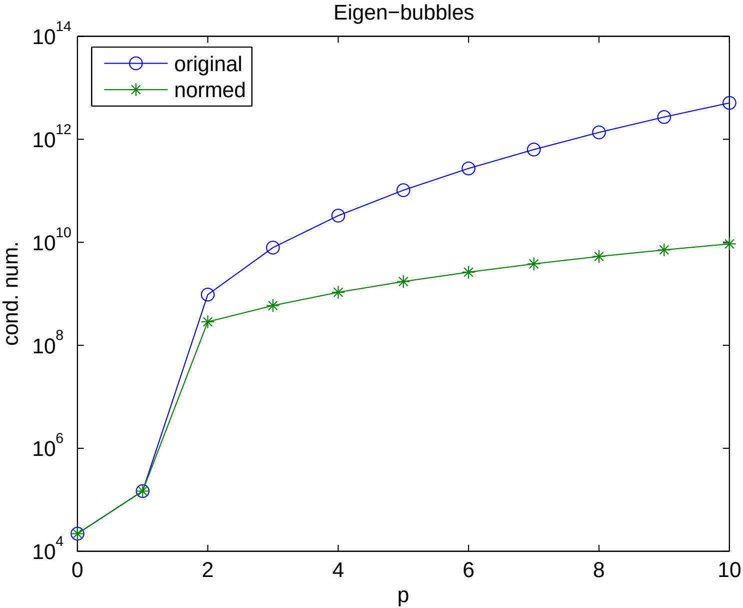
$$\hat{\psi} := \left|(\operatorname{curl}\hat{\psi}, \operatorname{curl}\hat{\psi}) - (\hat{\psi}, \hat{\psi})\right|^{-1/2} \hat{\psi}$$

$$(\operatorname{curl}\hat{\psi}, \operatorname{curl}\hat{\psi}) - (\hat{\psi}, \hat{\psi}) = \pm 1$$

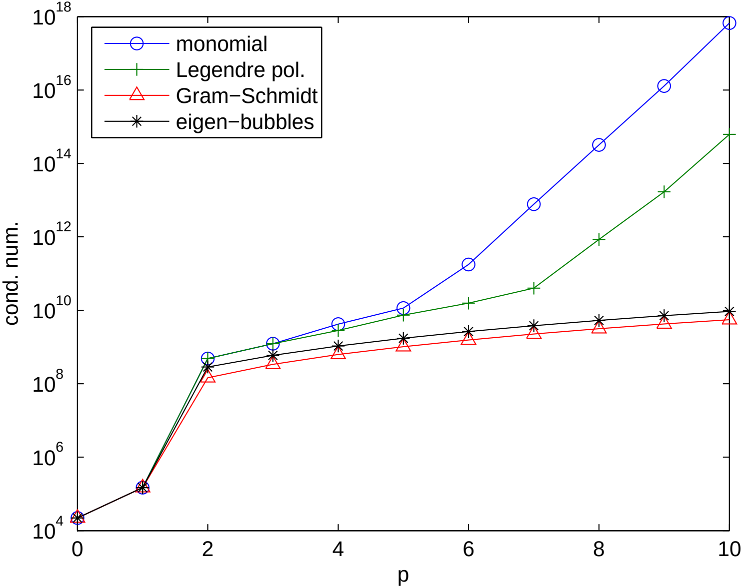
Normalization



Normalization

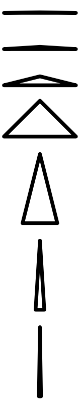


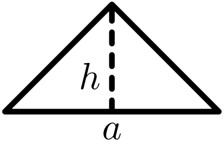
Comparison of Conditioning



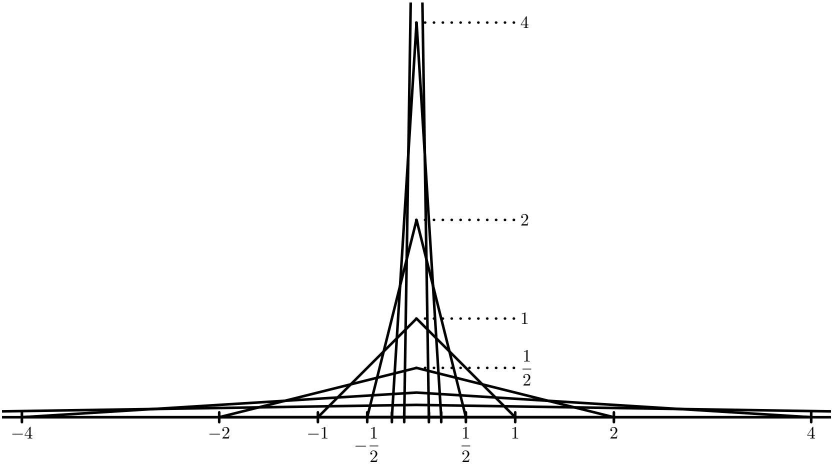
Influence of Elements' Geometry

1) $a = 16$ $\quad h = 1/8$
2) $a = 8$ $\quad h = 1/4$
3) $a = 4$ $\quad h = 1/2$
4) $a = 2$ $\quad h = 1$
5) $a = 1$ $\quad h = 2$
6) $a = 1/2$ $\quad h = 4$
7) $a = 1/4$ $\quad h = 8$

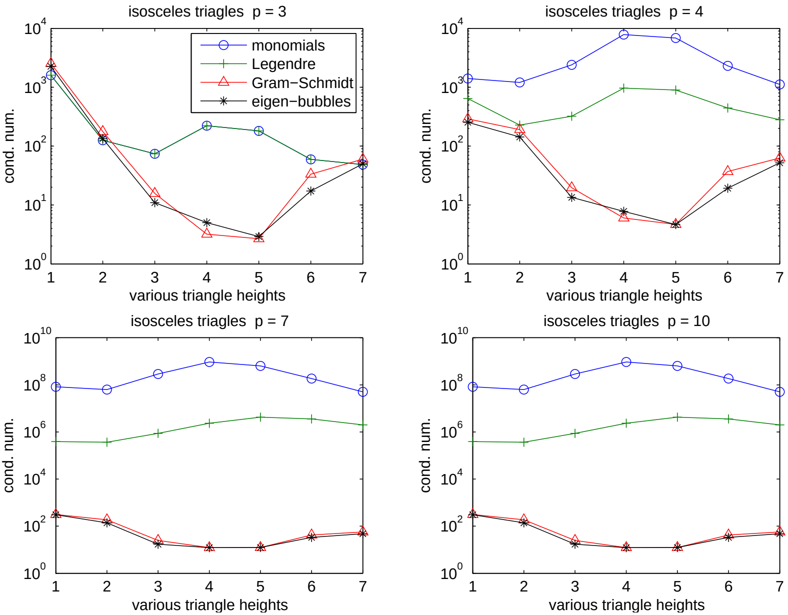




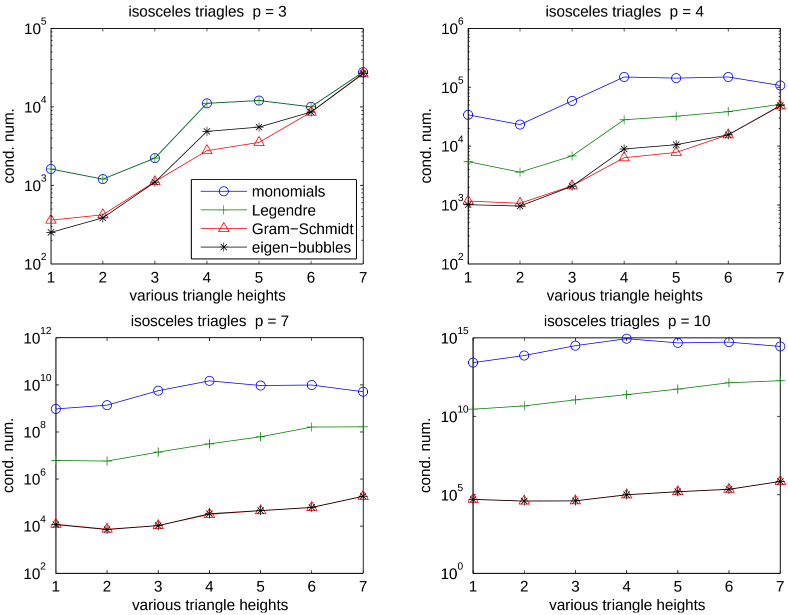
Influence of Elements' Geometry



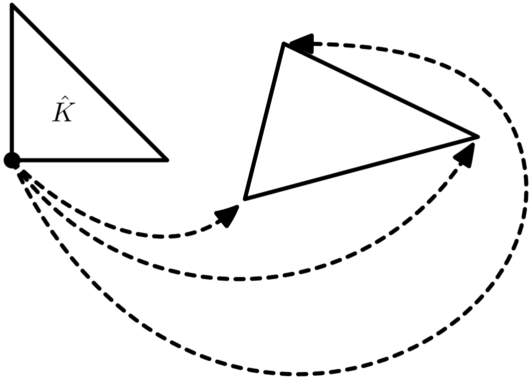
Influence of Elements' Geometry



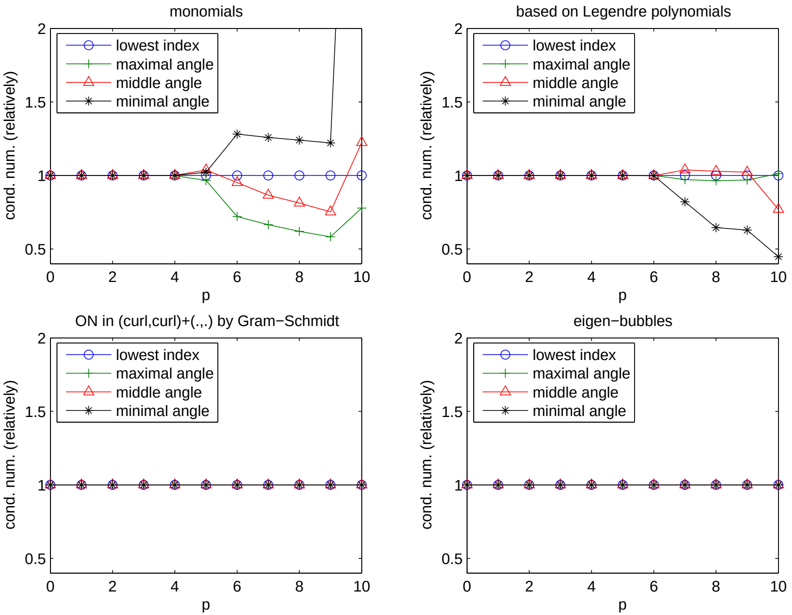
Influence of Elements' Geometry



Influence of Reference Maps



Influence of Reference Maps



Conclusions

- Condition number is relatively insensitive to the geometry of the elements.
- ON and eigen-bubbles have superior conditioning.
- ON and eigen-bubbles do not depend on reference maps.

Thank you for your attention

Tomáš Vejchodský (vejchod@math.cas.cz)

Pavel Šolín (solin@utep.edu)

Svatava Vyvialová (svyvialova@utep.edu)

Mathematical Institute, Academy of Sciences
Žitná 25, 115 67 Prague 1
Czech Republic

Department of Mathematical Sciences
University of Texas at El Paso
El Paso, TX 79968-0514
USA