Guaranteed and robust a posteriori error estimator for a singularly perturbed problem

Tomáš Vejchodský and Mark Ainsworth

Institute of Mathematics, Academy of Sciences
Žitná 25, 115 67 Prague 1
Czech Republic

Modelling 2009, June 22–26, 2009, Rožnov pod Radhoštěm

Outline

- Diffusion reaction problem: $$-\Delta u + \kappa^2 u = f \text{ in } \Omega$$ $$u = 0 \text{ on } \partial\Omega$$
- Finite elements: $u_h \in V_h$
- A posteriori error estimator: $e = u - u_h$
  - Guaranteed upper bound: $\|e\| \leq \eta$
  - Robust: $\exists C > 0,\ C \neq C(h,\kappa) : C\eta \leq \|e\|$

	upper bound	no constant	local (fast)	robust
Equlib res (1993)	$-$	$+$	$+$	$-$
Robust flux (1999)	$-$	$+$	$+$	$+$
Err majorant (1997)	$+$	$-$	$-$	$+$
NEW	$+$	$+$	$+$	$+$

Model Problem

- Classical formulation: $\kappa = \text{const.} > 0$

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega \subset \mathbb{R}^2$$
$$u = 0 \quad \text{on } \partial\Omega$$

- Weak formulation:
  $V = H_0^1(\Omega), \quad B(u,v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x + \int_\Omega \kappa^2 uv \, \mathrm{d}x$

$$u \in V : \quad B(u,v) = \int_\Omega fv \, \mathrm{d}x \quad \forall v \in V$$

- Linear triangular FEM:
  $V_h = \{v_h \in V : v_h|_K \in P^1(K), K \in \mathcal{T}_h\}$

$$u_h \in V_h : \quad B(u_h, v_h) = \int_\Omega f v_h \, \mathrm{d}x \quad \forall v_h \in V_h$$

Local Neumann Problems

- $e = u - u_h$
- Residual equation: $e \in V$:

$$B(e, v) = \int_\Omega f v \,\mathrm{d}x - B(u_h, v) \quad \forall v \in V$$

- Local Neumann problem: $\varepsilon_K \in V(K)$:

$$B_K(\varepsilon_K, v) = \int_K f v \,\mathrm{d}x - B_K(u_h, v) + \int_{\partial K} g_K v \,\mathrm{d}s \quad \forall v \in V(K)$$

- $V(K) = \{v \in H^1(K) : v = 0 \text{ on } \partial K \cap \partial\Omega\} \qquad K \in \mathcal{T}_h$
- $B_K(u, v) = \int_K \nabla u \cdot \nabla v \,\mathrm{d}x + \int_K \kappa^2 u v \,\mathrm{d}x$

Local Neumann Problems

- $e = u - u_h$
- Residual equation: $e \in V$:

$$B(e, v) = \int_\Omega fv \, \mathrm{d}x - B(u_h, v) \quad \forall v \in V$$

- Local Neumann problem:

$$\begin{aligned} -\Delta(\varepsilon_K + u_h) + \kappa^2(\varepsilon_K + u_h) &= f && \text{in } K \\ \nabla(\varepsilon_K + u_h) \cdot \mathbf{n}_K &= g_K && \text{on } \partial K \setminus \partial\Omega \\ \varepsilon_K + u_h &= 0 && \text{on } \partial K \cap \partial\Omega \end{aligned}$$

Properties of ε_K

Theorem 1: If $g_K|_\gamma + g_{K^*}|_\gamma = 0$ for $\gamma = \partial K \cap \partial K^*$

then $\|e\|^2 \le \sum_{K\in\mathcal{T}_h} \|\varepsilon_K\|_K^2$.

Notation: $\|v\|^2 = B(v,v) \qquad \|v\|_K^2 = B_K(v,v)$

Proof: $e = u - u_h$

$$B(e,v) = \sum_{K\in\mathcal{T}_h} \left(\int_K fv \,\mathrm{d}x - B_K(u_h,v) + \int_{\partial K} g_K v \,\mathrm{d}s \right)$$

$$= \sum_{K\in\mathcal{T}_h} B_K(\varepsilon_K, v) \le \left(\sum_{K\in\mathcal{T}_h} \|\varepsilon_K\|_K^2 \right)^{\frac{1}{2}} \|v\|$$

□

Properties of ε_K

Theorem 2: If $g_K = \partial u / \partial \mathbf{n}_K$ then $\|e\|^2 = \sum_{K \in \mathcal{T}_h} \|\varepsilon_K\|_K^2$.

Proof:

- $\kappa^2 > 0 \quad \Rightarrow \quad u = \varepsilon_K + u_h$

$$\begin{aligned} -\Delta(\varepsilon_K + u_h) + \kappa^2(\varepsilon_K + u_h) &= f \quad && \text{in } K \\ \nabla(\varepsilon_K + u_h) \cdot \mathbf{n}_K &= g_K && \text{on } \partial K \setminus \partial\Omega \\ \varepsilon_K + u_h &= 0 && \text{on } \partial K \cap \partial\Omega \end{aligned}$$

- $\kappa^2 = 0 \quad \Rightarrow \quad u = \varepsilon_K + u_h + C_K$ and $\|u - u_h\|_K = \|\varepsilon_K\|_K$

$\square$

Construction of fluxes g_K

Exists fast algorithm [M. Ainsworth, I. Babuška 1999]:

- $g_K|_\gamma + g_{K^*}|_\gamma = 0$
- $g_K|_\gamma \in P^1(\gamma), \quad \gamma \subset \partial K, \quad K \in \mathcal{T}_h,$
- robust

$$\|\varepsilon_K\|_K \preceq \|e\|_{\widetilde{K}} + \min(h_K, \kappa^{-1}) \, \|f - \Pi f\|_{L^2(\widetilde{K})}$$

Estimation of Local Errors

$$\varepsilon_K \in V(K), \quad v \in V(K),$$

$$B_K(\varepsilon_K, v) = \int_K fv \,\mathrm{d}x - B_K(u_h, v) + \int_{\partial K} g_K v \,\mathrm{d}s$$

Estimation of Local Errors

$$\varepsilon_K \in V(K), \quad v \in V(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

$$\begin{aligned} B_K(\varepsilon_K, v) = & \int_K f v \,\mathrm{d}x - B_K(u_h, v) \\ & + \int_{\partial K} g_K v \,\mathrm{d}s \\ & \underbrace{+ \int_K \mathrm{div}\, \mathbf{y}_K v \,\mathrm{d}x + \int_K \mathbf{y}_K \cdot \nabla v \,\mathrm{d}x - \int_{\partial K} \mathbf{y}_K \cdot \mathbf{n}_K v \,\mathrm{d}s}_{=0} \end{aligned}$$

Estimation of Local Errors

$$\varepsilon_K \in V(K), \quad v \in V(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

$$\begin{aligned} B_K(\varepsilon_K, v) = &\int_K f v \,\mathrm{d}x - B_K(u_h, v) \\ &+ \int_{\partial K} g_K v \,\mathrm{d}s \\ &+ \int_K \mathrm{div}\, \mathbf{y}_K v \,\mathrm{d}x + \int_K \mathbf{y}_K \cdot \nabla v \,\mathrm{d}x - \int_{\partial K} \mathbf{y}_K \cdot \mathbf{n}_K v \,\mathrm{d}s \end{aligned}$$

Estimation of Local Errors

$$\varepsilon_K \in V(K), \quad v \in V(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

$$\begin{aligned} B_K(\varepsilon_K, v) = & \int_K f v \,\mathrm{d}x - \int_K \nabla u_h \cdot \nabla v \,\mathrm{d}x - \int_K \kappa^2 u_h v \,\mathrm{d}x \\ & + \int_{\partial K} g_K v \,\mathrm{d}s \\ & + \int_K \mathrm{div}\, \mathbf{y}_K v \,\mathrm{d}x + \int_K \mathbf{y}_K \cdot \nabla v \,\mathrm{d}x - \int_{\partial K} \mathbf{y}_K \cdot \mathbf{n}_K v \,\mathrm{d}s \end{aligned}$$

Estimation of Local Errors

$$\varepsilon_K \in V(K), \quad v \in V(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

$$\begin{aligned} B_K(\varepsilon_K, v) = & \int_K f v \,\mathrm{d}x - \int_K \nabla u_h \cdot \nabla v \,\mathrm{d}x - \int_K \kappa^2 u_h v \,\mathrm{d}x \\ & + \int_{\partial K} g_K v \,\mathrm{d}s \\ & + \int_K \mathrm{div}\, \mathbf{y}_K v \,\mathrm{d}x + \int_K \mathbf{y}_K \cdot \nabla v \,\mathrm{d}x - \int_{\partial K} \mathbf{y}_K \cdot \mathbf{n}_K v \,\mathrm{d}s \end{aligned}$$

Estimation of Local Errors

$$\varepsilon_K \in V(K), \quad v \in V(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

$$B_K(\varepsilon_K, v) = \int_K f v \,\mathrm{d}x - \int_K \nabla u_h \cdot \nabla v \,\mathrm{d}x - \int_K \kappa^2 u_h v \,\mathrm{d}x + \int_{\partial K} g_K v \,\mathrm{d}s + \int_K \mathrm{div}\, \mathbf{y}_K v \,\mathrm{d}x + \int_K \mathbf{y}_K \cdot \nabla v \,\mathrm{d}x - \int_{\partial K} \mathbf{y}_K \cdot \mathbf{n}_K v \,\mathrm{d}s$$

$$r = f - \kappa^2 u_h$$

Estimation of Local Errors

$$\varepsilon_K \in V(K), \quad v \in V(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

$$\begin{aligned} B_K(\varepsilon_K, v) = & \int_K (r + \mathrm{div}\, \mathbf{y}_K) v \,\mathrm{d}x - \int_K \nabla u_h \cdot \nabla v \,\mathrm{d}x \\ & + \int_{\partial K} g_K v \,\mathrm{d}s \\ & + \int_K \mathbf{y}_K \cdot \nabla v \,\mathrm{d}x - \int_{\partial K} \mathbf{y}_K \cdot \mathbf{n}_K v \,\mathrm{d}s \end{aligned}$$

$$r = f - \kappa^2 u_h$$

Estimation of Local Errors

$$\varepsilon_K \in V(K), \quad v \in V(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

$$\begin{aligned} B_K(\varepsilon_K, v) = & \int_K (r + \mathrm{div}\,\mathbf{y}_K) v \,\mathrm{d}x - \int_K \nabla u_h \cdot \nabla v \,\mathrm{d}x \\ & + \int_{\partial K} g_K v \,\mathrm{d}s \\ & + \int_K \mathbf{y}_K \cdot \nabla v \,\mathrm{d}x - \int_{\partial K} \mathbf{y}_K \cdot \mathbf{n}_K v \,\mathrm{d}s \end{aligned}$$

$$r = f - \kappa^2 u_h$$

$$\mathbf{y}_K \cdot \mathbf{n}_K = g_K \quad \text{on } \partial K \setminus \partial\Omega$$

Estimation of Local Errors

$$\varepsilon_K \in V(K), \quad v \in V(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

$$B_K(\varepsilon_K, v) = \int_K (r + \mathrm{div}\, \mathbf{y}_K) v \,\mathrm{d}x - \int_K \nabla u_h \cdot \nabla v \,\mathrm{d}x$$

$$+ \int_K \mathbf{y}_K \cdot \nabla v \,\mathrm{d}x$$

$$r = f - \kappa^2 u_h$$

$$\mathbf{y}_K \cdot \mathbf{n}_K = g_K \quad \text{on } \partial K \setminus \partial\Omega$$

Estimation of Local Errors

$$\varepsilon_K \in V(K), \quad v \in V(K), \quad \mathbf{y}_K \in \mathbf{H}(\operatorname{div}, K),$$

$$B_K(\varepsilon_K, v) = \int_K \frac{1}{\kappa}(r + \operatorname{div} \mathbf{y}_K)\kappa v \,\mathrm{d}x$$

$$+ \int_K (\mathbf{y}_K - \nabla u_h) \cdot \nabla v \,\mathrm{d}x$$

$$r = f - \kappa^2 u_h$$

$$\mathbf{y}_K \cdot \mathbf{n}_K = g_K \quad \text{on } \partial K \setminus \partial\Omega$$

Estimation of Local Errors

$$\varepsilon_K \in V(K), \quad v \in V(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

$$B_K(\varepsilon_K, v) \le \left\| \frac{1}{\kappa}(r + \mathrm{div}\, \mathbf{y}_K) \right\|_{0,K} \|\kappa v\|_{0,K}$$

$$+ \int_K (\mathbf{y}_K - \nabla u_h) \cdot \nabla v \,\mathrm{d}x$$

$$r = f - \kappa^2 u_h$$

$$\mathbf{y}_K \cdot \mathbf{n}_K = g_K \quad \text{on } \partial K \setminus \partial\Omega$$

Estimation of Local Errors

$$\varepsilon_K \in V(K), \quad v \in V(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

$$B_K(\varepsilon_K, v) \le \frac{1}{2} \left\| \frac{1}{\kappa}(r + \mathrm{div}\, \mathbf{y}_K) \right\|_{0,K}^2 + \frac{1}{2} \|\kappa v\|_{0,K}^2$$

$$+ \int_K (\mathbf{y}_K - \nabla u_h) \cdot \nabla v \,\mathrm{d}x$$

$$r = f - \kappa^2 u_h$$

$$\mathbf{y}_K \cdot \mathbf{n}_K = g_K \quad \text{on } \partial K \setminus \partial\Omega$$

Estimation of Local Errors

$$\varepsilon_K \in V(K), \quad v \in V(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

$$B_K(\varepsilon_K, v) \le \frac{1}{2} \left\| \frac{1}{\kappa}(r + \mathrm{div}\, \mathbf{y}_K) \right\|_{0,K}^2 + \frac{1}{2} \|\kappa v\|_{0,K}^2$$

$$+ \|\mathbf{y}_K - \nabla u_h\|_{0,K} \|\nabla v\|_{0,K}$$

$r = f - \kappa^2 u_h$

$\mathbf{y}_K \cdot \mathbf{n}_K = g_K \quad$ on $\partial K \setminus \partial\Omega$

Estimation of Local Errors

$$\varepsilon_K \in V(K), \quad v \in V(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

$$B_K(\varepsilon_K, v) \leq \frac{1}{2} \left\| \frac{1}{\kappa}(r + \mathrm{div}\, \mathbf{y}_K) \right\|_{0,K}^2 + \frac{1}{2} \|\kappa v\|_{0,K}^2$$

$$+ \frac{1}{2} \|\mathbf{y}_K - \nabla u_h\|_{0,K}^2 + \frac{1}{2} \|\nabla v\|_{0,K}^2$$

$$r = f - \kappa^2 u_h$$
$$\mathbf{y}_K \cdot \mathbf{n}_K = g_K \quad \text{on } \partial K \setminus \partial\Omega$$

Estimation of Local Errors

$$\varepsilon_K \in V(K), \quad v \in V(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

$$B_K(\varepsilon_K, v) \leq \frac{1}{2} \left\| \frac{1}{\kappa}(r + \mathrm{div}\, \mathbf{y}_K) \right\|_{0,K}^2 + \frac{1}{2} \|\kappa v\|_{0,K}^2$$

$$+ \frac{1}{2} \|\mathbf{y}_K - \nabla u_h\|_{0,K}^2 + \frac{1}{2} \|\nabla v\|_{0,K}^2$$

$$r = f - \kappa^2 u_h$$
$$\mathbf{y}_K \cdot \mathbf{n}_K = g_K \quad \text{on } \partial K \setminus \partial\Omega$$
$$\frac{1}{2} \|\kappa v\|_{0,K}^2 + \frac{1}{2} \|\nabla v\|_{0,K}^2 = \frac{1}{2} |\!|\!| v |\!|\!|_K^2$$

Estimation of Local Errors

$$\varepsilon_K \in V(K), \quad v \in V(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

$$B_K(\varepsilon_K, v) \le \frac{1}{2} \left\| \frac{1}{\kappa}(r + \mathrm{div}\, \mathbf{y}_K) \right\|_{0,K}^2$$

$$+ \frac{1}{2} \|\mathbf{y}_K - \nabla u_h\|_{0,K}^2 + \frac{1}{2} |||v|||_K^2$$

$$r = f - \kappa^2 u_h$$

$$\mathbf{y}_K \cdot \mathbf{n}_K = g_K \quad \text{on } \partial K \setminus \partial\Omega$$

$$\frac{1}{2} \|\kappa v\|_{0,K}^2 + \frac{1}{2} \|\nabla v\|_{0,K}^2 = \frac{1}{2} |||v|||_K^2$$

Estimation of Local Errors

$$\varepsilon_K \in V(K), \quad v \in V(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

$$B_K(\varepsilon_K, \varepsilon_K) \leq \frac{1}{2} \left\| \frac{1}{\kappa}(r + \mathrm{div}\, \mathbf{y}_K) \right\|_{0,K}^2$$

$$+ \frac{1}{2} \|\mathbf{y}_K - \nabla u_h\|_{0,K}^2 + \frac{1}{2} \|\!|\varepsilon_K|\!\|_K^2$$

$$r = f - \kappa^2 u_h$$

$$\mathbf{y}_K \cdot \mathbf{n}_K = g_K \quad \text{on } \partial K \setminus \partial \Omega$$

Estimation of Local Errors

$$\varepsilon_K \in V(K), \quad v \in V(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

$$|||\varepsilon_K|||_K^2 \le \frac{1}{2} \left\| \frac{1}{\kappa}(r + \mathrm{div}\, \mathbf{y}_K) \right\|_{0,K}^2$$

$$+ \frac{1}{2} \|\mathbf{y}_K - \nabla u_h\|_{0,K}^2 + \frac{1}{2} |||\varepsilon_K|||_K^2$$

$$r = f - \kappa^2 u_h$$

$$\mathbf{y}_K \cdot \mathbf{n}_K = g_K \quad \text{on } \partial K \setminus \partial\Omega$$

Estimation of Local Errors

$$\varepsilon_K \in V(K), \quad v \in V(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

$$\frac{1}{2} |||\varepsilon_K|||_K^2 \leq \frac{1}{2} \left\| \frac{1}{\kappa}(r + \mathrm{div}\, \mathbf{y}_K) \right\|_{0,K}^2$$

$$+ \frac{1}{2} \|\mathbf{y}_K - \nabla u_h\|_{0,K}^2$$

$$r = f - \kappa^2 u_h$$

$$\mathbf{y}_K \cdot \mathbf{n}_K = g_K \quad \text{on } \partial K \setminus \partial\Omega$$

Estimation of Local Errors

$$\varepsilon_K \in V(K), \quad v \in V(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

$$|||\varepsilon_K|||_K^2 \leq \left\| \frac{1}{\kappa}(r + \mathrm{div}\, \mathbf{y}_K) \right\|_{0,K}^2$$

$$+ \; \|\mathbf{y}_K - \nabla u_h\|_{0,K}^2$$

$$r = f - \kappa^2 u_h$$

$$\mathbf{y}_K \cdot \mathbf{n}_K = g_K \quad \text{on } \partial K \setminus \partial\Omega$$

Estimation of Local Errors

$$\varepsilon_K \in V(K), \quad v \in V(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

- Local estimate

$$|||\varepsilon_K|||_K^2 \le \left\| \frac{1}{\kappa}(r + \mathrm{div}\,\mathbf{y}_K) \right\|_{0,K}^2 + \|\mathbf{y}_K - \nabla u_h\|_{0,K}^2 \equiv \eta_K^2(\mathbf{y}_K)$$

$$r = f - \kappa^2 u_h$$

$$\mathbf{y}_K \cdot \mathbf{n}_K = g_K \quad \text{on } \partial K \setminus \partial\Omega$$

Estimation of Local Errors

$$\varepsilon_K \in V(K), \quad v \in V(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

- Local estimate

$$\|\!|\varepsilon_K|\!\|_K^2 \leq \left\| \frac{1}{\kappa}(r + \mathrm{div}\, \mathbf{y}_K) \right\|_{0,K}^2 + \|\mathbf{y}_K - \nabla u_h\|_{0,K}^2 \equiv \eta_K^2(\mathbf{y}_K)$$

- Global estimate

$$\|\!|e|\!\|^2 \leq \sum_{K \in \mathcal{T}_h} \|\!|\varepsilon_K|\!\|_K^2 \leq \sum_{K \in \mathcal{T}_h} \eta_K^2(\mathbf{y}_K)$$

$$r = f - \kappa^2 u_h$$

$$\mathbf{y}_K \cdot \mathbf{n}_K = g_K \quad \text{on } \partial K \setminus \partial\Omega$$

Estimation of Local Errors

$\varepsilon_K \in V(K), \quad v \in V(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$

- Local estimate

$$|||\varepsilon_K|||_K^2 \leq \left\| \frac{1}{\kappa}(r + \mathrm{div}\, \mathbf{y}_K) \right\|_{0,K}^2 + \|\mathbf{y}_K - \nabla u_h\|_{0,K}^2 \equiv \eta_K^2(\mathbf{y}_K)$$

- Theorem 3: If $\mathbf{y}_K = \nabla(u_h + \varepsilon_K)$ then $|||\varepsilon_K|||_K = \eta_K(\mathbf{y}_K)$.
- Proof:
  - $f - \kappa^2 u_h + \mathrm{div}\, \mathbf{y}_K = f - \kappa^2 u_h + \Delta(u_h + \varepsilon_K) = \kappa^2 \varepsilon_K$
  - $\mathbf{y}_K - \nabla u_h = \nabla \varepsilon_K$

$\square$

$r = f - \kappa^2 u_h$

$\mathbf{y}_K \cdot \mathbf{n}_K = g_K \quad \text{on } \partial K \setminus \partial\Omega$

Choice of $\mathbf{y}_K$

(a) $\mathbf{y}_K$ uniquely given by g_K:

- $g_K|_\gamma \in P^1(\gamma) \quad \Rightarrow \quad \exists!\ \bar{\mathbf{y}}_K \in [P^1(K)]^2 : \ \bar{\mathbf{y}}_K \cdot \mathbf{n}_K = g_K$

(b) Minimize $\eta_K^2(\mathbf{y}_K)$ over $\mathbf{W}^2(K) \subset \mathbf{H}(\mathrm{div}, K)$

- $\mathbf{W}^2(K) := \bar{\mathbf{y}}_K + \mathbf{W}_0^2(K)$
- $\mathbf{W}_0^2(K) := \{\mathbf{y} \in [P^2(K)]^2 : \mathbf{y} \cdot \mathbf{n}_K = 0 \text{ on } \partial K \setminus \partial\Omega\}$
- $B^*(\mathbf{y}, \mathbf{w}) := \int_K \mathrm{div}\,\mathbf{y}\,\mathrm{div}\,\mathbf{w}\,\mathrm{d}x + \int_K \kappa^2 \mathbf{y} \cdot \mathbf{w}\,\mathrm{d}x$
- $\widetilde{\mathbf{y}}_K = \widetilde{\mathbf{y}}_K^0 + \bar{\mathbf{y}}_K, \quad \widetilde{\mathbf{y}}_K^0 \in \mathbf{W}_0^2(K)$
- Find $\widetilde{\mathbf{y}}_K^0 \in \mathbf{W}_0^2(K)$:
  $$B^*(\widetilde{\mathbf{y}}_K^0 + \bar{\mathbf{y}}_K, \mathbf{w}) = -\int_K f\,\mathrm{div}\,\mathbf{w}\,\mathrm{d}x \quad \forall \mathbf{w} \in \mathbf{W}_0^2(K)$$

Robustness for $\kappa \to \infty$

Theorem 4:
$f \in L^\infty(\Omega)$, $\nabla \varepsilon_K \in H(\mathrm{div}, K)$, regular family of triangulations
$\Rightarrow \exists C > 0$ (independent of h and κ):

$$\eta_K^2(\bar{\mathbf{y}}_K) \le C \left[\|\!|\varepsilon_K|\!\|_K^2 + \kappa^{-2} + \min(h^{-4}\kappa^{-4}, h^3\kappa^{-1}) \right]$$

Remark: If $\kappa \to \infty$: $\quad \|\!|e|\!\|^2 \simeq \sum_K \|\!|\varepsilon_K|\!\|_K^2 \simeq \kappa^{-2}$

$$\Rightarrow \quad I_{\mathrm{eff}}^2 = \frac{\eta^2(\bar{\mathbf{y}})}{\|\!|e|\!\|^2} \simeq \frac{\eta^2(\bar{\mathbf{y}})}{\kappa^{-2}} \preceq 1 + \min(h^{-4}\kappa^{-2}, h^3\kappa)$$

Data oscillations

$$\|\!|\varepsilon_K|\!\|_K^2 \leq \left(\left\| \frac{1}{\kappa}(\overline{f}_K - \kappa^2 u_h + \operatorname{div} \mathbf{y}_K) \right\|_{0,K} + \operatorname{osc}_K(f) \right)^2 + \left(\|\mathbf{y}_K - \nabla u_h\|_{0,K} + \operatorname{osc}_K(f) \right)^2 \equiv (\eta_K^{\mathrm{osc}}(\mathbf{y}_K))^2$$

$$\operatorname{osc}_K(f) = \min(h_K/\pi, \kappa^{-1}) \, \|f - \overline{f}_K\|_{0,K}$$

(a) $\overline{f}_K \in P^0(K)$: $\quad \displaystyle\int_K (f - \overline{f}_K)\overline{v} = 0 \quad \forall \overline{v} \in P^0(K)$

$$\Rightarrow \quad \overline{f}_K = |K|^{-1} \int_K f \, \mathrm{d}x$$

(b) $\overline{f}_K \in P^1(K)$: $\quad \displaystyle\int_K (f - \overline{f}_K)\overline{v} = 0 \quad \forall \overline{v} \in P^1(K)$

Numerical examples

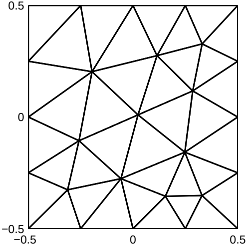
$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

Example (A)

$$\Omega = (-1/2, 1/2)^2$$

$$f = \cos(\pi x)\cos(\pi y)$$

$$u = \frac{\cos(\pi x)\cos(\pi y)}{\pi^2 + \kappa^2}$$

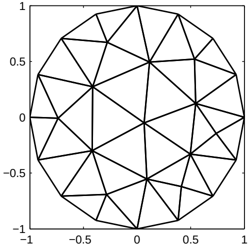


Example (B)

$$\Omega = \{(x,y) : r < 1\}$$

$$f = 1 \qquad r = \sqrt{x^2 + y^2}$$

$$u = \frac{1}{\kappa^2}\left(1 - \frac{I_0(\kappa r)}{I_0(\kappa)}\right)$$



Results – I_{eff}

$\eta_K(\bar{\mathbf{y}}_K)$ ... linear $\bar{\mathbf{y}}_K$

Example (A)

κ	no osc	const osc	linear osc
0	3.78	1.73	3.80
10^{-3}	3513.02	1.73	3478.19
10^{-2}	351.31	1.73	347.86
10^{-1}	35.16	1.73	34.85
1	3.78	1.79	3.80
10	1.60	4.68	1.85
10^2	1.52	10.30	2.14
10^3	1.37	10.51	2.00
10^4	1.35	10.51	1.99
10^5	1.35	10.50	1.98
10^6	1.35	10.50	1.98

Example (B)

κ	no osc
0	—
10^{-3}	1.05
10^{-2}	1.05
10^{-1}	1.05
1	1.14
10	1.85
10^2	1.64
10^3	1.66
10^4	1.67
10^5	1.67
10^6	1.67

Results – I_{eff}

$\eta_K(\widetilde{\mathbf{y}}_K)$ ... quadratic $\widetilde{\mathbf{y}}_K$, minimization

Example (A)

κ	no osc	const osc	linear osc
0	1.46	1.73	1.42
10^{-3}	493.75	1.72	1.41
10^{-2}	49.39	1.72	1.41
10^{-1}	5.12	1.72	1.41
1	1.46	1.75	1.42
10	1.49	4.14	1.75
10^2	1.24	10.24	1.86
10^3	1.17	10.39	1.79
10^4	1.17	10.38	1.78
10^5	1.17	10.38	1.78
10^6	1.17	10.38	1.78

Example (B)

κ	no osc
0	—
10^{-3}	1.05
10^{-2}	1.05
10^{-1}	1.05
1	1.05
10	1.54
10^2	1.37
10^3	1.41
10^4	1.42
10^5	1.42
10^6	1.42

Conclusions

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega \subset \mathbb{R}^2$$
$$u = 0 \quad \text{on } \partial\Omega$$

$$|||u - u_h|||^2 \leq \sum_{K \in \mathcal{T}_h} \left(\left\| \frac{1}{\kappa}(f - \kappa^2 u_h + \operatorname{div} \mathbf{y}_K) \right\|_{0,K}^2 + \|\mathbf{y}_K - \nabla u_h\|_{0,K}^2 \right)$$

- No constants
- Completely computable
- Guaranteed upper bound
- Elementwise local
- Robust for $\kappa \to \infty$, $\kappa \to 0$, $h \to 0$

Thank you for your attention

Tomáš Vejchodský and Mark Ainsworth

Institute of Mathematics, Academy of Sciences
Žitná 25, 115 67 Prague 1
Czech Republic

Modelling 2009, June 22–26, 2009, Rožnov pod Radhoštěm