On efficient solution of linear systems arising in *hp*-FEM

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ENUMATH'07, September 10–14 2007, Graz

Outline

- introduction, notation
- I. Static condensation of internal DOFs
- II. Partial orthogonalization of basis functions
- III. ILU-PCG
- Numerical examples

hp-FEM

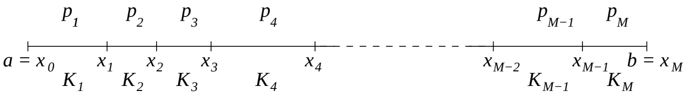
Weak

$$u \in V \quad a(u,v) = \mathcal{F}(v) \quad \forall v \in V$$

hp-FEM

$$u_{hp} \in V_{hp} \quad a(u_{hp},v_{hp}) = \mathcal{F}(v_{hp}) \quad \forall v_{hp} \in V_{hp}$$

$$V_{hp} = \{v_{hp} \in V : v_{hp}|_K \in P^{p_K}(K),\ K \in \mathcal{T}_{hp}\}$$

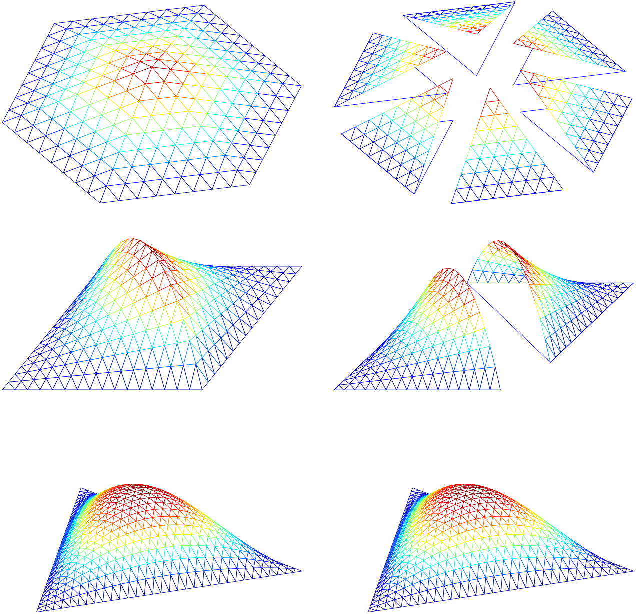


$$N = \dim V_{hp} \qquad \varphi_1, \dots, \varphi_N \qquad \text{basis functions}$$

$$N^K = \dim\left(P^{p_K}(K) \cap V_{hp}|_K\right) \qquad \varphi_1^K, \dots, \varphi_{N^K}^K \qquad \text{shape functions}$$

$$\varphi_i|_K = \varphi^K_{\iota_K^{-1}(i)} \qquad i = 1,2,\dots,N$$

hp-FEM basis and shape functions



Global and local stiffness matrices

- $\mathbb{A}\mathbb{Y} = \mathbb{F} \qquad \mathbb{A}_{ij} = a(\varphi_j, \varphi_i), \quad \mathbb{F}_i = \mathcal{F}(\varphi_i),$

$$i, j = 1, 2, \dots N$$

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  $$i, j = 1, 2, \dots N$$

- $a(\varphi_j, \varphi_i) = \sum_{K \in \mathcal{T}_{hp}} a_K(\varphi_j, \varphi_i) \qquad \mathcal{F}(\varphi_i) = \sum_{K \in \mathcal{T}_{hp}} \mathcal{F}_K(\varphi_i)$

  Example:

  $$a(\varphi_j, \varphi_i) = \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \, \mathrm{d}x \qquad \mathcal{F}(\varphi_i) = \int_\Omega f\varphi_i \, \mathrm{d}x$$

  $$a_K(\varphi_j, \varphi_i) = \int_K \nabla\varphi_j \cdot \nabla\varphi_i \, \mathrm{d}x \qquad \mathcal{F}_K(\varphi_i) = \int_K f\varphi_i \, \mathrm{d}x$$

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- $\mathbb{A}^K, \quad \mathbb{F}^K \qquad \mathbb{A}^K_{\ell m} = a_K(\varphi_{\iota_K(m)}, \varphi_{\iota_K(\ell)}), \quad \mathbb{F}^K_\ell = \mathcal{F}_K(\varphi_{\iota_K(\ell)})$

  $$\ell, m = 1, 2, \ldots N^K$$

  Connectivity mapping $\quad \iota_K : \{1, 2, \ldots, N^K\} \mapsto \{1, 2, \ldots, N\}$

Global and local stiffness matrices

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- $\mathbb{A}_{ij} = \sum_{K \in \mathcal{T}_{hp}} \mathbb{A}^K_{\iota_K^{-1}(i), \iota_K^{-1}(j)} \qquad \mathbb{F}_i = \sum_{K \in \mathcal{T}_{hp}} \mathbb{F}^K_{\iota_K^{-1}(i)}$

Enumeration – bubbles first

$$\underbrace{\varphi_1, \varphi_2, \dots, \varphi_M}_{\text{bubbles}}, \underbrace{\varphi_{M+1}, \dots, \varphi_N}_{\text{non-bubbles}} \qquad \mathbb{A}_{ij} = a(\varphi_j, \varphi_i)$$

$$\mathbb{A} = \begin{pmatrix} A & B^T \\ B & D \end{pmatrix} = \left(\begin{array}{cccc|c} \boxed{A^{K_1}} & & & & \\ & \boxed{A^{K_2}} & & & B^T \\ & & \boxed{A^{K_3}} & & \\ & & & \ddots & \\ \hline & & B & & D \end{array}\right)$$

$$\underbrace{\varphi_1^K, \varphi_2^K, \dots, \varphi_{M^K}^K}_{\text{bubbles}}, \underbrace{\varphi_{M^K+1}^K, \dots, \varphi_{N^K}^K}_{\text{non-bubbles}} \qquad \mathbb{A}_{\ell m}^K = a_K(\varphi_{\iota_K(m)}, \varphi_{\iota_K(\ell)})$$

$$\mathbb{A}^K = \begin{pmatrix} A^K & (B^K)^T \\ B^K & D^K \end{pmatrix}$$

I. Static condensation of internal DOFs

$$\mathbb{A}\mathbb{Y} = \mathbb{F}$$

$$\begin{pmatrix} A & B^T \\ B & D \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}$$

$$\mathbf{x} = A^{-1}(F - B^T\mathbf{y})$$

$$S\mathbf{y} = \widetilde{G}$$

$$S = D - BA^{-1}B^T$$

$$\widetilde{G} = G - BA^{-1}F$$

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$$\mathbb{A}^K\mathbb{Y}^K \neq \mathbb{F}^K$$

$$A^K\mathbf{x}^K + (B^K)^T\mathbf{y}^K = F^K$$
$$B^K\mathbf{x}^K + D^K\mathbf{y}^K \neq G^K$$

$$\mathbf{x}^K = (A^K)^{-1}\left(F^K - (B^K)^T\mathbf{y}^K\right)$$

$$S^K = D^K - B^K(A^K)^{-1}(B^K)^T$$
$$\widetilde{G}^K = G^K - B^K(A^K)^{-1}F^K$$

I. Static condensation of internal DOFs

$$\mathbb{A}\mathbb{Y} = \mathbb{F} \qquad\qquad \mathbb{A}^K\mathbb{Y}^K \neq \mathbb{F}^K$$

$$\begin{pmatrix} A & B^T \\ B & D \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix} \qquad \begin{aligned} A^K\mathbf{x}^K + (B^K)^T\mathbf{y}^K &= F^K \\ B^K\mathbf{x}^K + D^K\mathbf{y}^K &\neq G^K \end{aligned}$$

$$\begin{aligned} \mathbf{x} &= A^{-1}(F - B^T\mathbf{y}) \\ S\mathbf{y} &= \widetilde{G} \end{aligned} \qquad \mathbf{x}^K = (A^K)^{-1}\left(F^K - (B^K)^T\mathbf{y}^K\right)$$

$$\begin{aligned} S &= D - BA^{-1}B^T \\ \widetilde{G} &= G - BA^{-1}F \end{aligned} \qquad \begin{aligned} S^K &= D^K - B^K(A^K)^{-1}(B^K)^T \\ \widetilde{G}^K &= G^K - B^K(A^K)^{-1}F^K \end{aligned}$$

$$S_{ij} = \sum_{K\in\mathcal{T}_{hp}} S^K_{\iota_K^{-1}(M+i),\iota_K^{-1}(M+j)} \qquad \widetilde{G}_i = \sum_{K\in\mathcal{T}_{hp}} \widetilde{G}^K_{\iota_K^{-1}(M+i)}$$

$$i,j = 1,2,\ldots,N-M$$

Efficient algorithm

1. Build $\mathbb{A}^K$ and $\mathbb{F}^K$.
2. Compute $S^K = D^K - B^K(A^K)^{-1}(B^K)^T$
   and $\widetilde{G}^K = G^K - B^K(A^K)^{-1}F^K$.
3. Assemble S from S^K and $\widetilde{G}$ from $\widetilde{G}^K$.
4. Solve $S\mathbf{y} = \widetilde{G}$.
5. Disassemble $\mathbf{y}$ to $\mathbf{y}^K$, i.e., $\mathbf{y}^K_\ell = \mathbf{y}_{\iota_K(M^K+\ell)}$, $\ell = 1, \dots, N^K - M^K$.
6. Compute $\mathbf{x}^K = (A^K)^{-1}\Big(F_K - (B^K)^T\mathbf{y}^K\Big)$.
7. Concatenate $\mathbf{x}^K$ into $\mathbf{x} \in \mathbb{R}^M$, i.e., $\mathbf{x}_j = \mathbf{x}^K_{\iota_K^{-1}(j)}$, $j = 1, \dots, M$.

II. Partial orthogonalization

Original basis:

$$\underbrace{\varphi_1^b, \varphi_2^b, \dots, \varphi_M^b}_{\text{bubbles}}, \underbrace{\varphi_{M+1}^n, \dots, \varphi_N^n}_{\text{non-bubbles}}$$

$$\mathbb{A} = \begin{pmatrix} A & B^T \\ B & D \end{pmatrix} \qquad \mathbb{F} = \begin{pmatrix} F \\ G \end{pmatrix}$$

New basis:

$$\underbrace{\widetilde{\varphi}_1^b, \widetilde{\varphi}_2^b, \dots, \widetilde{\varphi}_M^b}_{\text{bubbles}}, \underbrace{\widetilde{\varphi}_{M+1}^n, \dots, \widetilde{\varphi}_N^n}_{\text{non-bubbles}}$$

$$\widetilde{\mathbb{A}} = \begin{pmatrix} A & 0 \\ 0 & S \end{pmatrix} \qquad \widetilde{\mathbb{F}} = \begin{pmatrix} F \\ \widetilde{G} \end{pmatrix}$$

$$S = D - BA^{-1}B^T$$
$$\widetilde{G} = G - BA^{-1}F$$

$$\widetilde{\varphi}_i^b = \varphi_i^b, \qquad i = 1, 2, \dots, M$$

$$\widetilde{\varphi}_{M+j}^n = \varphi_{M+j}^n - \sum_{k=1}^{M} Q_{jk} \varphi_k^b, \quad j = 1, 2, \dots, N - M$$

Q_{jk} are such that $\quad a(\widetilde{\varphi}_{M+j}^n, \varphi_i^b) = 0 \quad \dots \quad Q = BA^{-1}$

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$$S = D - BA^{-1}B^T$$

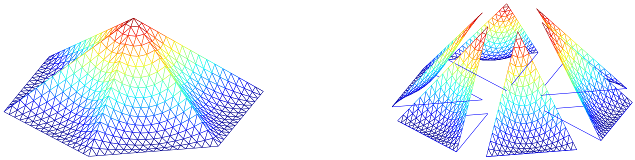
$$\widetilde{G} = G - BA^{-1}F$$

$$u_{hp} = \sum_{j=1}^{M} \widetilde{x}_j \widetilde{\varphi}_j^b + \sum_{j=1}^{N-M} \widetilde{y}_j \widetilde{\varphi}_{M+j}^n = \sum_{j=1}^{M} x_j \varphi_j^b + \sum_{j=1}^{N-M} y_j \varphi_{M+j}^n$$

$$\mathbf{x} = \widetilde{\mathbf{x}} - A^{-1}B^T\widetilde{\mathbf{y}} \quad \Leftrightarrow \quad \mathbf{x} = A^{-1}(F - B^T\mathbf{y}) \qquad A\widetilde{\mathbf{x}} = F$$

$$\mathbf{y} = \widetilde{\mathbf{y}} \qquad S\widetilde{\mathbf{y}} = \widetilde{G}$$

II. Partial orthogonalization is local



$$\operatorname{supp} \varphi^n_{M+j} = \operatorname{supp} \widetilde{\varphi}^n_{M+j} \quad \Longrightarrow \quad D_{ij} = 0 \quad \Leftrightarrow \quad S_{ij} = 0$$

$$\widetilde{\mathbb{A}}^K = \begin{pmatrix} A^K & 0 \\ 0 & S^K \end{pmatrix} \quad \xrightarrow{\text{assembling}} \quad \widetilde{\mathbb{A}} = \begin{pmatrix} A & 0 \\ 0 & S \end{pmatrix}$$

$$S^K = D^K - B^K (A^K)^{-1} (B^K)^T$$

III. ILU-PCG

$$\mathbb{A}\mathbb{Y} = \mathbb{F}$$

$$\underbrace{\widehat{\mathbb{L}}^{-1}\mathbb{A}\widehat{\mathbb{U}}^{-1}}_{\widehat{\mathbb{A}}}\,\underbrace{\widehat{\mathbb{U}}\mathbb{Y}}_{\widehat{\mathbb{Y}}} = \underbrace{\widehat{\mathbb{L}}^{-1}\mathbb{F}}_{\widehat{\mathbb{F}}}$$

$$\xrightarrow{\text{preconditioning}}$$

$$\widehat{\mathbb{A}}\widehat{\mathbb{Y}} = \widehat{\mathbb{F}}$$

$$\widehat{\mathbb{U}}\mathbb{Y} = \widehat{\mathbb{Y}}$$

$$\mathbb{A} = \mathbb{L}\mathbb{U} \quad \mathbb{L} = \begin{pmatrix} L_A & 0 \\ L_B & L_D \end{pmatrix} \quad \mathbb{U} = \begin{pmatrix} U_A & U_B \\ 0 & U_D \end{pmatrix}$$

$$\mathbb{A} \approx \widehat{\mathbb{L}}\widehat{\mathbb{U}} \quad \widehat{\mathbb{L}} = \begin{pmatrix} L_A & 0 \\ L_B & \widehat{L}_D \end{pmatrix} \quad \widehat{\mathbb{U}} = \begin{pmatrix} U_A & U_B \\ 0 & \widehat{U}_D \end{pmatrix}$$

$$S \approx \widehat{L}_D \widehat{U}_D$$

$$\begin{pmatrix} A & B^T \\ B & D \end{pmatrix} \xrightarrow{M \text{ steps}} \begin{pmatrix} L_A \backslash U_A & U_B \\ L_B & S \end{pmatrix} \xrightarrow{N-M \text{ steps}} \begin{pmatrix} L_A \backslash U_A & U_B \\ L_B & \widehat{L}_D \backslash \widehat{U}_D \end{pmatrix}$$

III. ILU-PCG

$$\mathbb{A}\mathbb{Y} = \mathbb{F}$$

$$\underbrace{\widehat{\mathbb{L}}^{-1}\mathbb{A}\widehat{\mathbb{U}}^{-1}}_{\widehat{\mathbb{A}}}\underbrace{\widehat{\mathbb{U}}\mathbb{Y}}_{\widehat{\mathbb{Y}}} = \underbrace{\widehat{\mathbb{L}}^{-1}\mathbb{F}}_{\widehat{\mathbb{F}}}$$

$\xrightarrow{\text{preconditioning}}$

$$\widehat{\mathbb{A}}\widehat{\mathbb{Y}} = \widehat{\mathbb{F}}$$

$$\widehat{\mathbb{U}}\mathbb{Y} = \widehat{\mathbb{Y}}$$

$$\mathbb{A} = \mathbb{L}\mathbb{U} \quad \mathbb{L} = \begin{pmatrix} L_A & 0 \\ L_B & L_D \end{pmatrix} \quad \mathbb{U} = \begin{pmatrix} U_A & U_B \\ 0 & U_D \end{pmatrix}$$

$$\mathbb{A} \approx \widehat{\mathbb{L}}\widehat{\mathbb{U}} \quad \widehat{\mathbb{L}} = \begin{pmatrix} L_A & 0 \\ L_B & \widehat{L}_D \end{pmatrix} \quad \widehat{\mathbb{U}} = \begin{pmatrix} U_A & U_B \\ 0 & \widehat{U}_D \end{pmatrix}$$

$$S \approx \widehat{L}_D\widehat{U}_D$$

$$\widehat{\mathbb{A}} = \widehat{\mathbb{L}}^{-1}\mathbb{A}\widehat{\mathbb{U}}^{-1} = \begin{pmatrix} I & 0 \\ 0 & \widehat{L}_D^{-1}S\widehat{U}_D^{-1} \end{pmatrix}$$

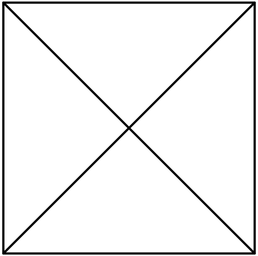
Conditioning

- $\text{cond}(S) \leq \text{cond}(\mathbb{A})$ [Mandel 1990]
- $\text{nnz}(S) = \text{nnz}(D) \leq \text{nnz}(\mathbb{A})$
- S is independent from bubble basis functions

Numerical Experiments

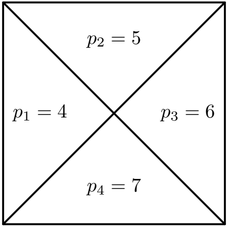
$$-\Delta u = f \quad \text{in } \Omega = (-1,1)^2$$
$$u = 0 \quad \text{on } \partial\Omega$$

$$u = \cos(x\pi/2)\cos(y\pi/2) \qquad f = u\pi^2/2$$



Numerical Experiments: *h*-FEM

ref. step	N	$N-M$	memory $[\times 10^3]$	solve CPU time [s] stat. c.	ILU-CG	N_{iter}	rel. err. [%]
0	50	16	2.7	0.004	0.005	3	1.2
1	225	89	11.0	0.012	0.017	5	$5.4_{\times 10}{-2}$
2	953	409	43.9	0.049	0.130	7	$3.4_{\times 10}{-3}$
3	3921	1745	175.7	0.389	1.665	11	$2.1_{\times 10}{-4}$
4	15905	7201	703.0	4.697	26.10	21	$1.3_{\times 10}{-5}$
5	64065	29249	2 811.9	71.10	415.5	40	$8.2_{\times 10}{-7}$



Numerical Experiments: p-FEM

p	N	$N-M$	memory $[\times 10^3]$	solve CPU time [s] stat. c.	ILU-CG	N_{iter}	rel. err. [%]
1	113	113	2.3	—	0.007	8	$1.0_{\times 10}{+1}$
2	481	481	9.2	—	0.049	10	$5.1_{\times 10}{-1}$
3	1105	849	25.6	0.102	0.105	12	$1.7_{\times 10}{-2}$
4	1985	1217	57.6	0.171	0.350	13	$4.1_{\times 10}{-4}$
5	3121	1585	112.9	0.302	0.987	14	$7.7_{\times 10}{-6}$
6	4513	1953	200.7	0.504	2.333	14	$1.3_{\times 10}{-7}$
7	6161	2321	331.8	0.808	4.933	15	$1.7_{\times 10}{-9}$
8	8065	2689	518.4	1.218	9.582	16	$2.3_{\times 10}{-11}$
9	10225	3057	774.4	1.773	17.407	16	$1.1_{\times 10}{-11}$

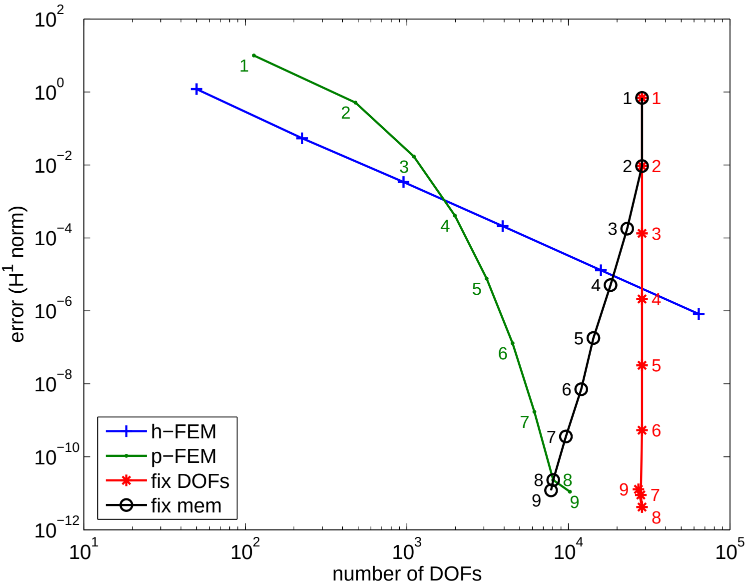
Numerical Experiments: fixed N_{DOFs}

p	N	$N-M$	memory $[\times 10^3]$	solve CPU time [s] stat. c.	ILU-CG	N_{iter}	rel. err. [%]
1	28561	28561	518	—	16.9	82	$6.9_{\times 10}{-1}$
2	28561	28561	518	—	28.1	83	$9.3_{\times 10}{-3}$
3	28561	22161	640	26.7	44.4	53	$1.3_{\times 10}{-4}$
4	28561	17761	810	21.2	59.8	41	$2.1_{\times 10}{-6}$
5	28561	14737	1 016	17.6	74.7	34	$3.2_{\times 10}{-8}$
6	28561	12561	1 254	15.0	89.4	29	$5.3_{\times 10}{-10}$
7	28085	11221	1 498	12.8	101.2	26	$8.9_{\times 10}{-12}$
8	28561	9661	1 823	11.7	118.5	24	$4.2_{\times 10}{-12}$
9	27145	9663	2 045	9.6	121.4	22	$1.3_{\times 10}{-11}$

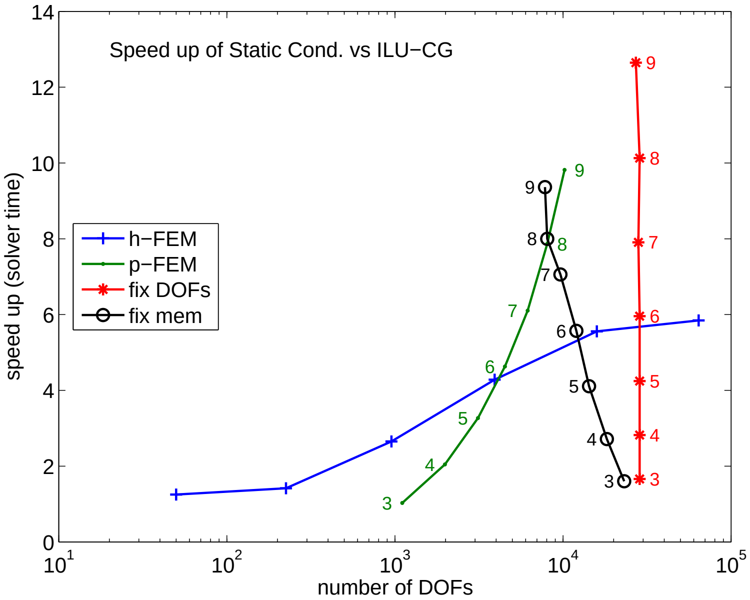
Numerical Experiments: fixed memory

p	N	$N-M$	memory $[\times 10^3]$	solve CPU time [s] stat. c.	ILU-CG	N_{iter}	rel. err. [%]
1	28561	28561	518	—	15.0	65	$6.9_{\times 10}{-1}$
2	28561	28561	518	—	26.8	62	$9.3_{\times 10}{-3}$
3	23113	17929	518	17.8	28.5	40	$1.8_{\times 10}{-4}$
4	18241	11329	518	8.8	23.9	28	$5.1_{\times 10}{-6}$
5	14281	7345	510	4.5	18.5	22	$1.8_{\times 10}{-7}$
6	12013	5253	530	2.8	15.6	19	$7.1_{\times 10}{-9}$
7	9661	3661	518	1.7	12.0	17	$3.6_{\times 10}{-10}$
8	8065	2689	518	1.2	9.6	16	$2.3_{\times 10}{-11}$
9	7813	2325	593	1.1	10.3	15	$1.2_{\times 10}{-11}$

Relative error comparison

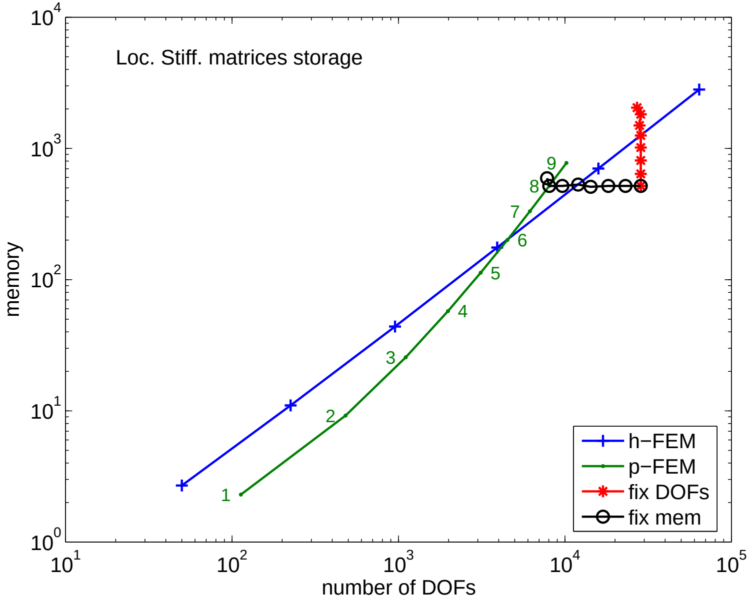


Speed up

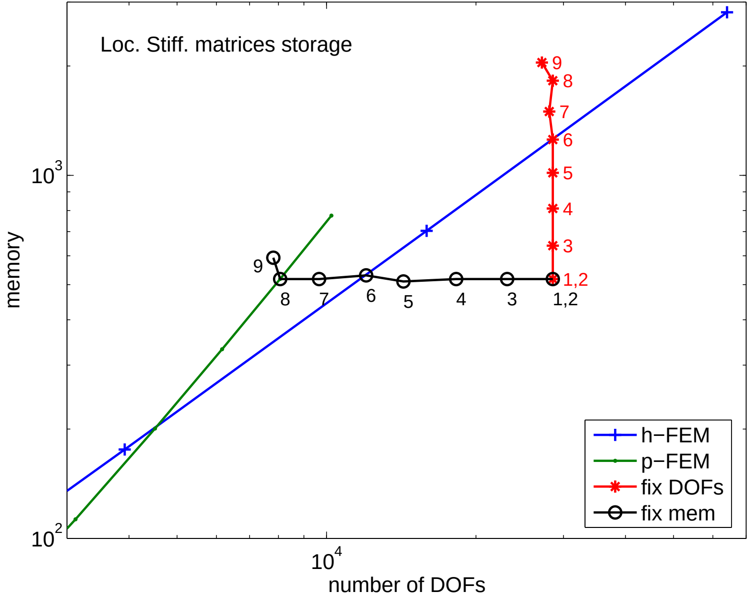


Speed up of Static Cond. vs ILU−CG
speed up (solver time)
number of DOFs
h−FEM
p−FEM
fix DOFs
fix mem
0
2
4
6
8
10
12
14
10^1
10^2
10^3
10^4
10^5
3
4
5
6
7
8
9

Memory requirements



Memory requirements – zoom



Thank you for your attention

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