

A posteriori error estimates in the finite element method

A survey of techniques

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March 5, 2009, KMA ZČU, Plzeň



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Poisson problem



- ▶ Classical formulation: find $u \in C^2(\Omega) \cap C(\bar{\Omega})$:

$$-\Delta u = f \quad \text{in } \Omega \quad u = 0 \quad \text{on } \partial\Omega$$

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- ▶ Weak formulation: $V = H_0^1(\Omega)$

$$u \in V : \quad \underbrace{\mathcal{B}(u, v)} = \underbrace{\mathcal{F}(v)} \quad \forall v \in V$$
$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx$$



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- ▶ Galerkin method $V_h \subset V \quad \dim V_h < \infty$

$$u_h \in V_h : \quad \mathcal{B}(u_h, v_h) = \mathcal{F}(v_h) \quad \forall v_h \in V_h \quad \Leftrightarrow \quad Ay = F$$

$$u_h(x) = \sum_{j=1}^N y_j \varphi_j(x) \quad \sum_{j=1}^N y_j \underbrace{\mathcal{B}(\varphi_j, \varphi_i)}_{A_{ij}} = \underbrace{\mathcal{F}(\varphi_i)}_{F_i}$$

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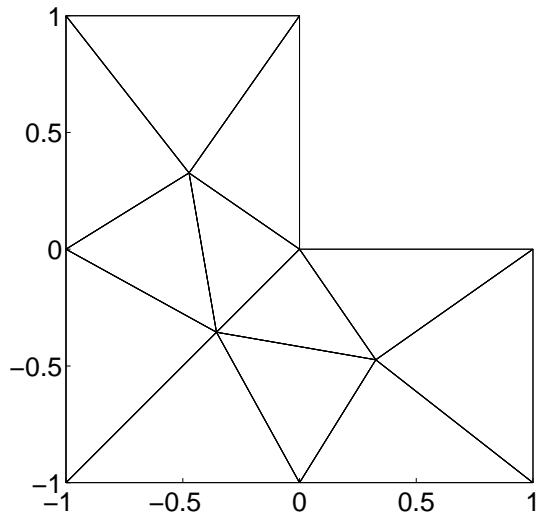
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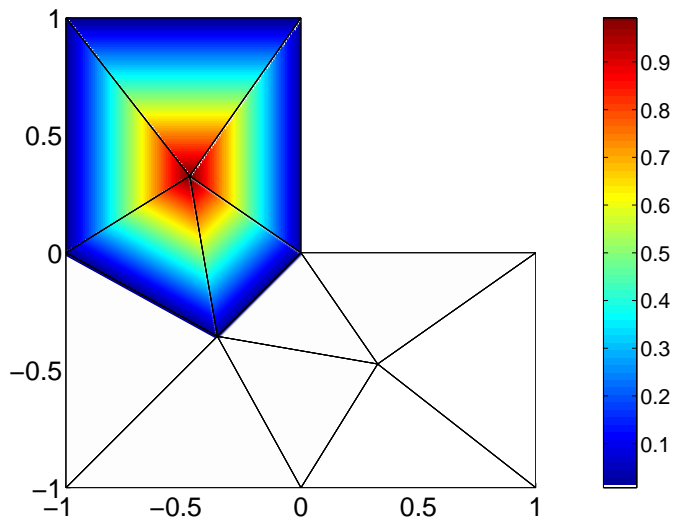
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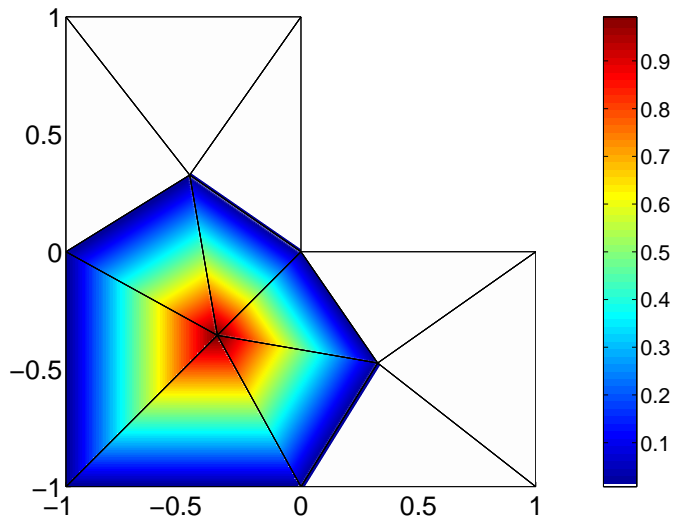
- ▶ FEM $V_h = \{v_h \in V : v_h|_K \in P^1(K) \quad \forall K \in \mathcal{T}_h\}$

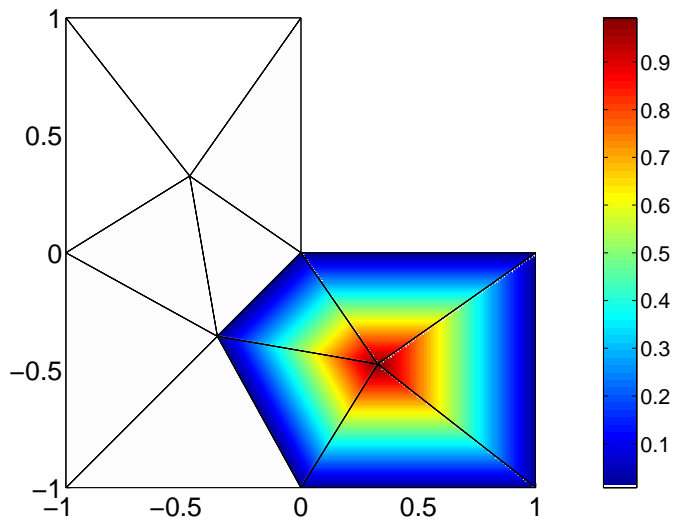
$$\varphi_1, \dots, \varphi_N \dots \text{FEM basis functions} \quad \varphi_i(x_j) = \delta_{ij}$$

Mesh 1











- ▶ Discretization error: $e = u - u_h$
- ▶ Energy norm: $\|u\|^2 = \mathcal{B}(u, u) = |u|_{H^1(\Omega)}^2$
- ▶ Céa's lemma: $\|e\| = \inf_{v_h \in V_h} \|u - v_h\| = \text{dist}(u, V_h)$
- ▶ Lagrange interpolation: $\pi_h^{\text{Lag}} : C(\bar{\Omega}) \mapsto V_h$
 $v \in H^2(\Omega) \cap C(\bar{\Omega}) \Rightarrow \|v - \pi_h^{\text{Lag}} v\|_{H^1(\Omega)} \leq C h |v|_{H^2(\Omega)}$
- ▶ $u \in H^2(\Omega) \cap C(\bar{\Omega}) \Rightarrow \|e\| \leq C h |u|_{H^2(\Omega)}$



A posteriori error estimates

Definition

- ▶ $\|e\| \approx \eta$ (or $\|e\| \leq \eta$, or $\eta \leq \|e\|$)
- ▶ $\eta = \eta(u_h, f, \Omega, \mathcal{T}_h, \dots)$

Properties

- ▶ Local:
$$\eta^2 = \sum_{K \in \mathcal{T}_h} \eta_K^2$$
- ▶ Guaranteed upper (lower) bound: $\|e\| \leq \eta$ ($\eta \leq \|e\|$)
- ▶ Asymptotic exactness:
$$\lim_{h \rightarrow 0} \frac{\eta}{\|e\|} = 1, \quad l_{\text{eff}} = \frac{\eta}{\|e\|}$$
- ▶ Efficient and reliable: $C_1 \eta \leq \|e\| \leq C_2 \eta$
- ▶ Robust: C_1 and C_2 are independent from quantities like coefficients in the equation, mesh aspect ratio etc.



$$\begin{aligned}u &\in V : & \mathcal{B}(u, v) &= \mathcal{F}(v) & \forall v \in V \\u_h &\in V_h : & \mathcal{B}(u_h, v_h) &= \mathcal{F}(v_h) & \forall v_h \in V_h\end{aligned}$$

► Residual: $\mathcal{R}(v) = \mathcal{F}(v) - \mathcal{B}(u_h, v) \quad \forall v \in V$

► Residual equation: $e \in V : \mathcal{B}(e, v) = \mathcal{R}(v) \quad \forall v \in V$

$$\left[\mathcal{B}(u, v) - \mathcal{B}(u_h, v) = \mathcal{F}(v) - \mathcal{B}(u_h, v) \quad \forall v \in V \right]$$

► Galerkin orthogonality: $\mathcal{B}(e, v_h) = 0 \quad \forall v_h \in V_h$

► $\|e\| = \sup_{0 \neq v \in V} \frac{|\mathcal{B}(e, v)|}{\|v\|} = \sup_{0 \neq v \in V} \frac{|\mathcal{R}(v)|}{\|v\|} = \|\mathcal{R}\|_{V^*}$



Explicit residual estimates

- Residual splitting: $\mathcal{R}(v) = \sum_{K \in \mathcal{T}_h} \int_K r v \, dx + \sum_{\ell} \int_{\ell} J_{\ell} v \, ds$
- $$r = f + \Delta u_h \quad J_{\ell} = (\nabla u_h^+ - \nabla u_h^-) \cdot \nu_{\ell}$$

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$$\begin{aligned} \mathcal{R}(v) &= \mathcal{F}(v) - \mathcal{B}(u_h, v) = \sum_{K \in \mathcal{T}_h} \left(\int_K f v \, dx - \int_K \nabla u_h \cdot \nabla v \, dx \right) \\ &= \sum_{K \in \mathcal{T}_h} \left(\int_K f v \, dx + \int_K \Delta u_h v \, dx - \int_{\partial K} \nabla u_h \cdot \nu_K v \, ds \right) \\ &= \sum_{K \in \mathcal{T}_h} \int_K r v \, dx + \sum_{\ell} \int_{\ell} (\nabla u_h^+ - \nabla u_h^-) \cdot \nu_{\ell} v \, ds \end{aligned}$$



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- ▶ Clément inter.: $\pi_h^{\text{Cl}} : V \mapsto V_h$ $\|v - \pi_h^{\text{Cl}} v\|_{0,K} \leq C_1 |K|^{1/2} \|\nabla v\|_{0,\omega_K}$
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$$\begin{aligned} \|e\|^2 &= \mathcal{B}(e, e) = \mathcal{R}(e) = \mathcal{R}(e - \pi_h^{\text{Cl}} e) \\ &= \sum_{K \in \mathcal{T}_h} \int_K r(e - \pi_h^{\text{Cl}} e) \, dx + \sum_{\ell} \int_{\ell} J_{\ell}(e - \pi_h^{\text{Cl}} e) \, ds \\ &\leq \sum_{K \in \mathcal{T}_h} C_1 \|r\|_{0,K} |K|^{1/2} \|\nabla e\|_{0,\omega_K} + \sum_{\ell} C_2 \|J_{\ell}\|_{0,\ell} |\ell|^{1/2} \|\nabla e\|_{0,\omega_{\ell}} \\ &\leq C_3 \left(\sum_{K \in \mathcal{T}_h} |K| \|r\|_{0,K}^2 + \sum_{\ell} |\ell| \|J_{\ell}\|_{0,\ell}^2 \right) + \varepsilon \|e\|^2 \end{aligned}$$

Explicit residual estimates



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$$\|e\|^2 \leq C_4 \underbrace{\left(\sum_{K \in \mathcal{T}_h} |K| \|r\|_{0,K}^2 + \sum_{\ell} |\ell| \|J_{\ell}\|_{0,\ell}^2 \right)}_{(\eta^{\text{expl}})^2}$$

Implicit residual estimates – Dirichlet



- ▶ Local Dirichlet problems:

$$e_K^{\text{Dir}} \in H_0^1(K) : \quad \mathcal{B}(e_K^{\text{Dir}}, v) = \mathcal{R}(v) \quad \forall v \in H_0^1(K)$$

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- ▶ Approximate local problems: $V_{0,h}(K) \subset H_0^1(K)$

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- ▶ $e^{\text{Dir}}|_K = e_K^{\text{Dir}} \quad \forall K \in \mathcal{T}_h; \quad V_0 = \{v \in V : v|_K \in H_0^1(K)\} \subset V$

- ▶ $e_h^{\text{Dir}}|_K = e_{K,h}^{\text{Dir}} \quad \forall K \in \mathcal{T}_h; \quad V_{0,h} = \{v \in V : v|_K \in V_{0,h}(K)\} \subset V_0$

- ▶ **Theorem:** $\|e_h^{\text{Dir}}\| \leq \|e^{\text{Dir}}\| \leq \|e\|$



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- ▶ **Theorem:** $\|e_h^{\text{Dir}}\| \leq \|e^{\text{Dir}}\| \leq \|e\|$

Proof:

- ▶ $e^{\text{Dir}} \in V_0 \quad v \in V_0$

$$\mathcal{B}(e - e^{\text{Dir}}, v) = \mathcal{R}(v) - \mathcal{R}(v) = 0$$

$$\Rightarrow \quad \mathcal{B}(e, e^{\text{Dir}}) = \|e^{\text{Dir}}\|^2$$

$$\begin{aligned} \Rightarrow \quad \|e - e^{\text{Dir}}\|^2 &= \|e\|^2 - 2\mathcal{B}(e, e^{\text{Dir}}) + \|e^{\text{Dir}}\|^2 \\ &= \|e\|^2 - \|e^{\text{Dir}}\|^2 \end{aligned}$$

- ▶ $\mathcal{B}(e^{\text{Dir}} - e_h^{\text{Dir}}, v_h) = \mathcal{R}(v_h) - \mathcal{R}(v_h) = 0 \quad \forall v_h \in V_{0,h}$



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- ▶ **Theorem:** $\|e_h^{\text{Dir}}\| \leq \|e^{\text{Dir}}\| \leq \|e\|$

- ▶ $\eta_K^{\text{Dir}} = \|e_{K,h}^{\text{Dir}}\|_K \quad (\eta^{\text{Dir}})^2 = \sum_{K \in \mathcal{T}_h} (\eta_K^{\text{Dir}})^2$

Implicit residual estimates – Neumann



- ▶ $e_K^{\text{Neu}} \in H_E^1(K) = \{v \in H^1(K) : v = 0 \text{ on } \partial K \cap \partial\Omega\}$:

$$\mathcal{B}_K(e_K^{\text{Neu}}, v) = \int_K f v \, dx - \mathcal{B}_K(u_h, v) + \int_{\partial K} g_K v \, ds \quad \forall v \in H_E^1(K)$$

$$\mathcal{B}_K(u, v) = \int_K \nabla u \cdot \nabla v \, dx$$

$$\begin{aligned} -\Delta(e_K^{\text{Neu}} + u_h) &= f && \text{in } K \\ \nabla(e_K^{\text{Neu}} + u_h) \cdot \nu_K &= g_K && \text{on } \partial K \setminus \partial\Omega \\ e_K^{\text{Neu}} + u_h &= 0 && \text{on } \partial K \cap \partial\Omega \end{aligned}$$

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- ▶ $g_K|_\ell \in P^1(\ell)$, $\ell \subset \partial K$, $K \in \mathcal{T}_h$, $g_K \approx \nabla u|_K \cdot \nu_K$ on ∂K
- ▶ $g_K|_\ell + g_{K^*}|_\ell = 0$ for $\ell = \partial K \cap \partial K^*$



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- ▶ **Theorem:** $\|e\|^2 \leq \sum_{K \in \mathcal{T}_h} \|e_K^{\text{Neu}}\|_K^2 \quad \|v\|_K^2 = \mathcal{B}_K(v, v)$

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- ▶ **Theorem:** $\|e\|^2 \leq \sum_{K \in \mathcal{T}_h} \|e_K^{\text{Neu}}\|_K^2 \quad \|v\|_K^2 = \mathcal{B}_K(v, v)$

Proof: $v \in V$

$$\begin{aligned} \mathcal{B}(e, v) &= \mathcal{R}(v) = \sum_{K \in \mathcal{T}_h} \left(\int_K f v \, dx - \mathcal{B}_K(u_h, v) + \int_{\partial K} g_K v \, ds \right) \\ &= \sum_{K \in \mathcal{T}_h} \mathcal{B}_K(e_K^{\text{Neu}}, v) \leq \sum_{K \in \mathcal{T}_h} \|e_K^{\text{Neu}}\|_K \|v\|_K \leq \left(\sum_{K \in \mathcal{T}_h} \|e_K^{\text{Neu}}\|_K^2 \right)^{\frac{1}{2}} \|v\| \end{aligned}$$



Implicit residual estimates – Neumann

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- ▶ **Theorem:** $\|e\|^2 \leq \sum_{K \in \mathcal{T}_h} \|e_K^{\text{Neu}}\|_K^2 \quad \|v\|_K^2 = \mathcal{B}_K(v, v)$

- ▶ **Remark:** Approximate Neumann problems:

$$e_{K,h}^{\text{Neu}} \in V_h^{\text{Neu}} \subset H_E^1(K) \Rightarrow \sum_{K \in \mathcal{T}_h} \|e_{K,h}^{\text{Neu}}\|_K^2 \leq \sum_{K \in \mathcal{T}_h} \|e_K^{\text{Neu}}\|_K^2$$

- ▶ $\eta_K^{\text{Neu}} = \|e_{K,h}^{\text{Neu}}\|_K \quad (\eta^{\text{Neu}})^2 = \sum_{K \in \mathcal{T}_h} (\eta_K^{\text{Neu}})^2$

In general: $\|e\| \not\leq \eta^{\text{Neu}}$



Hierarchical (residual) estimates

- ▶ $\widehat{V}_h = V_h \oplus Y_h \quad Y_h \subset V \quad V_h \cap Y_h = \{0\}$
- ▶ $\widehat{u}_h \in \widehat{V}_h: \mathcal{B}(\widehat{u}_h, \widehat{v}_h) = \mathcal{F}(\widehat{v}_h) \quad \forall \widehat{v}_h \in \widehat{V}_h$



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- ▶ $\bar{e}_h \in Y_h: \mathcal{B}(\bar{e}_h, y_h) = \mathcal{R}(y_h) \quad \forall y_h \in Y_h$



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- ▶ $\|e\| \approx \|\bar{e}_h\| \equiv \eta^{\text{Hie}}$

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- ▶ $\bar{e}_h \in Y_h: \quad \mathcal{B}(\bar{e}_h, y_h) = \mathcal{R}(y_h) \quad \forall y_h \in Y_h$
- ▶ $\|e\| \approx \|\bar{e}_h\| \equiv \eta^{\text{Hie}}$
- ▶ Saturation assumption:
 $\exists \beta < 1: \quad \|u - \widehat{u}_h\| \leq \beta \|u - u_h\|$
- ▶ Strengthened Cauchy-Schwarz inequality:
 $\exists \gamma < 1: \quad |\mathcal{B}(v_h, y_h)| \leq \gamma \|v_h\| \|y_h\| \quad \forall v_h \in V_h, y_h \in Y_h$
- ▶ $\|\bar{e}_h\| \leq \|\widehat{e}_h\| \leq \|e\| \leq \frac{1}{(1 - \beta^2)^{\frac{1}{2}}} \|\widehat{e}_h\| \leq \frac{1}{(1 - \beta^2)^{\frac{1}{2}}(1 - \gamma^2)^{\frac{1}{2}}} \|\bar{e}_h\|$

Error majorants

- ▶ Friedrichs inequality: $\|v\|_{0,\Omega} \leq C_\Omega \|\nabla v\|_{0,\Omega} \quad \forall v \in V = H_0^1(\Omega)$
- ▶ $\|e\| \leq C_\Omega \|f + \operatorname{div} \mathbf{y}\|_{0,\Omega} + \|\mathbf{y} - \nabla u_h\|_{0,\Omega} \quad \forall \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega)$



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Proof:

$$v \in V \quad \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega)$$

$$\begin{aligned} \mathcal{B}(e, v) &= \mathcal{R}(v) + \int_{\Omega} \mathbf{y} \cdot \nabla v \, dx + \int_{\Omega} \operatorname{div} \mathbf{y} v \, dx + \int_{\partial\Omega} \mathbf{y} \cdot \boldsymbol{\nu} v \, ds \\ &= \int_{\Omega} (f + \operatorname{div} \mathbf{y}) v \, dx + \int_{\Omega} (\mathbf{y} - \nabla u_h) \cdot \nabla v \, dx \\ &\leq \|f + \operatorname{div} \mathbf{y}\|_{0,\Omega} \|v\|_{0,\Omega} + \|\mathbf{y} - \nabla u_h\|_{0,\Omega} \|\nabla v\|_{0,\Omega} \\ &\leq \left(C_\Omega \|f + \operatorname{div} \mathbf{y}\|_{0,\Omega} + \|\mathbf{y} - \nabla u_h\|_{0,\Omega} \right) \|\nabla v\|_{0,\Omega} \\ &\leq \frac{1}{2} \left(C_\Omega \|f + \operatorname{div} \mathbf{y}\|_{0,\Omega} + \|\mathbf{y} - \nabla u_h\|_{0,\Omega} \right)^2 + \frac{1}{2} \|\nabla v\|_{0,\Omega}^2 \end{aligned}$$

Postprocessing

- ▶ Recovered gradient: $\nabla u_h \mapsto \mathcal{G}(u_h)$
- ▶ $\|e\| \approx \eta^{\text{post}} = \|\mathcal{G}(u_h) - \nabla u_h\|_{0,\Omega}$

Postprocessing

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- ▶ Superconvergence: $\|\nabla u - \mathcal{G}(u_h)\|_{0,\Omega} \leq C_1 h^{1+\epsilon}$
- ▶ Assumption: $\|e\| \geq C_2 h$
- ▶ Theorem: $\lim_{h \rightarrow 0} \frac{\eta^{\text{post}}}{\|e\|} = 1$

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Proof:

$$\frac{\eta^{\text{post}}}{\|e\|} \leq \frac{\|\nabla u - \nabla u_h\|_0}{\|e\|} + \frac{\|\nabla u - \mathcal{G}(u_h)\|_0}{\|e\|} \leq 1 + \frac{C_1 h^{1+\epsilon}}{C_2 h} \rightarrow 1$$

$$\frac{\eta^{\text{post}}}{\|e\|} \geq \frac{\|\nabla u - \nabla u_h\|_0}{\|e\|} - \frac{\|\nabla u - \mathcal{G}(u_h)\|_0}{\|e\|} \geq 1 - \frac{C_1 h^{1+\epsilon}}{C_2 h} \rightarrow 1$$



- ▶ Quantity of interest: $\Phi \in V^*$
- ▶ Adjoint problem: $z \in V : \mathcal{B}(v, z) = \Phi(v) \quad \forall v \in V$
- ▶ Approx. adjoint prob.: $z_h \in V_h : \mathcal{B}(v_h, z_h) = \Phi(v_h) \quad \forall v_h \in V_h$
- ▶ Error representation formula:

$$\Phi(e) = \mathcal{B}(e, z) = \mathcal{R}(z) = \mathcal{R}(z - z_h) = \mathcal{B}(u - u_h, z - z_h)$$

$$\begin{aligned} |\Phi(e)| &\leq \|u - u_h\| \|z - z_h\| \\ &\leq \eta^{\text{pri}} \eta^{\text{adj}} \end{aligned}$$



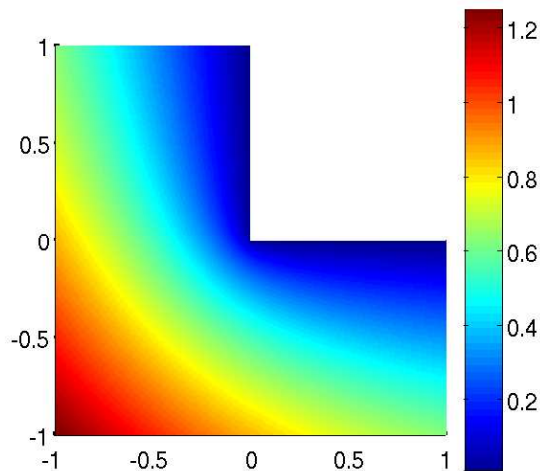
1. Construct the initial mesh \mathcal{T}_h .
2. Find u_h on \mathcal{T}_h .
3. Compute η_K for all $K \in \mathcal{T}_h$.
4. If $\sum_{K \in \mathcal{T}_h} \eta_K^2 \leq \text{TOL}^2 \Rightarrow \text{STOP}$.
5. If $\eta_K \geq \Theta \eta_{K,\max} \Rightarrow \text{mark } K$. $0 < \Theta < 1$
6. Refine marked elements and build the new mesh \mathcal{T}_h .
7. GO TO 2.

Example



$$-\Delta u = 0 \quad \text{in } \Omega$$
$$u = g_D \quad \text{on } \partial\Omega$$

$$u = r^{\frac{2}{3}} \sin \frac{2\theta - \pi}{3}$$

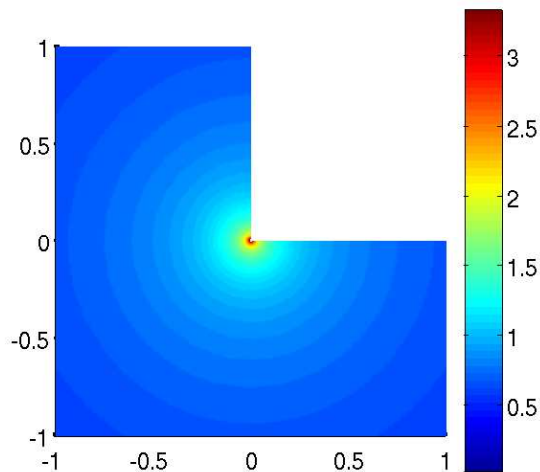


Example

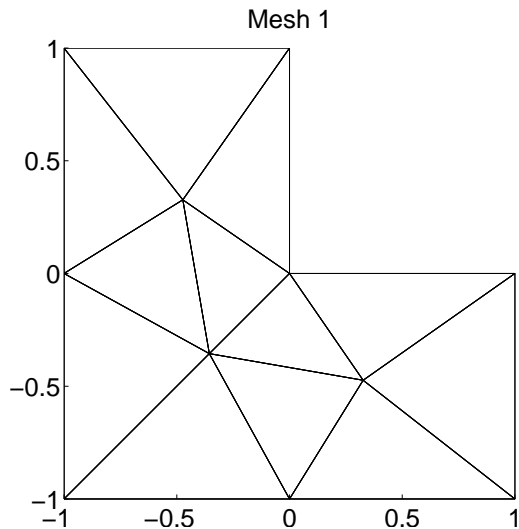


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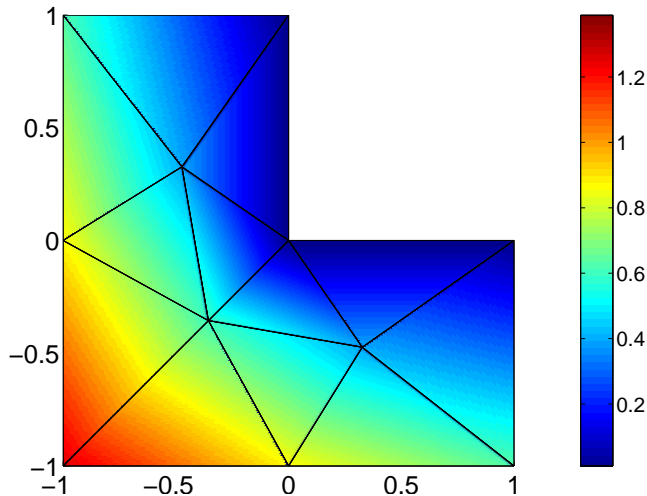
Adaptive algorithm



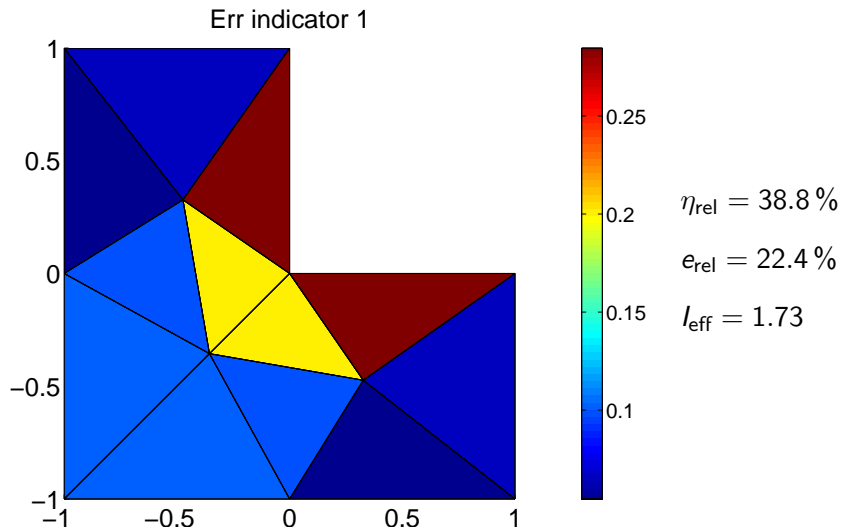
Adaptive algorithm



FEM solution 1



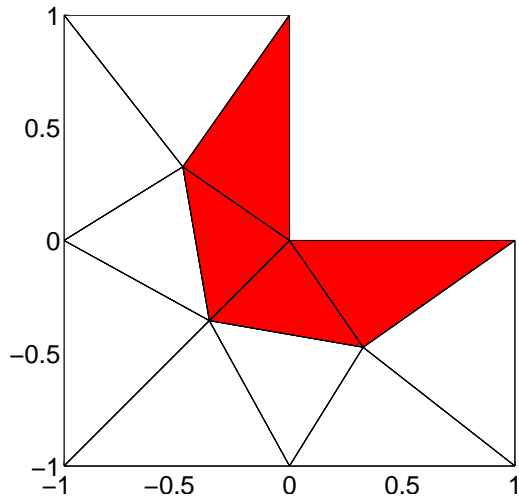
Adaptive algorithm



Adaptive algorithm



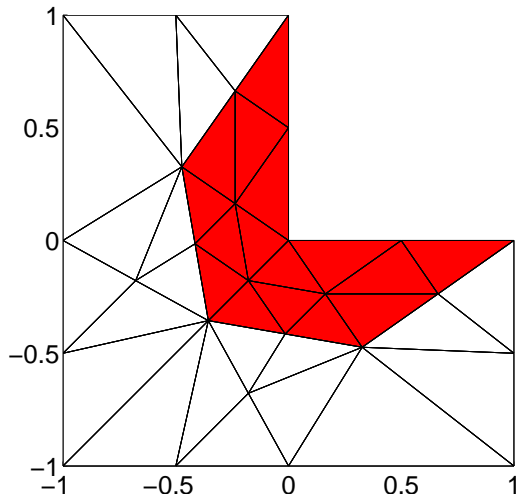
Marked elements 1



Adaptive algorithm



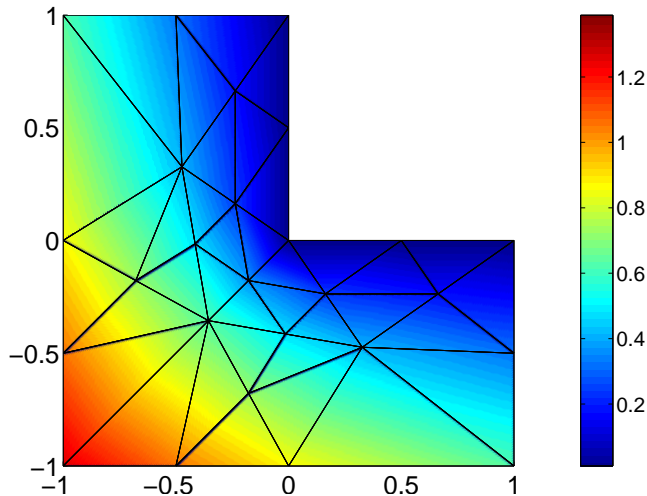
Refined mesh 1



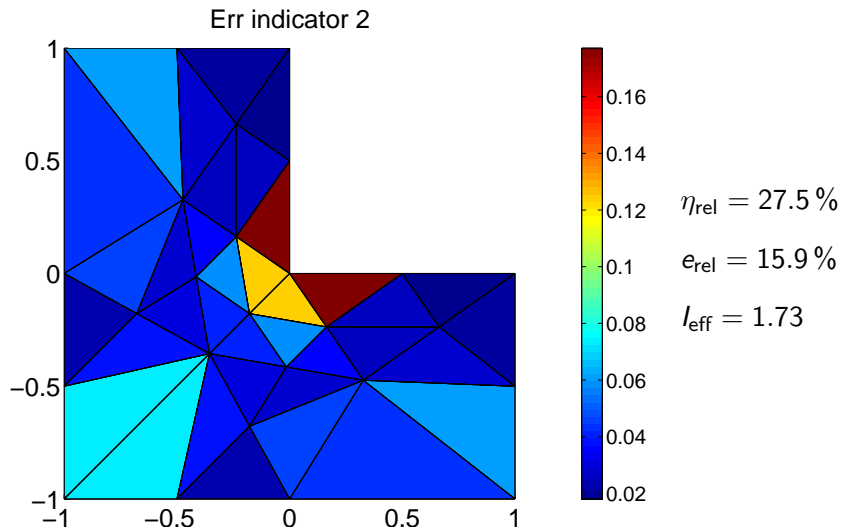
Adaptive algorithm



FEM solution 2



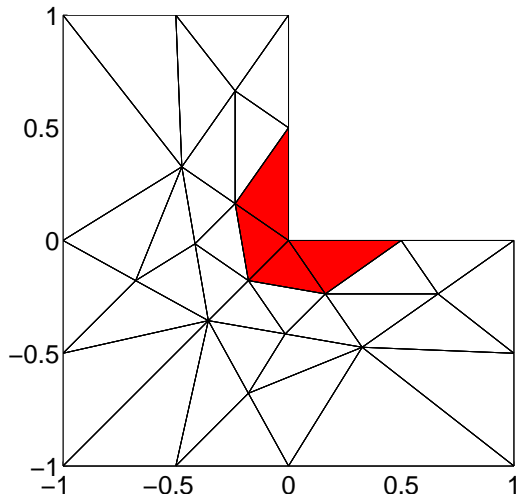
Adaptive algorithm



Adaptive algorithm



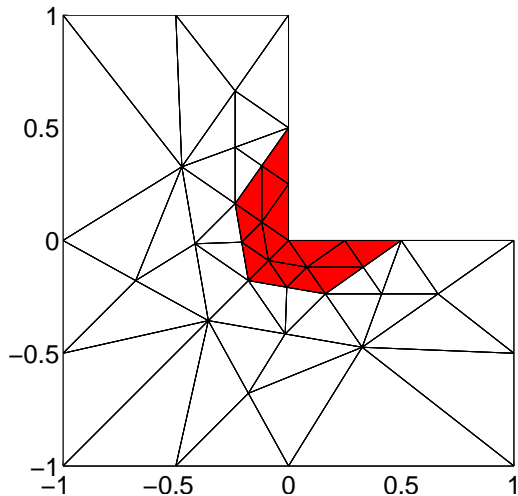
Marked elements 2



Adaptive algorithm



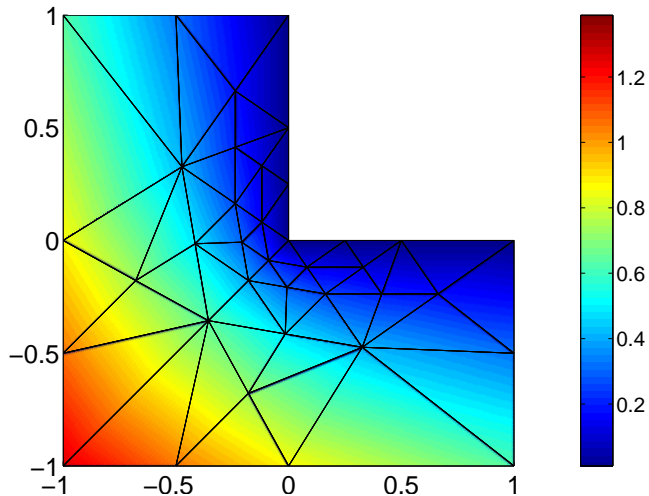
Refined mesh 2



Adaptive algorithm



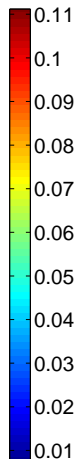
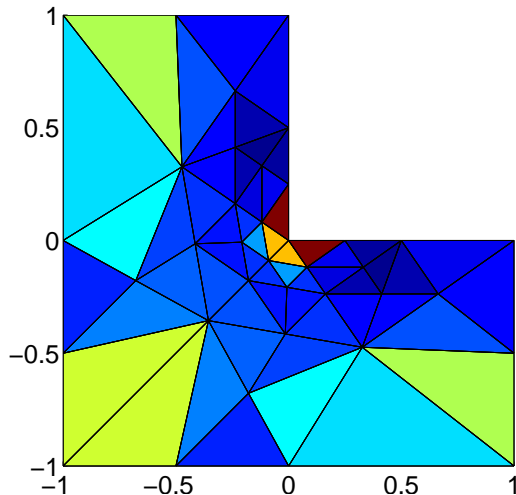
FEM solution 3



Adaptive algorithm



Err indicator 3



$$\eta_{\text{rel}} = 20.9\%$$

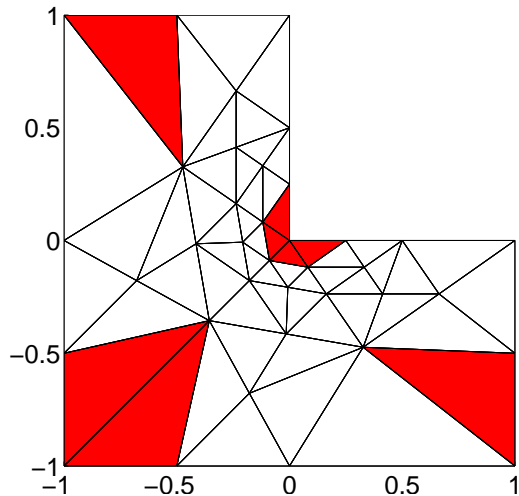
$$e_{\text{rel}} = 12.7\%$$

$$l_{\text{eff}} = 1.65$$

Adaptive algorithm



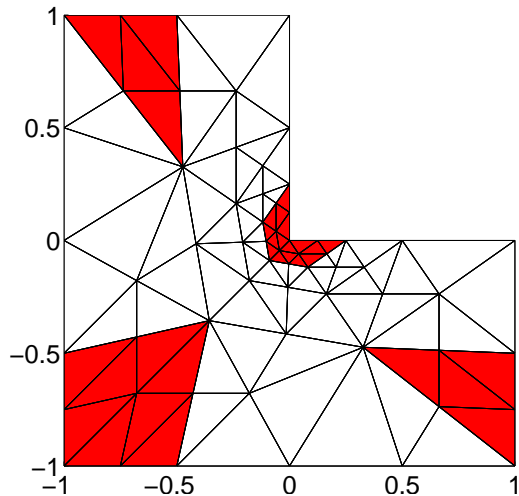
Marked elements 3



Adaptive algorithm



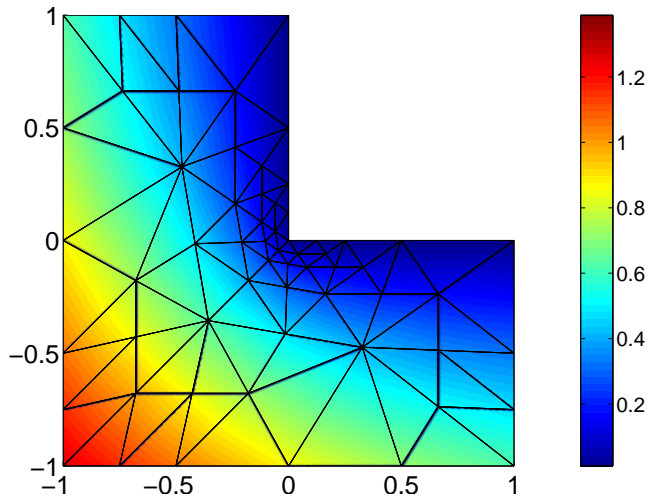
Refined mesh 3



Adaptive algorithm



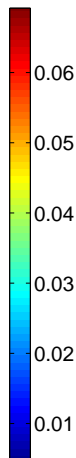
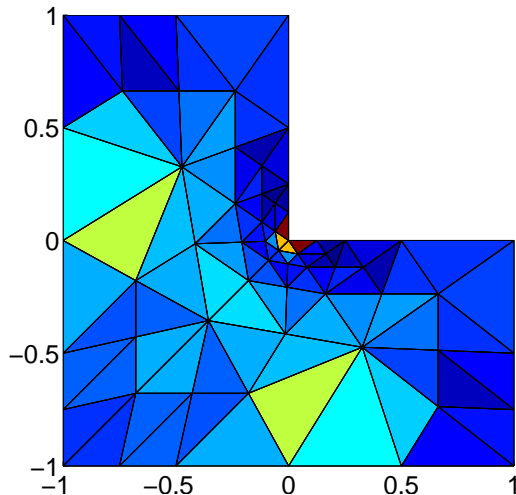
FEM solution 4



Adaptive algorithm



Err indicator 4



$$\eta_{\text{rel}} = 15.4\%$$

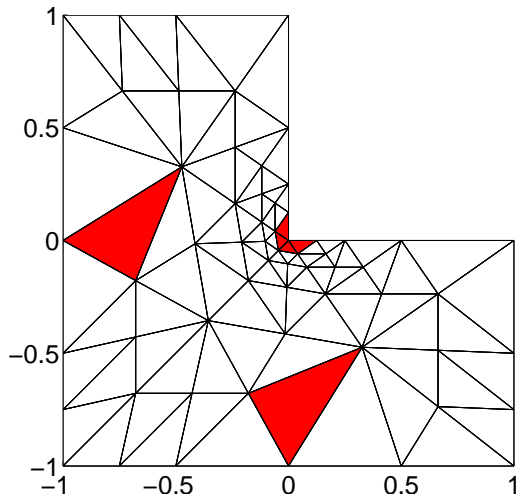
$$e_{\text{rel}} = 9.55\%$$

$$l_{\text{eff}} = 1.61$$

Adaptive algorithm



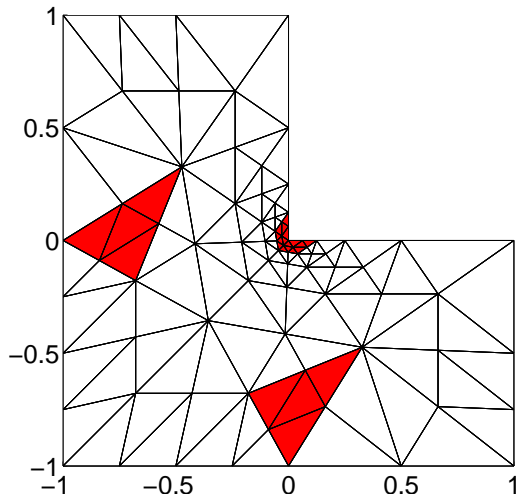
Marked elements 4



Adaptive algorithm



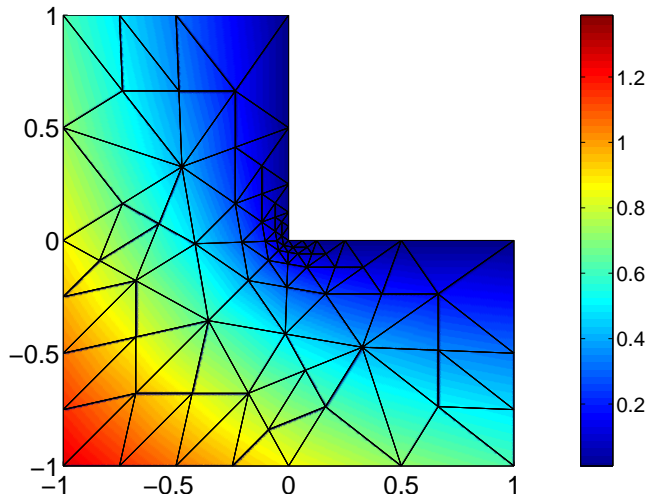
Refined mesh 4



Adaptive algorithm



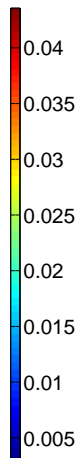
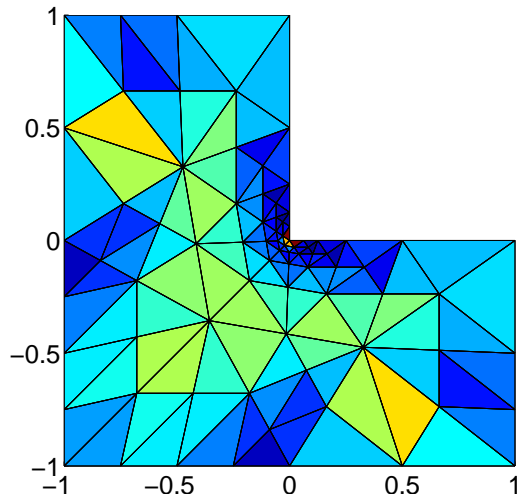
FEM solution 5



Adaptive algorithm



Err indicator 5

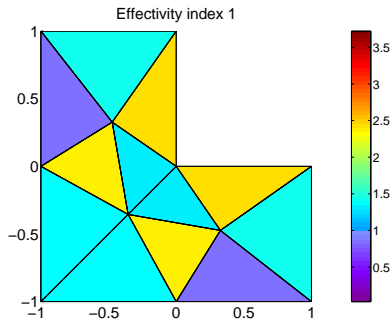
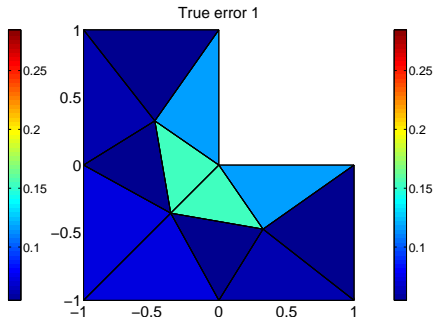
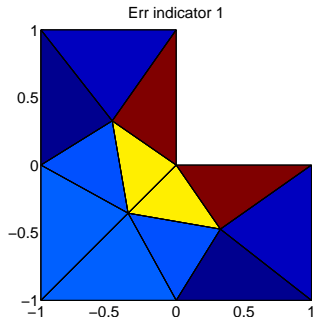


$$\eta_{\text{rel}} = 13.4\%$$

$$e_{\text{rel}} = 8.32\%$$

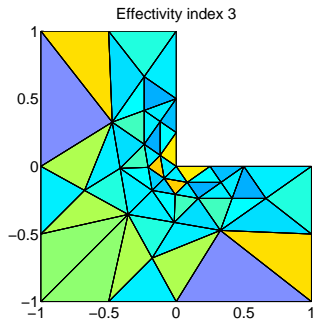
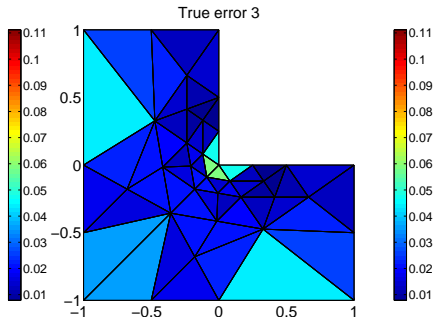
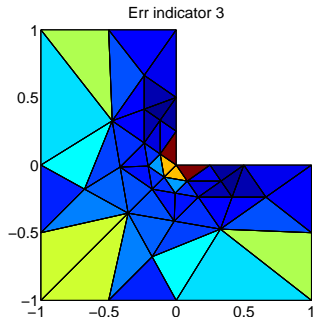
$$l_{\text{eff}} = 1.61$$

Effectivity index



$$l_{\text{eff}} = \frac{\eta}{\|e\|} = 1.73$$

Effectivity index



$$l_{\text{eff}} = \frac{\eta}{\|e\|} = 1.65$$



$$\begin{aligned} -\Delta u + \kappa^2 u &= f \quad \text{in } \Omega \subset \mathbb{R}^2 \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

$$\|e\|^2 \leq \sum_{K \in \mathcal{T}_h} \underbrace{\left(\left\| \frac{1}{\kappa} (r + \operatorname{div} \sigma_K) \right\|_{0,K}^2 + \|\sigma_K\|_{0,K}^2 \right)}_{\eta_K^2}$$

- ▶ $r = f + \Delta u_h - \kappa^2 u_h$
- ▶ $\sigma_K \in \mathbf{H}(\operatorname{div}, K)$
- ▶ $\sigma_K \cdot n_K = g_K - \frac{\partial u_h}{\partial n_K} \quad \text{on } \partial K$
- ▶ $g_K \approx \frac{\partial u}{\partial n_K} \quad \text{on } \partial K$

Thank you for your attention

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March 5, 2009, KMA ZČU, Plzeň