Discrete Green's function – a closer look

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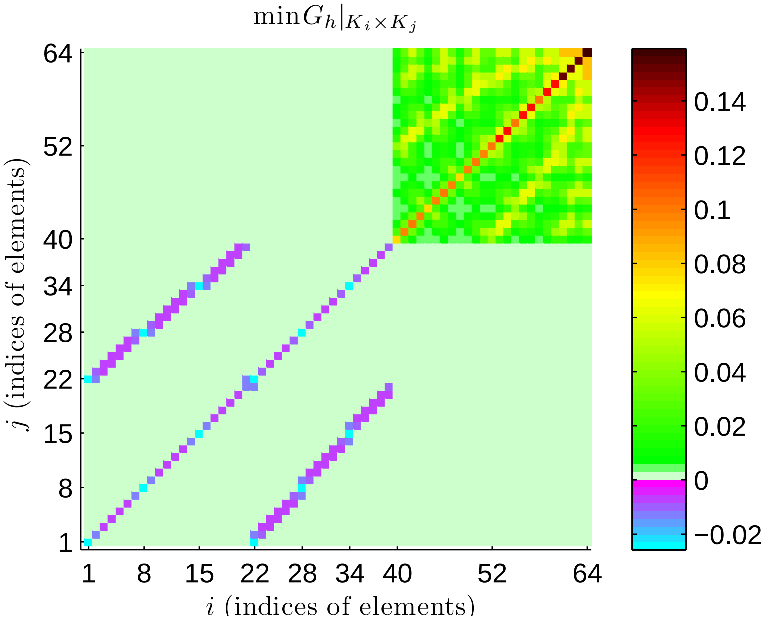
Discrete Green's function (DGF)



Model problem, p-FEM

- Classical formulation:

$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega$$

- p-FEM

$$V_h = \{v_h \in H_0^1(\Omega) : v_h|_K \in \mathbb{P}^p(K),\ \forall K \in \mathcal{T}_h\}$$

$$u_h \in V_h : \quad (\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h \in V_h$$

- DMP

$$f \geq 0 \text{ in } \Omega \quad \Rightarrow \quad u_h \geq 0 \text{ in } \Omega$$

Discrete Green's function (DGF)

- Definition: $y \in \Omega$

$$G_{h,y} \in V_h : \quad (\nabla v_h, \nabla G_{h,y}) = v_h(y) \quad \forall v_h \in V_h$$

$$G_h(x,y) = G_{h,y}(x), \quad x \in \Omega,\ y \in \Omega.$$

- Representation formula:

$$u_h(y) = \int_\Omega G_h(x,y) f(x)\,\mathrm{d}x$$

- Expression in a basis:
  $\varphi_1, \varphi_2, \ldots, \varphi_N$ — basis in V_h
  $A_{ij} = (\nabla \varphi_j, \nabla \varphi_i) \quad i,j = 1,2,\ldots,N$

$$G_h(x,y) = \sum_{i=1}^{N} \sum_{j=1}^{N} \varphi_i(y) (A^{-1})_{ij} \varphi_j(x)$$

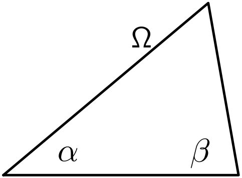
Characterization of the DMP

Theorem

$$DMP \quad \Leftrightarrow \quad G_h \geq 0 \ in \ \Omega^2$$

Experiments

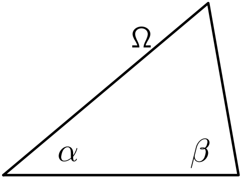
$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

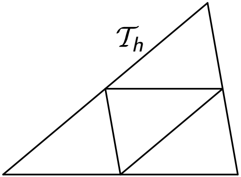


Input parameters: α, β, p

Experiments

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

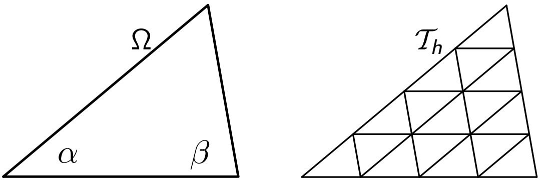




Input parameters: α, β, p

Experiments

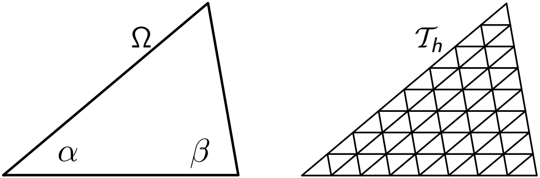
$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$



Input parameters: α, β, p

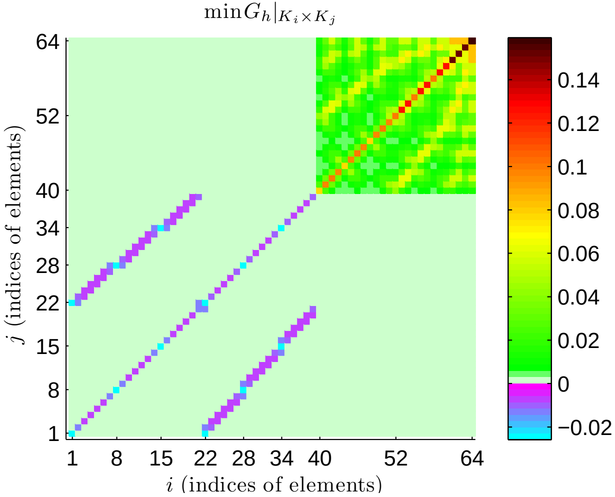
Experiments

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

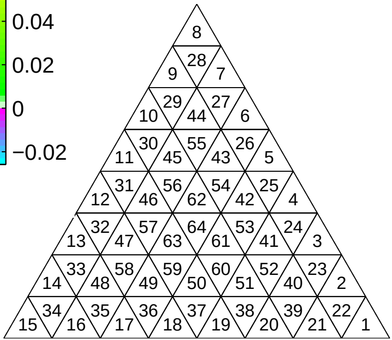


Input parameters: α, β, p

Visualization of DGF: min $G_h|_{K_i \times K_j}$

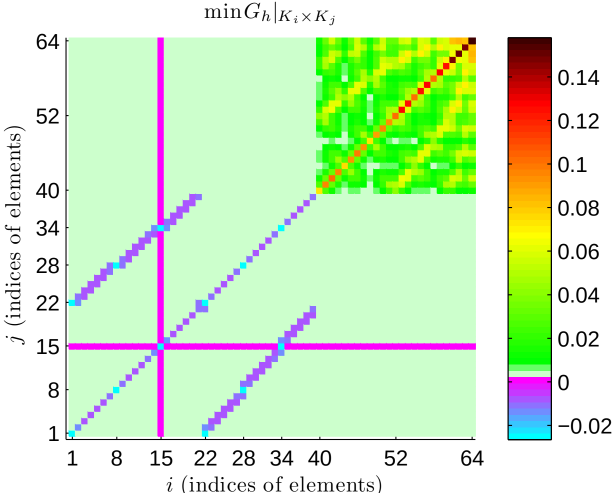


$\alpha = 60^\circ$
$\beta = 60^\circ$
$p = 3$

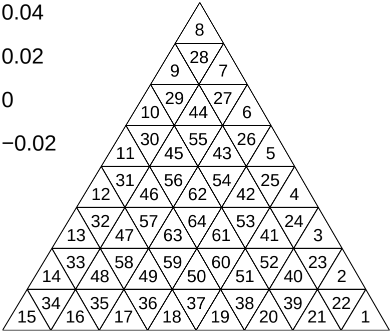


$$\Omega = \bigcup_i K_i, \qquad \Omega^2 = \bigcup_{i,j} K_i \times K_j$$

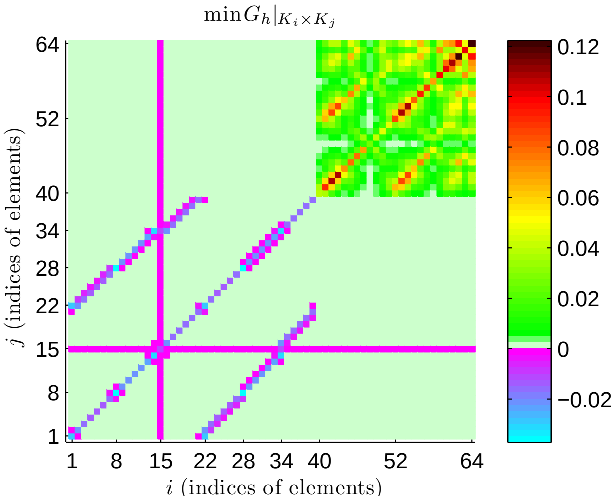
Visualization of DGF: min $G_h|_{K_i \times K_j}$



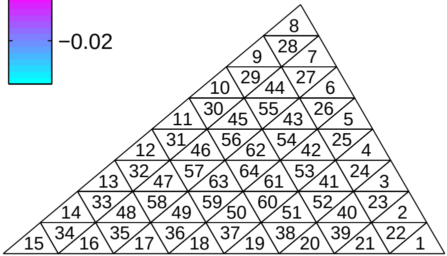
$\alpha = 59°$
$\beta = 60°$
$p = 3$



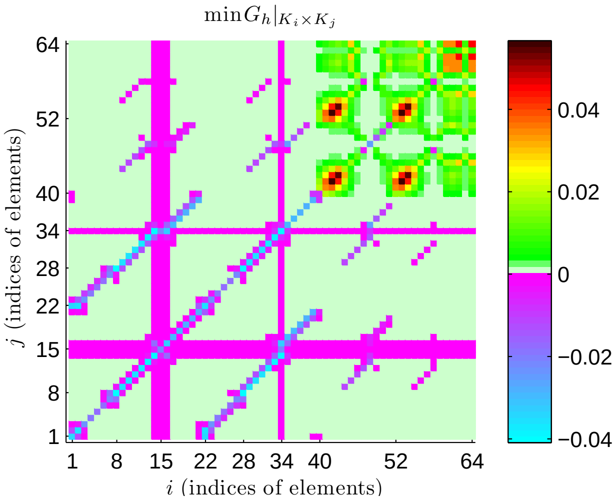
Visualization of DGF: min $G_h|_{K_i \times K_j}$



$\alpha = 40^\circ$
$\beta = 60^\circ$
$p = 3$



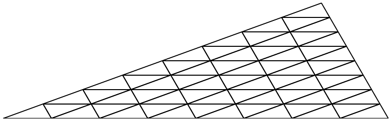
Visualization of DGF: min $G_h|_{K_i \times K_j}$



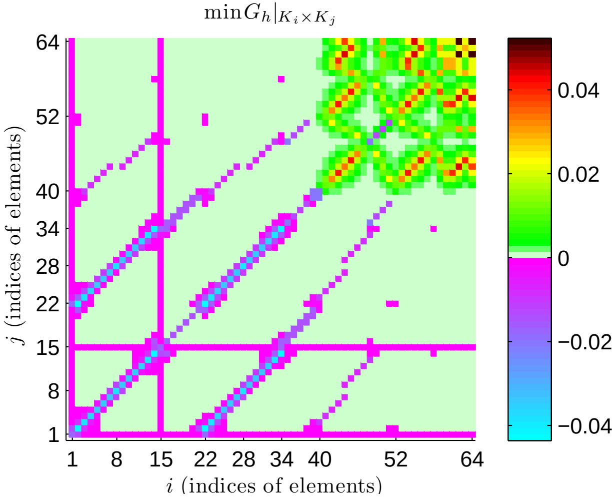
$\alpha = 20°$

$\beta = 60°$

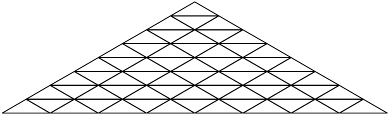
$p = 3$



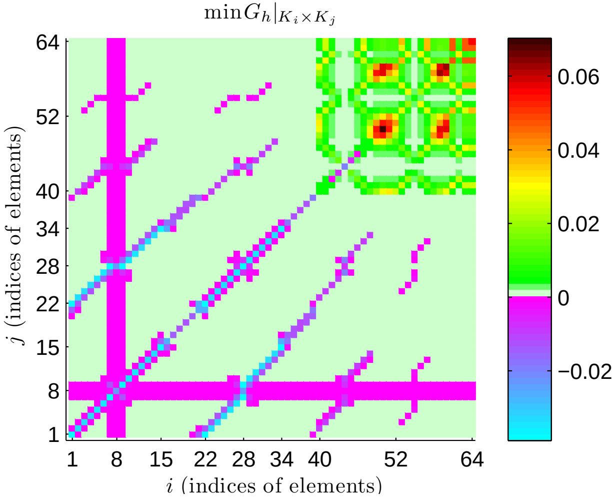
Visualization of DGF: min $G_h|_{K_i \times K_j}$



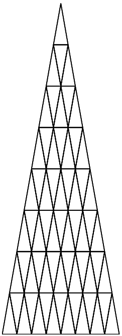
$\alpha = 30°$
$\beta = 30°$
$p = 3$



Visualization of DGF: $\min G_h|_{K_i \times K_j}$



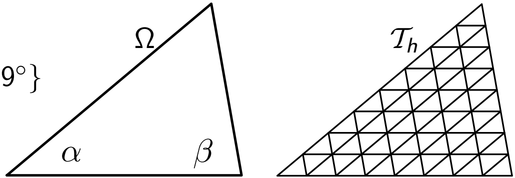
$\alpha = 80^\circ$
$\beta = 80^\circ$
$p = 3$



Numerical experiment

$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega$$

- $p \in \{1, 2, \ldots, 6\}$
- $\alpha, \beta \in \{1^\circ, 2^\circ, \ldots, 179^\circ\}$

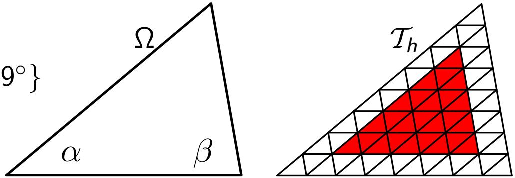


- Boundary region $\Omega_{\mathcal{B}} = \cup\{K \in \mathcal{T}_h : K \cap \partial\Omega \neq \emptyset\}$
- Interior region $\Omega_{\mathcal{I}} = \cup\{K \in \mathcal{T}_h : K \cap \partial\Omega = \emptyset\}$
- Legend:
  - $G_h \geq 0$ in Ω^2 $\Rightarrow$ 
  - $G_h \geq 0$ in $\Omega \times \Omega_{\mathcal{I}}$ $\Rightarrow$ 
  - $G_h \geq 0$ in $\Omega_{\mathcal{I}}^2$ $\Rightarrow$ 
  - $G_h \ngeq 0$ in $\Omega_{\mathcal{I}}^2$ $\Rightarrow$ 

Numerical experiment

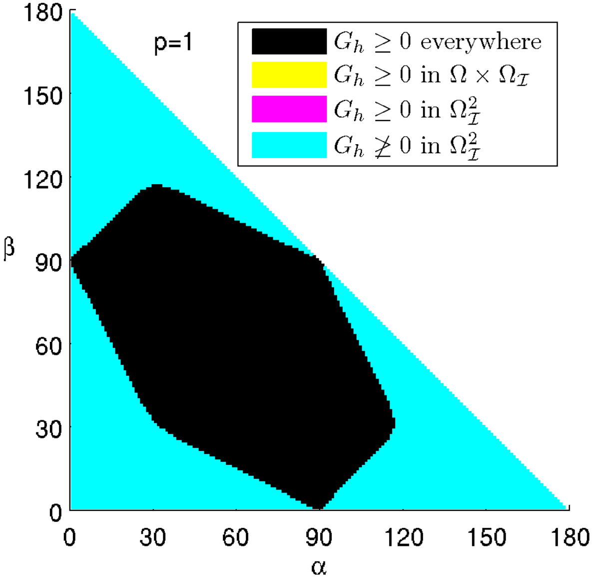
$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega$$

- $p \in \{1, 2, \dots, 6\}$
- $\alpha, \beta \in \{1^\circ, 2^\circ, \dots, 179^\circ\}$



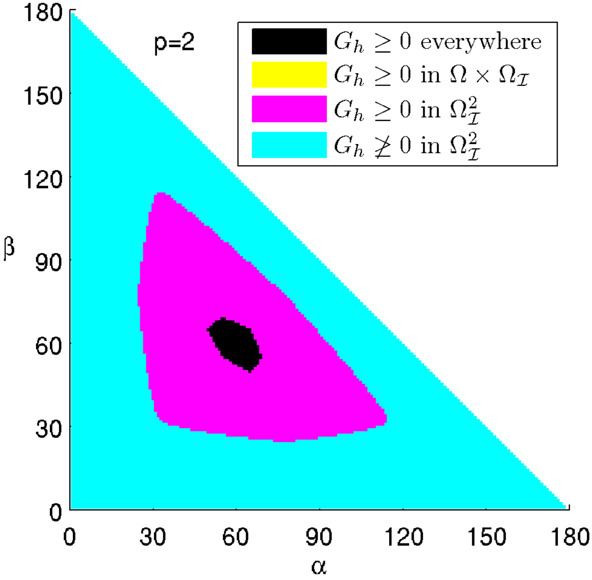
- Boundary region $\Omega_{\mathcal{B}} = \cup\{K \in \mathcal{T}_h : K \cap \partial\Omega \neq \emptyset\}$
- Interior region $\Omega_{\mathcal{I}} = \cup\{K \in \mathcal{T}_h : K \cap \partial\Omega = \emptyset\}$
- Legend:
  - $G_h \geq 0$ in Ω^2 $\Rightarrow$ (black)
  - $G_h \geq 0$ in $\Omega \times \Omega_{\mathcal{I}}$ $\Rightarrow$ (yellow)
  - $G_h \geq 0$ in $\Omega_{\mathcal{I}}^2$ $\Rightarrow$ (magenta)
  - $G_h \not\geq 0$ in $\Omega_{\mathcal{I}}^2$ $\Rightarrow$ (cyan)

Results $p = 1$



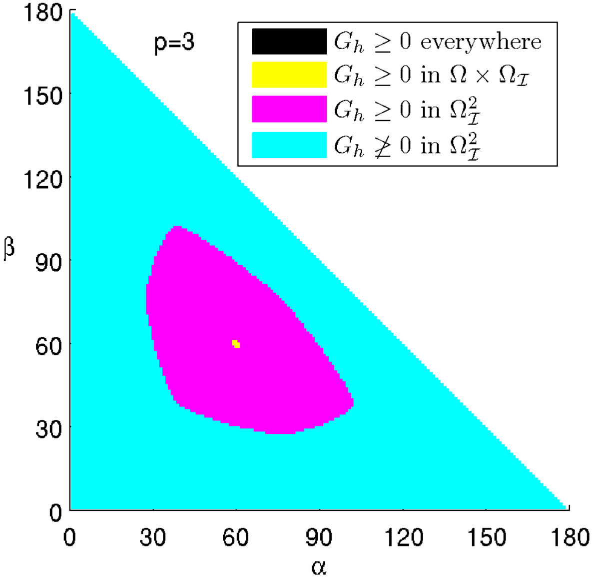
p=1
$G_h \geq 0$ everywhere
$G_h \geq 0$ in $\Omega \times \Omega_{\mathcal{I}}$
$G_h \geq 0$ in $\Omega_{\mathcal{I}}^2$
$G_h \ngeq 0$ in $\Omega_{\mathcal{I}}^2$
β
α
0
30
60
90
120
150
180

Results $p = 2$

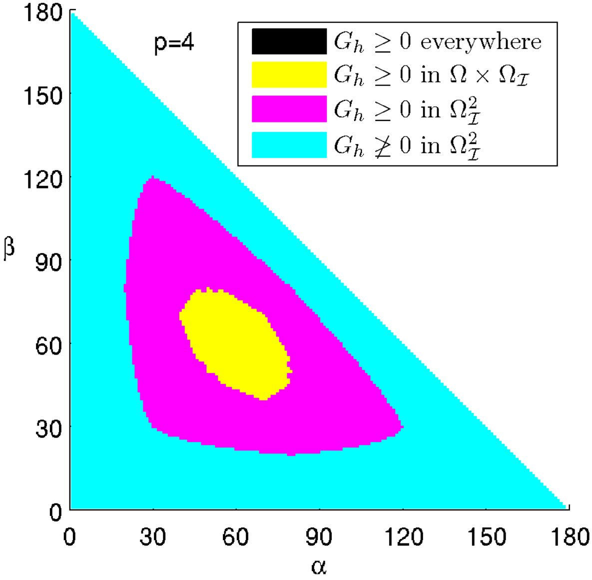


p=2
$G_h \geq 0$ everywhere
$G_h \geq 0$ in $\Omega \times \Omega_{\mathcal{I}}$
$G_h \geq 0$ in $\Omega_{\mathcal{I}}^2$
$G_h \not\geq 0$ in $\Omega_{\mathcal{I}}^2$
β
α
0
30
60
90
120
150
180

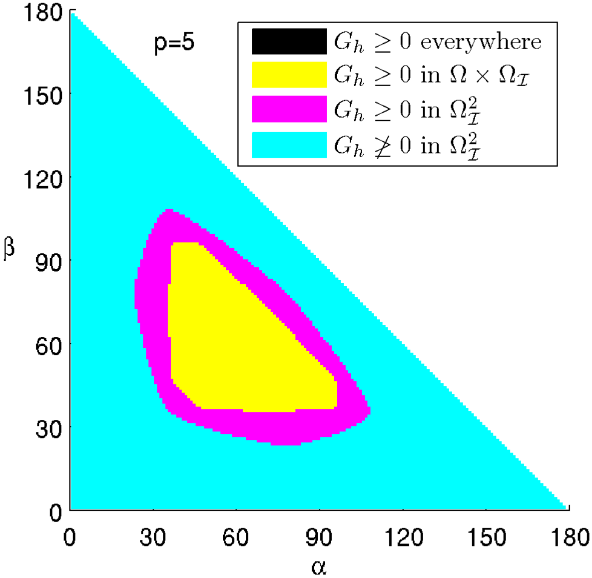
Results $p = 3$


p=3
$G_h \geq 0$ everywhere
$G_h \geq 0$ in $\Omega \times \Omega_{\mathcal{I}}$
$G_h \geq 0$ in $\Omega_{\mathcal{I}}^2$
$G_h \not\geq 0$ in $\Omega_{\mathcal{I}}^2$
β
α
0
30
60
90
120
150
180

Results $p = 4$

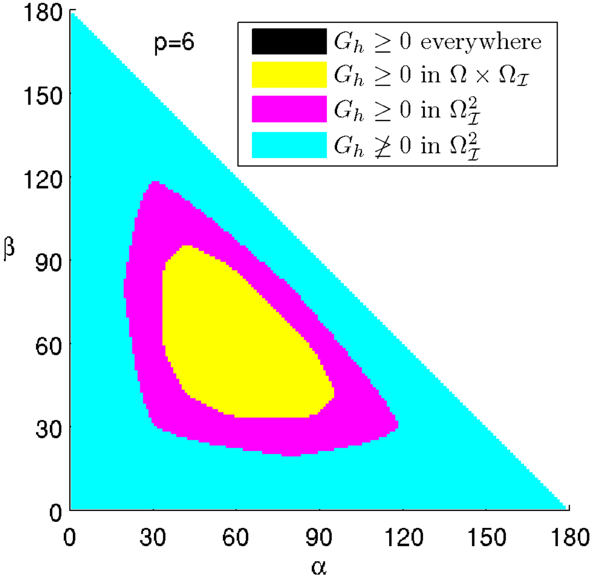

p=4
$G_h \geq 0$ everywhere
$G_h \geq 0$ in $\Omega \times \Omega_{\mathcal{I}}$
$G_h \geq 0$ in $\Omega_{\mathcal{I}}^2$
$G_h \ngeq 0$ in $\Omega_{\mathcal{I}}^2$
β
α
0
30
60
90
120
150
180

Results $p = 5$



p=5
$G_h \geq 0$ everywhere
$G_h \geq 0$ in $\Omega \times \Omega_{\mathcal{I}}$
$G_h \geq 0$ in $\Omega_{\mathcal{I}}^2$
$G_h \not\geq 0$ in $\Omega_{\mathcal{I}}^2$
β
α
0
30
60
90
120
150
180

Results $p = 6$



p=6
$G_h \geq 0$ everywhere
$G_h \geq 0$ in $\Omega \times \Omega_{\mathcal{I}}$
$G_h \geq 0$ in $\Omega_{\mathcal{I}}^2$
$G_h \not\geq 0$ in $\Omega_{\mathcal{I}}^2$
β
α
0
30
60
90
120
150
180

Theorems about interior region

Theorem
Let $G_h \geq 0$ *in* $\Omega \times \Omega_{\mathcal{I}}$ *then*

$$f \geq 0 \ in \ \Omega \quad \Rightarrow \quad u_h \geq 0 \ in \ \Omega_{\mathcal{I}}.$$

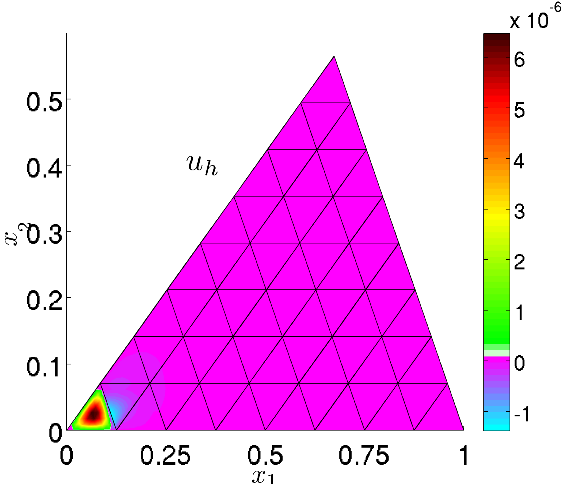
Theorem
Let $G_h \geq 0$ *in* $\Omega_{\mathcal{I}}^2$ *then*

$$f \geq 0 \ in \ \Omega_{\mathcal{I}} \quad and \quad f = 0 \ in \ \Omega_{\mathcal{B}} \quad \Rightarrow \quad u_h \geq 0 \ in \ \Omega_{\mathcal{I}}.$$

Pathological example

$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega$$

Parameters: $\alpha = 40^\circ$, $\beta = 60^\circ$, $p = 3$, $f = \begin{cases} 1 & \text{for } x_1 \leq 0.03, \\ 0 & \text{otherwise} \end{cases}$



Thank you for your attention

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