Computing the constant in Friedrichs' inequality

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Motivation

Classical formulation:

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

Weak formulation: $V = H_0^1(\Omega)$

$$u \in V : \quad a(u,v) = \mathcal{F}(v) \quad \forall v \in V$$

Error bound: $u_h \in V$

$$|||u - u_h||| \le C_{\mathrm{F}} \, \|f + \operatorname{div} \mathbf{y}\|_0 + \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega)$$

Notation:

- $a(u,v) = (\nabla u, \nabla v)$
- $\mathcal{F}(v) = (f, v)$
- $(\varphi, \psi) = \int_\Omega \varphi \psi \, \mathrm{d}x$
- Energy norm: $|||e|||^2 = a(e,e) = (\nabla e, \nabla e) = \|\nabla e\|_0^2$

Friedrichs' inequality

Standard:

$$\|v\|_0 \leq C_{\mathrm{F}} \|\nabla v\|_0 \quad \forall v \in H_0^1(\Omega)$$

Generalization:

$$\|v\|_0 \leq C_{\mathrm{F}} |||v||| \quad \forall v \in V$$

Variants:

- $|||v|||^2 = \|\nabla u\|_{\mathcal{A}} = (\mathcal{A}\nabla u, \nabla u)$
- $|||v|||^2 = \|\nabla u\|_{\mathcal{A}}^2 + \|u\|_c^2 = (\mathcal{A}\nabla u, \nabla u) + (cu, u)$
- $V = H_0^1(\Omega)$ [FIGURE OMEGA]
- $V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma\}$ [FIGURE OMEGA]

Relation with eigenvalues

Friedrichs' inequality:

$$\|v\|_0 \le C_{\mathrm{F}} \|\nabla v\|_0 \quad \forall v \in V \quad \Rightarrow \quad C_{\mathrm{F}} = \sup_{v \in V} \frac{\|v\|_0}{\|\nabla v\|_0}$$

Laplace eigenvalue problem

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega, \qquad u_i = 0 \text{ on } \partial\Omega$$

Theorem: $C_{\mathrm{F}}^2 = \dfrac{1}{\lambda_1}$ where $\lambda_1 = \min_i \lambda_i$.

Proof:
Weak formulation: $u_i \in V : \quad (\nabla u_i, \nabla v) = \lambda_i (u_i, v) \quad \forall v \in V$

$$\lambda_1 = \inf_{v \in V} \frac{\|\nabla v\|_0^2}{\|v\|_0^2} \quad \Leftrightarrow \quad \frac{1}{\lambda_1} = \sup_{v \in V} \frac{\|v\|_0^2}{\|\nabla v\|_0^2}$$

□

Rayleigh–Ritz approximation of λ_1

Weak formulation:

$$u_i \in V : \quad (\nabla u_i, \nabla v) = \lambda_i (u_i, v) \quad \forall v \in V$$

Rayleigh–Ritz method: $V^h \subset V$, $\dim V^h < \infty$

$$u_i^h \in V^h : \quad (\nabla u_i^h, \nabla v^h) = \lambda_i^h (u_i^h, v^h) \quad \forall v^h \in V^h$$

Theorem: $\lambda_1 \leq \lambda_1^h$

Proof:

$$\lambda_1 = \inf_{v \in V} \frac{\|\nabla v\|_0^2}{\|v\|_0^2} \leq \inf_{v^h \in V^h} \frac{\|\nabla v\|_0^2}{\|v\|_0^2} = \lambda_1^h$$

□

Corollary: $C_{\mathrm{F}}^h \leq C_{\mathrm{F}}$

Lower bound on λ_1

Method of *a priori-a posteriori inequalities*.

Theorem (Kuttler and Sigillito, 1978):

- Let H be a separable Hilbert space.
- Let $A : H \mapsto H$ be a symmetric operator with dense domain $D(A)$.
- Other technical assumptions on A.
- Let λ_* and $u_* \in D(A)$ be arbitrary.
- Consider $w \in D(A)$ such that $Aw = Au_* - \lambda_* u_*$.

Then

$$\min_i \left| \frac{\lambda_i - \lambda_*}{\lambda_i} \right| \leq \frac{\|w\|_H}{\|u_*\|_H}.$$

Ussage: $H = L^2(\Omega)$, $A = -\Delta \quad \Rightarrow$

$$\min_i \left| \frac{\lambda_i - \lambda_*}{\lambda_i} \right| \leq \frac{\|w\|_0}{\|u_*\|_0} \leq C_{\mathrm{F}} \frac{\|\nabla w\|_0}{\|u_*\|_0}$$

Algorithm

$$\min_i \left| \frac{\lambda_i - \lambda_*}{\lambda_i} \right| \leq \frac{\|w\|_0}{\|u_*\|_0} \leq C_{\mathrm{F}} \frac{\|\nabla w\|_0}{\|u_*\|_0}$$

Theorem

$$\|\nabla w\|_0 \leq \|\nabla u_* - \mathbf{q}\|_0 + C_{\mathrm{F}} \|\lambda_* u_* + \operatorname{div} \mathbf{q}\|_0 \quad \forall \mathbf{q} \in \mathbf{H}(\operatorname{div}, \Omega),$$

- Compute Rayleigh–Ritz approximations λ_1^h and u_1^h
- Set $\lambda_* = \lambda_1^h$ and $u_* = u_1^h$
- Find approximate minimizer $\mathbf{q}_h \in \mathbf{H}(\operatorname{div}, \Omega)$.
- Put $\alpha = \|\nabla u_h - \mathbf{q}_h\|_0 / \|u_h\|_0$, $\beta = \|\lambda_h u_h + \operatorname{div} \mathbf{q}_h\|_0 / \|u_h\|_0$
- $C_{\mathrm{F}} = \dfrac{1}{\sqrt{\lambda_1}} \quad \Rightarrow \quad \dfrac{\lambda_h - \lambda_1}{\lambda_1} \leq \dfrac{1}{\sqrt{\lambda_1}} \left(\alpha + \dfrac{1}{\sqrt{\lambda_1}} \beta \right)$
- $\Rightarrow 0 \leq X^2 + \alpha X + \beta - \lambda_h$, where $X = \sqrt{\lambda_1}$
- Solution:
  $X_2^2 \leq \lambda_1, \quad$ where $X_2 = \left(\sqrt{\alpha^2 + 4(\lambda_h - \beta)} - \alpha \right) / 2.$
- Thus, $C_{\mathrm{F}} \leq 1/X_2$

Computing $\mathbf{q}_h \in \mathbf{H}(\mathrm{div}, \Omega)$

$$
\begin{aligned}
&\|\nabla u_* - \mathbf{q}\|_0 + C_{\mathrm{F}} \|\lambda_* u_* + \mathrm{div}\,\mathbf{q}\|_0 \\
&\approx \left\|\nabla u_1^h - \mathbf{q}\right\|_0 + (\lambda_1^h)^{-1/2} \left\|\lambda_1^h u_1^h + \mathrm{div}\,\mathbf{q}\right\|_0 \\
&\le (1+\varrho^{-1}) \left\|\nabla u_1^h - \mathbf{q}\right\|_0^2 + \frac{1+\varrho}{\lambda_h} \left\|\lambda_1^h u_1^h - \mathrm{div}\,\mathbf{q}\right\|_0^2
\end{aligned}
$$

Minimize over $W_h \subset \mathbf{H}(\mathrm{div}, \Omega)$:

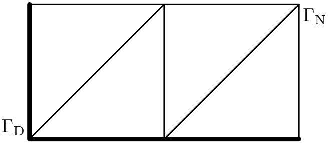
Find $\mathbf{q}_h \in W_h$:

$$
(\mathrm{div}\,\mathbf{q}_h, \mathrm{div}\,\boldsymbol{\psi}_h) + \frac{\lambda_h}{\varrho}(\mathbf{q}_h, \boldsymbol{\psi}_h) = \frac{\lambda_h}{\varrho}(\nabla u_h, \boldsymbol{\psi}_h) - (\lambda_h u_h, \mathrm{div}\,\boldsymbol{\psi}_h) \qquad \forall \boldsymbol{\psi}_h \in W_h
$$

Solve by standard Raviart-Thomas finite elements.

Example 1

$$-\Delta u = f \text{ in } (0,2)\times(0,1)$$
$$u = 0 \text{ on } \Gamma_D$$
$$\mathbf{n}^\top \nabla u = 0 \text{ on } \Gamma_N$$



$$f = \frac{5\pi^2}{16} u$$
$$u = \sin\frac{\pi x_1}{4} \sin\frac{\pi x_2}{2}$$

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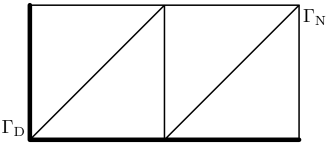
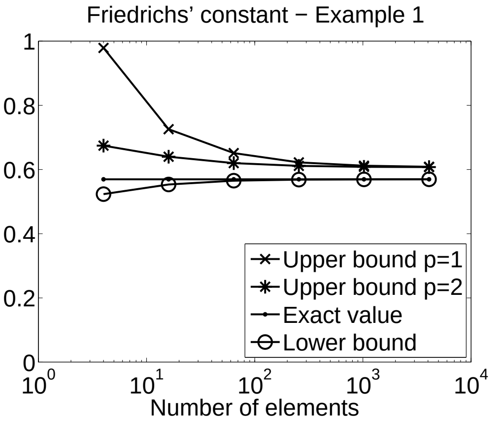


$$f = \frac{5\pi^2}{16} u$$
$$u = \sin\frac{\pi x_1}{4} \sin\frac{\pi x_2}{2}$$

$$C_{\rm F} = \frac{4}{\sqrt{5}\pi} \doteq 0.5694$$
$$C_{\rm F}^{\rm low} = 0.5693$$
$$C_{\rm F}^{\rm up} = 0.6004$$



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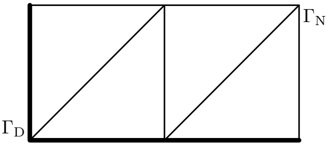
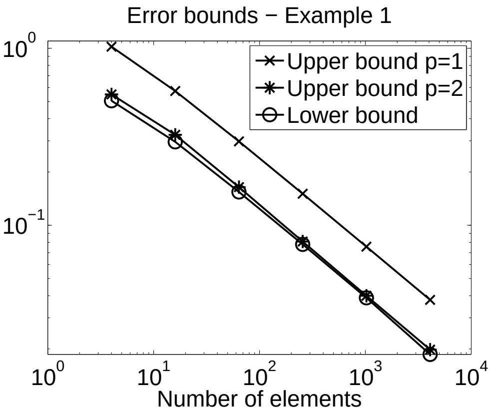


Lower bound:
reference solution

Upper bound:
error majorant



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$$-\Delta u = f \text{ in } (0,2)\times(0,1)$$
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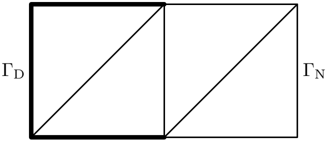

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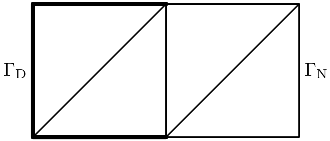
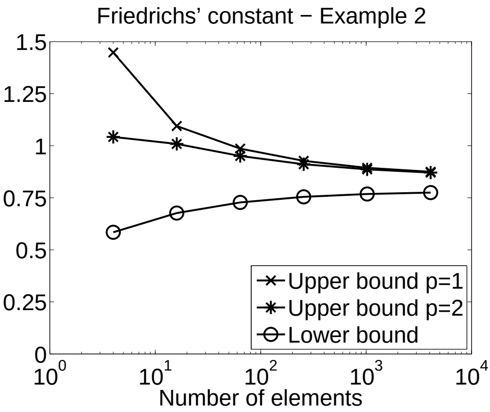


$$f = \frac{5\pi^2}{16} \sin\frac{\pi x_1}{4} \sin\frac{\pi x_2}{2}$$

$C_{\mathrm{F}} = ?$
$C_{\mathrm{F}}^{\mathrm{low}} = 0.7750$
$C_{\mathrm{F}}^{\mathrm{up}} = 0.8712$



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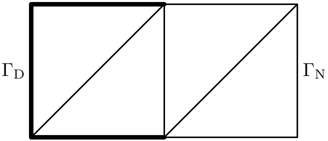

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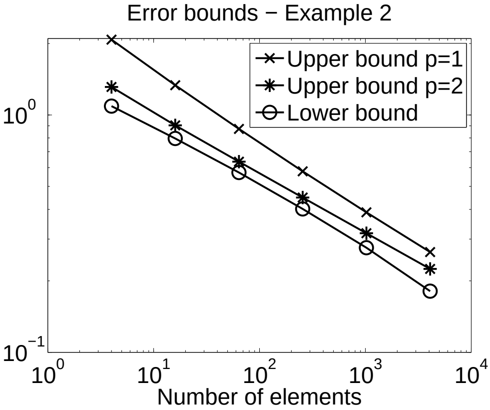

Conclusions

- Practical method
- Guaranteed upper bound on Friedrichs' constant
- Easy to generalize to similar inequalities
- Computationally demanding
- Exact representaion of the domain Ω
- $\Rightarrow$ curved elements
- $\Rightarrow$ Splines!

Thank you for your attention

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