Discrete maximum principles

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Defence of habilitation thesis
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Habilitation thesis

Setting:

- linear elliptic second-order PDE
- finite element method (FEM)
  - linear (lowest-order)
  - higher-order
- discrete maximum principles (DMP)

Goals:

- General theory of DMP
- Unified survey of known results
- New results (higher-order FEM)

Habilitation thesis

[A1] A. Hannukainen, S. Korotov, and T. Vejchodský: *J. Comput. Appl. Math.* **226** (2009), 275–287. **IF 0.943**

[A2] S. Korotov and T. Vejchodský: In: *Numerical mathematics and advanced applications ENUMATH 2009*, Springer, Berlin, 2010, pp. 533–541.

[A3] T. Vejchodský and P. Šolín: *Math. Comp.* **76** (2007), 1833–1846. **IF 1.230**

[A4] T. Vejchodský and P. Šolín: *Adv. Appl. Math. Mech.* **1** (2009), 201–214.

[A5] T. Vejchodský and P. Šolín: *J. Numer. Math.* **15** (2007), 233–243. **IF 0.586**

[A6] T. Vejchodský: *Appl. Numer. Math.* **60** (2010), 486–500. **IF 1.279**

[A7] T. Vejchodský: In: *Numerical mathematics and advanced applications ENUMATH 2009*, Springer, Berlin, 2010, pp. 901–909.

[A8] P. Šolín and T. Vejchodský: *J. Comput. Appl. Math.* **209** (2007), 54–65. **IF 0.943**

Introduction

Example 1: $-u'' = f$ in $(0,2)$, $\quad u(0) = u(2) = 0$

Conservation of nonnegativity: $f \geq 0 \quad \Rightarrow \quad u \geq 0$

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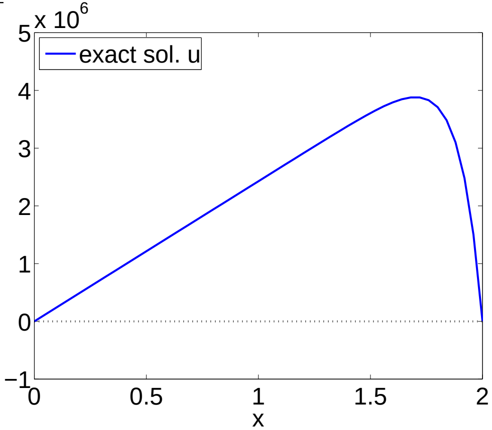
- $u = \dfrac{2 + (\mathrm{e}^{20} - 1)x - 2\mathrm{e}^{10x}}{200}$
- $f = \mathrm{e}^{10x}$



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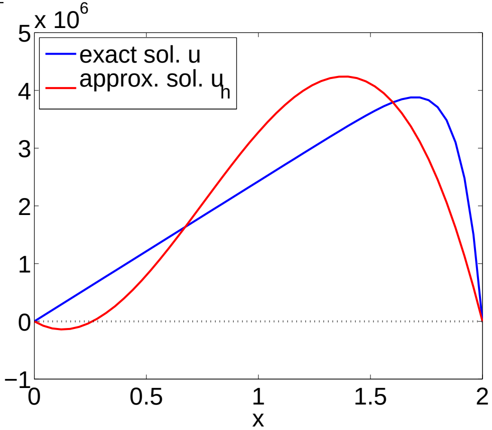
- $u = \dfrac{2 + (\mathrm{e}^{20} - 1)x - 2\mathrm{e}^{10x}}{200}$
- $f = \mathrm{e}^{10x}$
- $u_h \in \mathbb{P}_0^3(0,2)$



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Example 2: $-\Delta u = f$ in $\Omega = (0,4) \times (0,2), \quad u = 0$ on $\partial\Omega$

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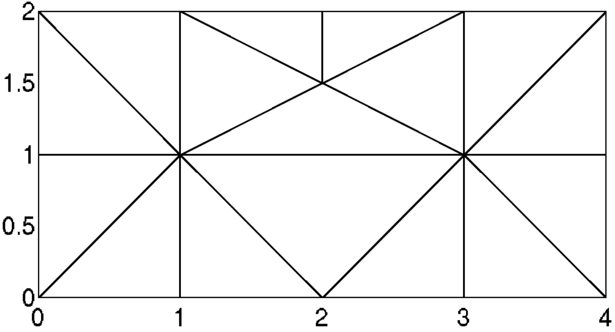
$$f(x_1, x_2) = \begin{cases} 1 & \text{for } x_1 < 1 \\ 0 & \text{for } x_1 \geq 1 \end{cases} \qquad u_h \text{ by linear FEM}$$



Brandts, Korotov, Křížek, Šolc, SIAM Review 51 (2009), 317–335

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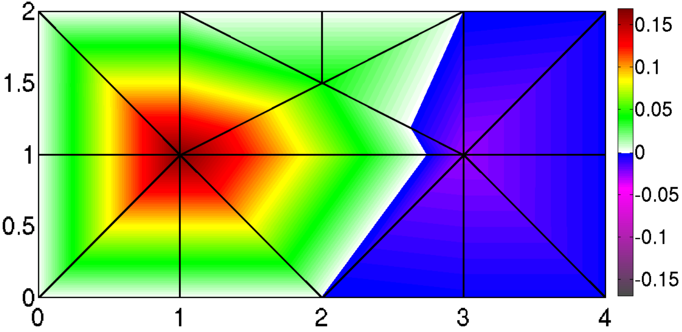
u_h by linear FEM



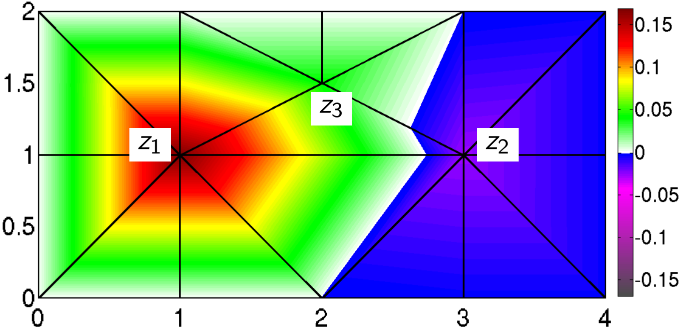
Negative values 10× magnified.

Brandts, Korotov, Křížek, Šolc, SIAM Review 51 (2009), 317–335

Introduction

Example 2:

$$Az = b \quad \Leftrightarrow \quad \overbrace{\begin{pmatrix} + & - & + \\ - & + & + \\ + & + & + \end{pmatrix}}^{A^{-1}} \overbrace{\begin{pmatrix} + \\ 0 \\ 0 \end{pmatrix}}^{b} = \overbrace{\begin{pmatrix} + \\ - \\ + \end{pmatrix}}^{z}$$



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Model problem

$$
\begin{aligned}
-\operatorname{div}(\mathcal{A}\nabla u) + \mathbf{b}\cdot\nabla u + cu &= f && \text{in } \Omega \\
u &= g_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}} \\
\alpha u + (\mathcal{A}\nabla u)\cdot\mathbf{n} &= g_{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}}
\end{aligned}
$$

Maximum Principle:

$$f \le 0 \text{ and } g_{\mathrm{N}} \le 0 \quad \Rightarrow \quad \max_{\overline{\Omega}} u \le \max_{\Gamma_{\mathrm{D}}} \max\{0, u\}$$

Minimum Principle:

$$f \ge 0 \text{ and } g_{\mathrm{N}} \ge 0 \quad \Rightarrow \quad \min_{\overline{\Omega}} u \ge \min_{\Gamma_{\mathrm{D}}} \min\{0, u\}$$

Conservation of Nonnegativity:

$$f \ge 0,\ g_{\mathrm{D}} \ge 0, \text{ and } g_{\mathrm{N}} \ge 0 \quad \Rightarrow \quad u \ge 0$$

Comparison Principle:

$$f \ge \widetilde{f},\ g_{\mathrm{D}} \ge \widetilde{g}_{\mathrm{D}}, \text{ and } g_{\mathrm{N}} \ge \widetilde{g}_{\mathrm{N}} \quad \Rightarrow \quad u \ge \widetilde{u}$$

Finite element method

Weak form.: $u - \overline{g}_{\mathrm{D}} \in V: \quad a(u,v) = \mathcal{F}(v) \quad \forall v \in V$

FEM form.: $u_h - \overline{g}_{\mathrm{D},h} \in V_h: \quad a(u_h,v_h) = \mathcal{F}(v_h) \quad \forall v_h \in V_h$

$$V_h \subset V, \ \dim V_h < \infty$$

Discrete Maximum Principle (DMP):

$$f \le 0 \text{ and } g_{\mathrm{N}} \le 0 \quad \Rightarrow \quad \max_{\overline{\Omega}} u_h \le \max_{\Gamma_{\mathrm{D}}} \max\{0, u_h\}$$

Discrete Minimum Principle:

$$f \ge 0 \text{ and } g_{\mathrm{N}} \ge 0 \quad \Rightarrow \quad \min_{\overline{\Omega}} u_h \ge \min_{\Gamma_{\mathrm{D}}} \min\{0, u_h\}$$

Discrete Conservation of Nonnegativity:

$$f \ge 0, \ g_{\mathrm{D},h} \ge 0, \ \text{and } g_{\mathrm{N}} \ge 0 \quad \Rightarrow \quad u_h \ge 0$$

Discrete Comparison Principle:

$$f \ge \widetilde{f}, \ g_{\mathrm{D},h} \ge \widetilde{g}_{\mathrm{D},h}, \ \text{and } g_{\mathrm{N}} \ge \widetilde{g}_{\mathrm{N}} \quad \Rightarrow \quad u_h \ge \widetilde{u}_h$$

Discrete Green's function

Definition: $\mathbf{y} \in \Omega, \quad G_{h,\mathbf{y}} \in V_h : \quad a(v_h, G_{h,\mathbf{y}}) = v_h(\mathbf{y}) \quad \forall v_h \in V_h$

Notation: $G_h(\mathbf{x}, \mathbf{y}) = G_{h,\mathbf{y}}(\mathbf{x}) \quad \forall (\mathbf{x}, \mathbf{y}) \in \Omega^2$

Theorem: DMP $\quad \Leftrightarrow$
(i) $G_h(\mathbf{x}, \mathbf{y}) \geq 0 \quad \forall (\mathbf{x}, \mathbf{y}) \in \Omega^2$
(ii) $g_{\mathrm{D},h}(\mathbf{y}) - (\Pi_h^0 g_{\mathrm{D},h})(\mathbf{y}) \geq 0 \quad \forall g_{\mathrm{D},h} \in V_h^\partial,\ g_{\mathrm{D},h} \geq 0$ in Ω, $\mathbf{y} \in \Omega$

Theorem: For linear FEM:
(i) $\quad \Leftrightarrow \quad A^{-1} \geq 0$
(ii) $\quad \Leftrightarrow \quad -A^{-1} A^\partial \geq 0$

Results for linear FEM

Systematic overview of known results.

$$
\begin{aligned}
-(\mathcal{A}u')' + bu' + cu &= f \qquad \text{in } \Omega = (a,b) \\
u(a) &= g_{\mathrm{D}} \\
\alpha u(b) + \mathcal{A}u'(b) &= g_{\mathrm{N}}
\end{aligned}
$$

Original contributions:

- Complete characterization in 1D
- New general sufficient conditions for
  - simplicial FEM, arbitrary dimension
  - block FEM, arbitrary dimension [A2]
- Right triangular prismatic FEM [A1]

Results for linear FEM

Systematic overview of known results.

$$
\begin{aligned}
-\operatorname{div}(\lambda \nabla u) + cu &= f && \text{in } \Omega \subset \mathbb{R}^d \\
u &= g_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}} \\
\alpha u + (\lambda \nabla u) \cdot \mathbf{n} &= g_{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}}
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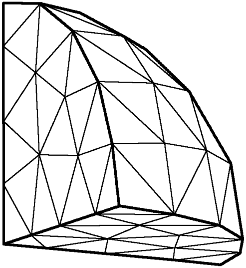

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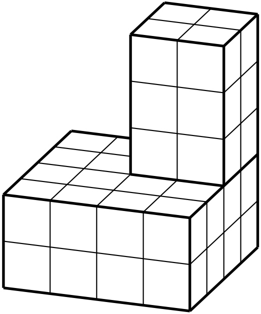

Results for linear FEM

Systematic overview of known results.

$$-\Delta u + cu = f \quad \text{in } \Omega \subset \mathbb{R}^3$$
$$u = 0 \quad \text{on } \partial\Omega$$

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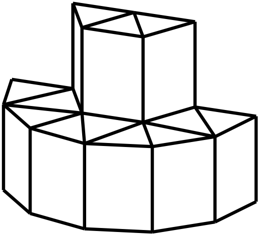

Results for higher-order FEM

Mostly original results.

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-(\mathcal{A}u')' &= f \quad \text{in } \Omega = (a,b) \\
u(a) &= g_{\mathrm{D}} \\
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- 1D diffusion equation [A3]–[A5]
- 1D diffusion-reaction equation [A6]
- 2D numerical tests – negative conclusions [A7]
- 1D weaker concept [A8]

Results for higher-order FEM

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Results for higher-order FEM

Mostly original results.

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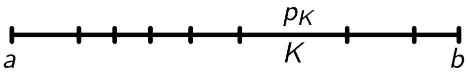
1D diffusion equation

Problem: $-u'' = f$ in (a, b), $\quad u(a) = u(b) = 0$

FEM: $u_{hp} \in V_{hp} = \{v_{hp} \in C_0[a, b] : v_{hp} \in \mathbb{P}^{p_K}(K),\ K \in \mathcal{T}_h\}$

Theorem: $h_K \leq H^*_{\mathrm{rel}}(p_K)(b-a)\ \forall K \in \mathcal{T}_h \quad \Rightarrow \quad u_{hp}$ satisfies DMP



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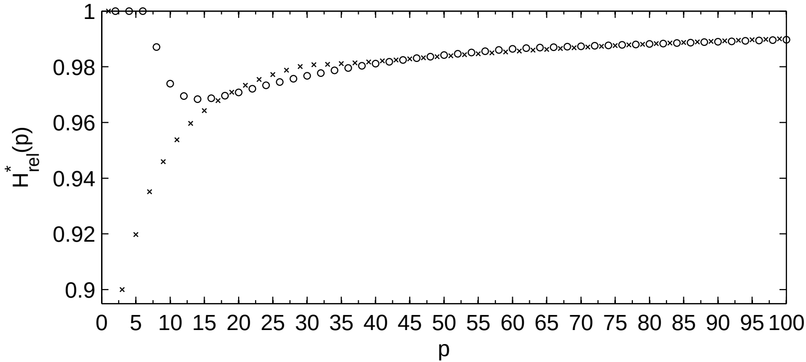

Conclusions

Setting:

- linear elliptic second-order PDE
- finite element method (FEM)
  - linear
  - higher-order
- discrete maximum principles (DMP)

Output:

- general theory
- unified presentation
- general overview of DMP results
- original contributions (higher-order FEM)
- 8 original papers

Thank you for your attention

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Parabolic problems

$$\begin{aligned} \frac{\partial u}{\partial t} - \operatorname{div}(\mathcal{A}\nabla u) + \mathbf{b} \cdot \nabla u + cu &= f && \text{in } \Omega \\ u &= g_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}} \\ \alpha u + (\mathcal{A}\nabla u) \cdot \mathbf{n} &= g_{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}} \\ u(x,t) &= u_0(x) && \text{for } t = 0 \end{aligned}$$

Conservation of Nonnegativity:

$f \geq 0,\ g_{\mathrm{D}} \geq 0,\ g_{\mathrm{N}} \geq 0,$ and $u_0 \geq 0 \quad \Rightarrow \quad u \geq 0$

Parabolic problems

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u &= 0 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \\ u(x,t) &= u_0(x) && \text{for } t = 0 \end{aligned}$$

Conservation of Nonnegativity: $u_0 \geq 0 \quad \Rightarrow \quad u \geq 0$

Method of lines:

$$u_h(x,t) = \sum_{i=1}^{N} z_i(t)\varphi_j(x) \quad \Rightarrow \quad \begin{aligned} M\dot{\mathbf{z}}(t) + A\mathbf{z}(t) &= \mathbf{b} \\ \mathbf{z}(0) &= \mathbf{z}_0 \end{aligned}$$

- Semidiscretization: DMP fails on any mesh. [Vejchodský, 2004]
- Full-discretization: DMP satisfied for $\Delta t > \tau$ [Faragó]
- Mass-lumping, FDM: DMP satisfied under the same conditions as in the elliptic case.