Milan Tvrdý

Correction and addition to my paper "The normal form and the stability of solutions of a system of differential equations in the complex domain"

Czechoslovak Mathematical Journal, Vol. 22 (1972), No. 1, 176–179

Persistent URL: http://dml.cz/dmlcz/101084

Czechoslovak Mathematical Journal, 22 (97) 1972, Praha

CORRECTION AND ADDITION TO MY PAPER "THE NORMAL FORM AND THE STABILITY OF SOLUTIONS OF A SYSTEM OF DIFFERENTIAL EQUATIONS IN THE COMPLEX DOMAIN"

MILAN TVRDÝ, Praha

(Received April 29, 1971)

1. Correction. The correct form of the condition (Q_4) in Theorem 3,4 of [1] (pp. 53–59) is the following.

Given an arbitrary $p \in \mathscr{P}(\lambda)$, there exists such a complex number α_p that

$$\{\eta_k\}_p = \alpha_p \lambda_k \quad (k = 1, 2, \ldots, l) . \tag{3,16}$$

Notation 3,4 is now unnecessary and the proof of the theorem is to be modified in an obvious manner: $\nu = 0$ and hence $\mathscr{Q}_{k,\nu}(\lambda) = \mathscr{P}_k(\lambda)$, $\mathscr{S}_{k,\nu}(\lambda) = \mathsf{E}[p \in \mathscr{P}_k(\lambda) : : p_k \geqq 0] = \tilde{\mathscr{P}}_k(\lambda) = \tilde{\mathscr{P}}(\lambda)$ $(k = 1, 2, \ldots, l)$, $\mathscr{M}'_\nu = \emptyset$, $\mathscr{S}'_\nu(\lambda) = \emptyset$ and $\beta(y) \equiv 0$. Instead of (3,20) we have

$$\{g_k\}_p [(p, \lambda) - \lambda_k] + \{Y_k\}_p =$$

$$= \{\tilde{X}_k(g)\}_p + \sum_{j=l+2}^{n} \varepsilon_j (p_j + 1) \{g_k\}_{\tilde{p}(j)} - \sum_{\substack{\omega+\sigma=p \\ \omega \in \mathscr{M}_3 \\ \sigma \in \tilde{\mathscr{P}}(\lambda)}} \left(\sum_{j=1}^{l} p_j \lambda_j \right) \alpha_\sigma \{g_k\}_\omega -$$

$$- \sum_{\substack{\omega+\sigma=p \\ \omega,\sigma \in \mathscr{M}_2}} \left(\sum_{j=l+1}^{n} (\omega_j + 1) \{g_k\}_{\hat{\omega}(j)} \{Y_j\}_\sigma \right) \quad \text{for} \quad p \in \mathscr{M}_2 , \quad k = 1, 1, \ldots, l ,$$

$$\{Y_k\}_p = \{\tilde{X}_k(g)\}_p \quad \text{for} \quad p \in \mathscr{M}_2 , \quad k = l + 1, l + 2, \ldots, n .$$

Under the original assumption (Q_4) the implication (3,20) ⇒ (3,21) is false. (The author is indebted to A. D. BRJUNO who discovered this error.) Even by Theorem 2 of A. D. Brjuno from [2] divergence can occur in the case.

Theorem 3,4 is now a special case of Theorem 1 from [2]. (The remark at the beginning of sec. 3,4 in [1] concerns only [3], not [2].) Corollary (p. 59) is no more a direct consequence of Theorem 3,4, but it can be proved in a quite similar way as Theorem 3,4. (Under assumptions of this Corollary the relation (3,19′) holds also for $j = 1, 2, \ldots, l$ and the implication (3,20) ⇒ (3,21) in the original form is true.)

Finally let us note that the proofs of all results of A. D. Brjuno will be given in [4] and [5].

2. Addition. The following simple generalization of the well-known Cartan's Uniqueness Theorem is in a close connection with Theorem 4,2 A from [1] (pp. 66–67). The proof of Theorem 4,2 A could be based on it and on the method of L. Reich from [6] and [7].

In the following we make use of notations and conventions from [8], in particular of those introduced in chapters I–III.

Proposition. *Let* D *be a bounded domain in the space* C_n *of* n *complex variables and let* $\varrho_1, \varrho_2, \ldots, \varrho_n$ *be such complex numbers that*

$$1 = |\varrho_1| = |\varrho_2| = \ldots = |\varrho_m| > |\varrho_{m+1}| \geqq \ldots \geqq |\varrho_n| > 0 .$$

Then the mapping T

$$x_j' = \varrho_j x_j + [\textit{higher powers}] \quad (j = 1, 2, \ldots, n) \tag{1}$$

is formally similar to the mapping

$$\begin{aligned} y_j' &= \varrho_j y_j & (j = 1, 2, \ldots, m), \\ y_j' &= \varrho_j y_j + [\textit{higher powers}] & (j = m + 1, m + 2, \ldots, n), \end{aligned} \tag{2}$$

whenever T *maps* D *into* D.

Proof. By [6] any mapping (1) is formally similar to a mapping of the form

$$y_j = \varrho_j y_j + \sum_{|p| \geqq 2} \{V_j\}_p y^p = \varrho_j y_j + \sum_{r \geqq 2} \mathscr{V}_{j,r}(y) \quad (j = 1, 2, \ldots, n), \tag{3}$$

where $p = (p_1, p_2, \ldots, p_n)$, $|p| = p_1 + p_2 + \ldots + p_n$, $y^p = y_1^{p_1} y_2^{p_2} \ldots y_n^{p_n}$, $\mathscr{V}_{j,r}(y)$ is a polynomial consisting of all terms in (3) of the order r and $\{V_j\}_p = 0$ whenever $1 \leqq j \leqq m$ and $\varrho^p \neq \varrho_j$. (Certainly if $1 \leqq j \leqq m$ and $|p_{m+1}| + |p_{m+2}| + \ldots + |p_n| > 0$, then $\{V_j\}_p = 0$.)

Let us order the coefficients $\{V_j\}_p$ ($|p| \geqq 2$, $j = 1, 2, \ldots, m$) in the usual way. ($\{V_j\}_p \prec \{V_k\}_q$ iff the first nonzero number in the set $\{|q| - |p|, k - j, q_1 - p_1, \ldots, q_n - p_n\}$ is positive.) Let $\{V_k\}_q$ be the first unvanishing coefficient. Then there is a polynomial transformation U ($z_j' = z_j + \tilde{U}_j(z)$, $j = 1, 2, \ldots, n$, where $\tilde{U}_j$ are finite polynomials) such that $V = U^{-1}TU$ has the form

$$y_j' = \varrho_j y_j + [\textit{powers of degree higher than } |\mathbf{q}|] \qquad (j = 1, 2, \ldots, k-1),$$

$$y_k' = \varrho_k y_k + \{V_k\}_{\mathbf{q}} y^{\mathbf{q}} + \mathscr{W}_{k,\mathbf{q}}(y) + [\textit{higher powers}],$$

$$y_j = \varrho_j y_j + [\textit{powers of degree higher than } |\mathbf{q}| - 1] \quad (j = k+1, k+2, \ldots, m),$$

$$y_j = \varrho_j y_j + [\textit{higher powers}] \qquad (j = m+1, m+2, \ldots, n).$$

($\mathscr{W}_{k,\mathbf{q}}(y)$ is a polynomial of the variables $y_1, y_2, \ldots, y_m$ which contains terms of degree $|\mathbf{q}|$ and not preceding $\{V_k\}_{\mathbf{q}}$ in the given ordering.)

Let s be an arbitrary natural number and let the mapping $\mathbf{T}^s$ be given by

$$y_j^{(s)} = \sum_{|\mathbf{p}| \geqq 1} \{T_j^s\}_{\mathbf{p}} y^{\mathbf{p}} \quad (j = 1, 2, \ldots, n)$$

$(\mathbf{T}^2(y) = \mathbf{T}(\mathbf{T}(y)),\ \mathbf{T}^s(y) = \mathbf{T}(\mathbf{T}^{s-1}(y)))$.

Let $\mathbf{T}(\mathbf{D}) \subset \mathbf{D}$. Then given an arbitrary $\mathbf{p}$ with $|\mathbf{p}| \geqq 1$, there exists a real number $C_{\mathbf{p}}$ such that

$$|\{T_j^s\}_{\mathbf{p}}| \leqq C_{\mathbf{p}} \quad (j = 1, 2, \ldots, n;\ s = 1, 2, \ldots),$$

i.e. $\{\mathbf{T}^s\}$ $(s = 1, 2, \ldots)$ is weakly bounded. By [8] (I, §3, p. 12) $\{\mathbf{V}^s\} = \{\mathbf{U}^{-1}\mathbf{T}^s\mathbf{U}\}$ $(s = 1, 2, \ldots)$ is weakly bounded, too. But in $\mathbf{V}^2$

$$y_j^{(2)} = \varrho_j^2 y_j + [\textit{powers of degree higher than } |\mathbf{q}|] \quad (j = 1, 2, \ldots, k-1),$$

$$y_k^{(2)} = \varrho_k^2 y_k + \varrho_k \{V_k\}_{\mathbf{q}} y^{\mathbf{q}} + \varrho_k \mathscr{W}_{k,\mathbf{q}}(y) + \{V_k\}_{\mathbf{q}} \varrho^{\mathbf{q}} y^{\mathbf{q}} + \mathscr{W}_{k,\mathbf{q}}(\varrho_1 y_1, \ldots, \varrho_m y_m) +$$
$$+ [\textit{higher powers}] = \varrho_k^2 y_k + 2\varrho_k \{V_k\}_{\mathbf{q}} y^{\mathbf{q}} + \mathscr{W}_{k,\mathbf{q}}^{(2)}(y) + [\textit{higher powers}],$$

where $\mathscr{W}_{k,\mathbf{q}}^{(2)}$ contains only terms of degree $|\mathbf{q}|$ and not preceding $\{V_k\}_{\mathbf{q}}$. Generally in $\mathbf{V}^s$

$$y_k^{(s)} = \varrho_k^s y_k + s\varrho_k^{s-1}\{V_k\}_{\mathbf{q}} y^{\mathbf{q}} + \varrho_k^{s-1}\mathscr{W}_{k,\mathbf{q}}(y) + \varrho_k^{s-2}\mathscr{W}_{k,\mathbf{q}}(\varrho_1 y_1, \ldots, \varrho_m y_m) + \ldots$$
$$\ldots + \mathscr{W}_{k,\mathbf{q}}(\varrho_1^{s-1} y_1, \ldots, \varrho_m^{s-1} y_m) + [\textit{higher powers}] =$$
$$= \varrho_k^s y_k + s\varrho_k^{s-1}\{V_k\}_{\mathbf{q}} y^{\mathbf{q}} + \mathscr{W}_{k,\mathbf{q}}^{(s)}(y) + [\textit{higher powers}],$$

where $\mathscr{W}_{k,\mathbf{q}}^{(s)}$ contains only terms of degree $|\mathbf{q}|$ and not preceding $\{V_k\}_{\mathbf{q}}$. It is clear that the set $\{s\varrho_k^{s-1}\{V_k\}_{\mathbf{q}}\}$ $(s = 1, 2, \ldots)$ is bounded iff $\{V_k\}_{\mathbf{q}} = 0$.

References

[1] *M. Tvrdý:* The normal form and the stability of solutions of a system of differential equations in the complex domain. (Czech. Math. Journ. *20 (95)*, 1970, 39—73.)

[2] *А. Д. Брюно:* О рассходимости преобразований дифференциальных уравнений к нормальной фоме, (ДАН СССР *174* : 5, 1967, 1003—1006; English transl. Soviet Math. Dokl. *8* : 3, 1967, 692—695.)

[3] *А. Д. Брюно:* О сходимости преобразований дифференциальных уравнений к нормальной форме, (ДАН СССР *165* : 5, 1966, 987—989; English transl. Soviet Math. Dokl. *6*, 1965, 1536.)
[4] *А. Д. Брюно:* Аналитическая форма дифференциальных уравнений I. (Тр. Моск. Мат. Общ, *25*, 1971, 119—259.)
[5] *А. Д. Брюно:* Аналитическая форма дифференциальных уравнений II. (Тр. Моск. Мат. Общ, *26*, 1972.)
[6] *L. Reich:* Das Typenproblem bei formal-biholomorphen Abbildungen mit anziehendem Fixpunkt. (Math. Ann. *179*, 1969, 227—250.)
[7] *L. Reich:* Biholomorphe Abbildungen mit anziehendem Fixpunkt und analytische Differentialgleichungssysteme in Nähe einer Gleichgewichtslage. (Math. Ann. *181*, 1969, 163—172.)
[8] *S. Bochmer, W. T. Martin:* Several Complex Variables. (Princeton University Press, 1948.)

Author's address: Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV v Praze).