ANALYTIC APPROXIMATIONS OF UNIFORMLY CONTINUOUS OPERATORS

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Abstract. It is shown that if a separable real Banach space admits a separating analytic function with an additional property (K) (concerning uniform behavior of radiuses of convergence) then every uniformly continuous operator into any real Banach space can be approximated by analytic operators. In particular, the result applies to c_0.

In our present note, we present a modification of the following theorem due to Kurzweil [K]:

Theorem 1. *Let X be a real separable Banach space which admits a separating polynomial. Then for every real Banach space Y, every continuous (nonlinear) operator $F\colon X \to Y$ and $\varepsilon > 0$ there exists a real analytic operator $H\colon X \to Y$ such that*

$$\sup_{x\in X} |H(x) - F(x)| < \varepsilon.$$

More precisely, we replace the assumption of a separating polynomial by that of a separating analytic function with uniformly bounded (from below) radius of convergence at each point (condition (K)). This is a strictly weaker condition, but the class of approximated operators is limited to uniformly continuous ones. In particular, every uniformly continuous operator from c_0 is uniformly approximable by analytic operators, but the general case of continuous operators remain open. The space c_0 is a critical example in this context. Recall that by a result of Deville ([D]), every Banach space admitting a separating C^∞ Fréchet smooth function, which does not contain a copy of c_0 admits a separating polynomial. Thus the impossibility of approximations of all continuous functions (operators) on c_0 would turn Theorem 1 into a characterization.

In the second part of the note we investigate the nonsuperreflexive spaces satisfying condition (K). We show that any Banach space with DP (Dunford-Pettis) property satisfying condition (K) is isomorphic to a subspace of c_0. More generally, this is true for every Banach space on which all scalar polynomials are weakly sequentially continuous.

The main result of this paper, Theorem 3, was obtained independently by both authors. We have realized that at a Conference on Infinite Dimensional Analysis, held in Madrid

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in December 1997. In order to avoid unnecessary duplicity, we decided to publish a joint paper. The second part of the note was influenced by some discussions with Gilles Godefroy, to whom we would like to express our thanks. We would also like to thank the Department of Analysis of Universidad Complutense de Madrid for excellent working conditions and for organizing the meeting.

General background on smoothness in Banach spaces can be found in [DGZ]. We refer to [AO] for facts on real analytic functions on real Banach spaces and their complex extensions. Given a real Banach space $(X, \|\cdot\|)$ we denote by $(\widetilde{X}, \|\cdot\|)$ the complexified version of X, which is (as a real space) isomorphic to $X \oplus X$. Given a real analytic operator $q(x)$ defined on some open subset $\mathcal{O} \subset X$ with values in Y, there exists some open (complex) neighbourhood $\mathcal{U}$ of $\mathcal{O}$ in $\widetilde{X}$, and a unique complex analytic operator $\tilde{q}\colon \mathcal{U} \to \widetilde{Y}$ such that $\tilde{q}|_{\mathcal{O}} = q$.

Given $x \in X, d > 0$ we denote by $\widetilde{B}(x,d) = \{x \in \widetilde{X}; \|x - z\| \le d\}$.

For the convenience of the reader we try to keep a similar notation to that in [K].

Definition 2. Let X be a separable real Banach space. We say that X *satisfies the condition* (K) if there exists a real analytic function q on X such that the following conditions are satisfied:

(i) $q \ge 0$ on X;
(ii) $q^{-1}([0,1])$ is a bounded open neighbourhood of the origin;
(iii) there exists $d > 0$ such that for each $x \in X$, the Taylor series of $\tilde{q}$ converges uniformly on $\widetilde{B}(x,d) \subset \widetilde{X}$ to $\tilde{q}$.

Theorem 3. *Let X be a separable real Banach space with property (K). Let Y be an arbitrary real Banach space, $F\colon X \to Y$ be a uniformly continuous operator, and $\varepsilon > 0$. Then there exists a real analytic operator $H\colon X \to Y$ such that $\sup_{x\in X} \|F - H\| \le \varepsilon$.*

Proof. Let us assume that the function q on X and $d > 0$ verify the property (K) of X. Let us start by introducing some notation:

Given $y \in X, r > 0$ we define sets:

$$K(y,r) = \{x \in X; \quad q(x-y) < r\},$$
$$C(y,r) = \{x \in X; \quad q(x-y) > r\}.$$

By standard scaling arguments, we may assume that $\varepsilon = 1$ and $x \in K\left(y, \frac{1}{2}\right)$ implies $\|F(x) - F(y)\| < \frac{1}{4}$.

Choose sequences $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ such that $\bigcup_{i\in\mathbb{N}} K\left(x_i, \frac{1}{4}\right) = X$ and $\bigcup_{i\in\mathbb{N}} B\left(y_i, \frac{d}{2}\right) = X$. Set $S_{l,m} = K\left(x_l, \frac{1}{4}\right) \cap \bigcup_{i=1}^{m} B\left(y_i, \frac{d}{2}\right)$.

It is easy to see that $\bigcup_{m\in\mathbb{N}} S_{l,m} = K\left(x_l, \frac{1}{4}\right)$. Given $n \in \mathbb{N}$,

$$V_n = \sup\{|\tilde{q}(y_i + z - x_j)| + 1; \quad z \in \widetilde{B}(0,d), \quad 1 \le i,j \le n\}$$

exists and is finite.

Choose a sequence $\varepsilon_n \searrow 0, \varepsilon_1 < \frac{1}{20}$, and define the open coverings $\{D_n\}_{n\in\mathbb{N}}$, $\{D_n^*\}_{n\in\mathbb{N}}$ of X as follows:

$$D_n = \bigcap_{i=1}^{n-1} C\left(x_i, \frac{1}{4} - \varepsilon_n\right) \cap K\left(x_n, \frac{1}{4}\right)$$
$$D_n^* = \bigcap_{i=1}^{n-1} C\left(x_i, \frac{1}{4} - 3\varepsilon_n\right) \cap K\left(x_n, \frac{1}{4} + 2\varepsilon_n\right).$$

Both of the above coverings are easily shown to be locally finite, $D_n \subset D_n^*$, $n \in \mathbb{N}$.

We proceed by defining subsets $T_n \subset \mathbb{R}^n$ and analytic functions $\tilde{\phi}_n$ on $X + \widetilde{B}(0,d) \subset \widetilde{X}$, $n \in \mathbb{N}$ as follows:

$$T_n = \left\{[\tau_1, \dots, \tau_n]; \frac{1}{4} - 2\varepsilon_n \le \tau, \text{ for } i < n \text{ and } -1 \le \tau_n \le \frac{1}{4} + \varepsilon_n\right\}$$
$$\phi_n(z) = (\|F(x_n)\| + 1) \cdot \nu_n \cdot \int\limits_{T_n} \cdots \int e^{-t_n\left(\sum\limits_{i=1}^{n} a_i(\tilde{q}(z-x_i)-\tau_i)^2\right)} d\tau_1 \dots d\tau_n,$$

where ν_n is the normalizing factor:

$$\frac{1}{\nu_n} = \int\limits_{\mathbb{R}^n} \cdots \int e^{-t_n\left(\sum\limits_{i=1}^{n} a_i\tau_i^2\right)} d\tau_1 \dots d\tau_n.$$

We proceed by choosing the values $a_i, t_i > 0$.

The values a_i are chosen so that

$$a_i < \frac{1}{2^i \cdot 9 \cdot V_i^2}. \tag{0}$$

The values t_i are chosen sufficiently large for the following conditions to be satisfied:

$$(\|F(x_n)\| + 1)^2 e^{-t_n \cdot \frac{1}{n}} < \frac{1}{2^n} \tag{1}$$
$$|\phi_n(x) - \|F(x_n)\| - 1| < \frac{1}{2} \text{ for } x \in D_n \tag{2}$$
$$|\phi_n(x)| < \frac{1}{2^{n+3}(\|F(x_n) + 1)} \text{ for } x \notin D_n^*. \tag{3}$$

Condition (1) is clear.

To see (2) and (3), it is enough to realize that

$$\{[q(x-x_1),\dots,q(x-x_n)];\quad x\in D_n\}+\{(\alpha_1,\dots,\alpha_n);|\alpha_i|\le\varepsilon_n\}\subset T_n,$$
$$T_n\cap\left\{\{[q(x-x_1),\dots,q(x-x_n)];\quad x\notin D_n^*\}+\left\{(\alpha_1,\dots,\alpha_n);|\alpha_i|\le\frac{\varepsilon_n}{2}\right\}\right\}=\emptyset,$$

and

$$\lim_{t_n\to\infty}\frac{\int\limits_{\mathbb{R}^n}\cdots\int^{-t_n\left(\sum\limits_{i=1}^{n}a_i\tau_i^2\right)}d\tau_1\dots d\tau_n}{\int\limits_{[-\frac{\varepsilon_n}{2},\frac{\varepsilon_n}{2}]^n}\cdots\int e^{-t_n\left(\sum\limits_{i=1}^{n}a_i\tau_i^2\right)}d\tau_1\dots d\tau_n}=1.$$

Let

$$\phi(x)=\sum_{i=1}^{\infty}\phi_i(x)$$
$$H^*(x)=\sum_{i=1}^{\infty}F(x_i)\phi_i(x).$$

We proceed by proving that ϕ and H^* are real analytic on X. As a uniform limit of complex analytic functions is complex analytic, this follows from the next statement. For every $x_0\in X$ there exists $\delta>0$ and $n_0\in\mathbb{N}$ such that

$$(\|F(x_n)\|+1)|\phi_n(x_0+z)|\le\frac{1}{2^n}\tag{4}$$

whenever $n>n_0, z\in\widetilde{B}(0,\delta)$.

Fix a point x_0. There exist $j_0,m_0\in\mathbb{N}$ such that $x_0\in S_{j_0,m_0}$. Thus for some $\alpha>0$ and $n'\in\mathbb{N}$:

$$\left(\frac{1}{4}-2\varepsilon_{n'}\right)-q(x_0-x_{j_0})>\alpha.\tag{5}$$

We need the following estimates.

Let $\xi\in\mathbb{C},\xi=\xi_1+\imath\xi_2,\xi_1,\xi_2\in\mathbb{R}$, $n,j\in\mathbb{N}$. Then

$$\begin{aligned}\int\limits_{\mathbb{R}}|e^{-t_na_j(\xi-\tau)^2}|d\tau&\le\int\limits_{\mathbb{R}}|e^{-t_na_j[(\xi_1-\tau)^2-\xi_2^2+2\xi_2(\xi_1+\tau)\imath]}|d\tau\\&\le e^{t_na_j\cdot\xi_2^2}\cdot\int\limits_{\mathbb{R}}e^{-t_na_jr^3}\ d\tau.\end{aligned}\tag{6}$$

Let $i>\max\{j_0,m_0,n'\}, z\in\widetilde{B}\left(0,\frac{d}{2}\right)$. Then $|\tilde{q}(x_0+z-x_i)|<V_i$, so by (6):

$$\int\limits_{\mathbb{R}}|e^{-t_na_i(\tilde{q}(x_0+z-x_i)-\tau)^2}|d\tau\le e^{t_na_iV_i^2}\cdot\int\limits_{\mathbb{R}}e^{-t_na_i\tau^2}\ d\tau.\tag{7}$$

Finally, by (5) choose $\frac{d}{2} > \delta_1 > 0$, small enough in order to have

$$\begin{aligned}\left(\frac{1}{4} - 2\varepsilon_{n'}\right) - \mathrm{Re}\ \tilde{q}(x_0 + z - x_{j_0}) \geq \alpha \\ |\mathrm{Im}\ \tilde{q}(x_0 + z - x_{j_0})| < \frac{\alpha}{2}\end{aligned} \tag{8}$$

whenever $z \in \widetilde{B}(0, \delta_1)$. Thus

$$\begin{aligned}\int\limits_{\frac{1}{4}-2\varepsilon_{n'}}^{\infty} |e^{-t_n a_{j_0}(\tilde{q}(x_0+z-x_{j_0})-\tau)^2}|d\tau &\leq \int\limits_{0}^{\infty} e^{-t_n a_{j_0}[(\alpha+\tau)^2 - \frac{\alpha^2}{5}]}d\tau \\ &\leq e^{-t_n a_{j_0}\frac{\alpha^2}{2}} \int\limits_{\mathbb{R}} e^{-t_n a_{j_0}\tau^2}\ d\tau.\end{aligned} \tag{9}$$

By (0) there exist $n_0 > \max\left\{j_0, m_0, n', \frac{4}{a_{j_0}\cdot\alpha^2}\right\}$ such that

$$\sum_{i=n_0}^{\infty} a_i V_i^2 < \frac{a_{j_0}\alpha^2}{8}. \tag{10}$$

By continuity, there exists $0 < \delta < \delta_1$ such that

$$\sum_{j=1}^{n_0-1} a_j [\mathrm{Im}(\tilde{q}(x_0 + z - x_j))]^2 < \frac{a_{j_0}\alpha^2}{8} \tag{11}$$

whenever $z \in \widetilde{B}(0, \delta)$.

Consequently, by (6)–(11):

$$\begin{aligned}&|\phi_n(x_0 + z)| \leq \\ &\leq (\|F(x_n)\| + 1) \cdot \prod_{\substack{i=1 \\ i\neq j_0}}^{n} \frac{\int_{\mathbb{R}} |e^{-t_n a_i(\tilde{q}(x_0+z-x_i)-\tau_i)^2}|d\tau_i}{\int_{\mathbb{R}} e^{-t_n a_i \tau_i^2} d\tau_i} \cdot \frac{\int_{\frac{1}{4}-2\varepsilon_n}^{\infty} |e^{-t_n a_{j_0}(\tilde{q}(x_0+z-x_j)-\tau_{j_0})^2}|d\tau_{j_0}}{\int_{\mathbb{R}} e^{-t_n a_{j_0}\tau_{j_0}^2} d\tau_{j_0}} \\ &\leq (\|F(x_n)\| + 1) \cdot \prod_{\substack{i=1 \\ i\neq j_0}}^{n_0-1} e^{t_n a_i(\mathrm{Im}\ \tilde{q}(x_0+z-x_i))^2} \cdot \prod_{i=n_0}^{n} e^{t_n a_i V_i^2} \cdot e^{-t_n a_{j_0}\frac{\alpha^2}{2}} \leq \\ &\leq (\|F(x_n)\| + 1)e^{-t_n a_{j_0}\frac{\alpha^2}{4}}.\end{aligned} \tag{12}$$

whenever $n \geq n_0, z \in \widetilde{B}(0, \delta)$.

Condition (1) and the choice of n_0 immediately imply (4).

We proceed by showing that the analytic operator $H(x) = \frac{H^*(x)}{\phi(x)}$ satisfies $\sup_{x\in X} \|F - H\| > 1$. Fix $x \in X$

$$F(x) - H(x) = F(x)\sum_{i=1}^{\infty} \frac{\phi_i(x)}{\phi(x)} - \sum_{i=1}^{\infty} \frac{F(x_i)\phi_i(x)}{\phi(x)}$$
$$= \frac{1}{\phi(x)} \sum_{i=1}^{\infty} (F(x)\phi_i(x) - F(x_i)\phi_i(x)).$$

Set $I_1 = \{i; x \in D_i^*\}, I_2 = \{i; x \notin D_i^*\}$.

Then

$$\|F(x)-H(x)\| \le \frac{1}{\phi(x)} \sum_{i\in I_1} \|F(x)-F(x_i)\|\phi_i(x) + \frac{\|F(x)\|}{\phi(x)} \sum_{i\in I_2} \phi_i(x) + \frac{1}{\phi(x)} \sum_{i\in I_2} \|F(x_i)\|\phi_i(x).$$

If $i \in I_1$ then $x \in K\left(x_i, \frac{1}{2}\right)$ and

$$\|F(x) - F(x_i)\| < \frac{1}{4}.$$

Moreover, $x \in D_l$ for some l, so by (2)

$$\|F(x) - F(x_l)\| < \frac{1}{4}, \quad \phi_l(x) > \|F(x_l)\| + \frac{1}{2}$$

and

$$\phi(x) \ge \phi_l(x) > \|F(x)\|, \quad \phi(x) > \frac{1}{2}.$$

The above inequalities together with (3) imply

$$\|F(x) - H(x)\| \le \frac{1}{4} + \frac{1}{8} + \frac{2}{8} < 1.$$

$\diamondsuit$

It is perhaps worth noticing that the same proof as above yields the following conclusion: *Let X be a separable Banach space satisfying $(K), \{x_n\}_{n\in\mathbb{N}}$ be an ε-separated sequence, $\varepsilon > 0$ and $F\colon \{x_n\}_{n\in\mathbb{N}} \to Y$ be an arbitrary function. Then for every $\delta > 0$ there exists an analytic function $H\colon X \to Y$ such that $\sup_n |F(x_n) - H(x_n)| < \delta$.*

Examples.

It is well-known that c_0 does not admit a separating polynomial ([B]). However, the space c_0 satisfies the condition (K). Indeed, the function $q(x) = \sum_{i=1}^{\infty} x_i^{2i}$ constructed in [FPWZ] is easily shown to satisfy (K), where $d<1$. Clearly, condition (K) is inherited by subspaces and finite direct sums. Under suitable circumstances, it can also pass to infinite direct sums. Assume all members of a sequence $\{(X_n,\|\cdot\|)_n\}$ satisfy the condition (K) with q_n and d_n. Suppose $d=\inf_{n\in\mathbb{N}} d_n>0$, there exists a sequence $\alpha_n\in\mathbb{N}$ such that $\sup_{n\in\mathbb{N}}|\tilde{q}_n(\widetilde{B}(0,d))|^{\alpha_n}<1$ and $\sup_{n\in\mathbb{N}}\operatorname{diam} q_n^{-1}([0,1]) < +\infty$. Then $\left(\sum_{n=1}^{\infty}\oplus X_n\right)_{c_0}$ satisfies the condition (K) with $q((x_1,x_2,\dots))=\sum_{n=1}^{\infty} q_n^{2n\alpha_n}(x_n)$ and $\frac{d}{2}$. Thus for example $\left(c_0\oplus\sum_{n=1}^{\infty}\oplus\ell_{2n}\right)_{c_0}$ satisfies (K).

By the result of Deville [D], a space admitting a separating C^∞ Frećhet smooth function (in particular every space satisfying (K)) is saturated by spaces from $\{\ell_p,\text{p-even}\}\cup\{c_0\}$. While a full classification of spaces satisfying (K) is probably hopeless, there may be a chance for the extremal cases when either $c_0\not\hookrightarrow X$ or $\ell_p\not\hookrightarrow X$ for every p-even. The first case leads to spaces with a separating polynomial ([D]). We devote the rest of the paper to the investigation of the second case.

Definition 4. We say that X *satisfies condition (S)* if there exists a continuous function $b(x)\geq 0$ on $X, b^{-1}([0,1))$ is a bounded open neighbourhood of the origin, and $\alpha>0$ such that for every $x_0\in X$ and every $x_n\xrightarrow{w}0$, $\|x_n\|\leq\alpha$, we have $b(x_0+x_n)\to b(x_0)$.

Lemma 5. *Let X satisfy condition (S). Then there exists an equivalent norm $|||\cdot|||$ on X, and $\beta>0$ such that for each $x_0\in S_{(X,|||\cdot|||)}$, $x_n\xrightarrow{w}0$, $|||x_n|||\leq\beta$, we have $|||x_0+x_n|||\to 1$.*

Proof. The proof is a variant of an argument in [FWZ]. Choose $\phi(t)$ an increasing continuous function on $[0,1),\phi(0)=0$, $\lim_{t\to 1}\phi(t)=+\infty$. Put $\tilde{b}(x)=\phi(b(x))$ for $x\in b^{-1}([0,1))$ and $\tilde{b}(x)=+\infty$ otherwise. Given $x_0\in b^{-1}([0,1))$ and $x_n\xrightarrow{w}0$, $\|x_n\|\leq\alpha$, we have again $\tilde{b}(x_0+x_n)\to\tilde{b}(x_0)$. Define

$$U(x)=\inf\left\{\sum_{i=1}^{k}\xi_i\tilde{b}(x_i),\xi_i\geq 0,\sum_{i=1}^{k}\xi_i=1,\sum_{i=1}^{k}\xi_i x_i=x\right\}.$$

It is standard to check that $U(x)$ is finite and convex on an open set $C=\operatorname{conv}(b^{-1}([0,1)))$, and $U(x)=+\infty$ on $X\backslash C$.

Fix any $x_0\in C, x_n\xrightarrow{w}0,\|x_n\|\leq\alpha,\varepsilon>0$. There exist $\xi_i\geq 0,\sum_{i=1}^{k}\xi_i=1$, $y_i\in b^{-1}([0,1))$, $\sum_{i=1}^{k}\xi_i y_i=x_0$ such that $U(x_0)+\varepsilon\geq\sum_{i=1}^{k}\xi_i\tilde{b}(y_i)$. Also

$$\lim_{n\to\infty}\sum_{i=1}^{k}\xi_i\tilde{b}(y_i+x_n)=\sum_{i=1}^{k}\xi_i\tilde{b}(y_i)\leq U(x_0)+\varepsilon,$$

so $\limsup_{n\to\infty}U(x_0+x_n)\leq\limsup_{n\to\infty}\sum_{i=1}^{k}\xi_i\tilde{b}(y_i+x_n)\leq U(x_0)+\varepsilon$ for any $\varepsilon>0$. Thus $\limsup_{n\to\infty}U(x_0+x_n)\leq U(x_0)$. The same argument applies to a sequence $\{-x_n\}$, so $\limsup_{n\to\infty}$

$U(x_0 - x_n) \leq U(x_0)$. For convexity of U we finally obtain

$$\lim_{n\to\infty} U(x_0 + x_n) = U(x_0).$$

Putting $|||\cdot|||$ to be the Minkowski functional of $U^{-1}([0,1])$ we obtain an equivalent norm on X satisfying $|||x_0 + x_n||| \to 1$ whenever $x_0 \in S_{(X,|||\cdot|||)}$ and $x_n \xrightarrow{w} 0$, $\|x_n\| \leq \alpha$. Lastly, we choose $\beta > 0$ such that $\beta B_{(X,|||\cdot|||)} \subset \alpha B_{(X,\|\cdot\|)}$. ♢

Lemma 6. *Let $(X, \|\cdot\|)$ be a separable Banach space, $\ell_1 \not\hookrightarrow X$. Suppose there exists $\alpha > 0$, such that for every $x_0 \in S_X$, $x_n \xrightarrow{w} 0$, $\|x_n\| \leq \alpha$ we have $\|x_0 + x_n\| \to 1$. Then X is isomorphic to a subspace of c_0.*

Proof. By [GKL] (see also [KW]) it is enough to show that $\|\cdot\|$ is c-Lipschitz UKK^* norm for some $c \in (0,1]$. More precisely, we need to show that $\|\cdot\|^*$ satisfies the following:

$$\|x^*\| + c \limsup_{n\to\infty} \|x_n^*\| \leq \limsup_{n\to\infty} \|x^* + x_n^*\| \text{ whenever } x^* \in X^* \text{ and } x_n^* \xrightarrow{w^*} 0. \tag{13}$$

It is easy to see that it suffices to prove (13) for all x^* from a dense subset of S_{X^*}, in particular (using the Bishop-Phelps theorem) for all norm attaining functionals from S_{X^*}. Let x^* be such a functional, $x^*(x) = 1$, $x \in S_X$ and $x_n^* \xrightarrow{w^*} 0$. We may in addition assume that $\lim \|x^* + x_n^*\|$ and $\lim \|x^*\| = L$ exist. There exists a sequence $\{x_n\} \subset B_X$ such that $x_n^*(x_n) > \frac{5}{6}L$. By Rosenthal's theorem, we may assume that $\{x_n\}$ is weakly Cauchy, and using the fact that $x_n^* \xrightarrow{w^*} 0$, we can pass to yet another subsequence of x_n^* and replace x_n by $y_n = \frac{1}{2}(x_{n_1} - x_{n_2})$, where $n_1 \neq n_2$, $n_1, n_2 \to \infty$ as $n \to \infty$. Thus finally we may assume that $x_n^*(y_n) > \frac{1}{3}L$, where $y_n \xrightarrow{w} 0$, $y_n \in B_X$.

We obtain the following estimate:

$$\begin{aligned}
\limsup_{n\to\infty} \|x^* + x_n^*\| &\geq \limsup_{n\to\infty} \langle x^* + x_n^*, x + \alpha y_n)\rangle \\
&\geq \limsup_{n\to\infty} x^*(x) + x_n^*(x) + \alpha x^*(y_n) + \alpha x_n^*(y_n) \\
&\geq 1 + \frac{1}{3}\alpha L = \|x^*\| + \frac{1}{3}\alpha \limsup_{n\to\infty} \|x_n^*\|.
\end{aligned}$$

♢

Proposition 7. *Let X be a Banach space satisfying (K). Suppose all scalar polynomials on X map weakly null sequences into sequences convergent to zero. Then X is isomorphic to a subspace of c_0.*

Proof. It was proved in [AHV] that given a Banach space X, every scalar polynomial maps weakly null sequences into sequences convergent to zero if and only if every scalar polynomial maps weakly Cauchy sequences into norm convergent ones. Thus by results of [HT], X is a separable Asplund space. The function $q(\cdot)$ on X from the definition of

(K) is a uniform limit of polynomials on each $B(x,d)$, so in particular for every $x \in X$, $q(x+y_n) \to q(x)$ whenever $y_n \xrightarrow{w} 0$, $\|y_n\| \le d$. Lemma 6 finishes the proof. $\diamondsuit$

By a result of [P] or [R], spaces with DP property (in particular all $C(K)$ spaces and all subspaces of c_0) satisfy the above condition on sequential continuity of polynomials.

It can be shown ([HT]) that replacing the assumption (K) in Proposition 7 by an existence of a separating analytic function on some open bounded set in X, one obtains that X is a separable polyhedral space.

Recall a result of [LP] by which every $C(K)$ space which is isomorphic to a subspace of c_0 is isomorphic to c_0. Thus we have the following:

Corollary 8. *Let $X \cong C(K)$ satisfy condition (K). Then X is isomorphic to c_0.*

This Corollary should be compared with [DFH] where it is shown that every separable polyhedral Banach space (e.g. $C(K)$ where K is scattered) admits a separating analytic convex function defined on some bounded convex open set.

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